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Paths in quantum Cayley trees and L^2-cohomology

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Abstract

We study existence, uniqueness and triviality of path cocycles in the quantum Cayley graph of universal discrete quantum groups. In the orthogonal case we find that the unique path cocycle is trivial, in contrast with the case of free groups where it is proper. In the unitary case it is neither bounded nor proper. From this geometrical result we deduce the vanishing of the first L^2-Betti number of $A_o(I_n)$.

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1. Introduction

Consider the unital $*$-algebras $\mathscr{S} = \mathscr{A}_o(I_n)$, $\mathscr{A}_u(I_n)$ introduced by Wang [24] and defined by n^2 generators u_{ij} and relations as follows:

$$\mathscr{A}_u(I_n) = \langle u_{ij} \mid (u_{ij})_{ij} \text{ and } (u_{ij}^*)_{ij} \text{ are unitary} \rangle,$$
$$\mathscr{A}_o(I_n) = \mathscr{A}_u(I_n) / \langle u_{ij} = u_{ij}^* \text{ for all } i, j \rangle.$$

These algebras naturally become Hopf algebras with coproduct given by $\delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$. It results from the general theory of Woronowicz [25] that they admit unique Haar integrals $h : \mathscr{S} \to \mathbb{C}$. We denote by H the corresponding Hilbert space completions, and by S_{red} the operator norm completions of the images of the natural "regular" representations $\lambda : \mathscr{S} \to B(H)$.

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Since the introduction of $\mathscr{A}_o(I_n)$ and $\mathscr{A}_u(I_n)$ and the seminal work of Banica [4,3] on their theory of corepresentations, strong relations and similarities with the free group algebras $\mathbb{C}F_n$ have been noted:

– canceling the generators u_{ij} with $i \neq j$ yields natural quotient maps $\mathscr{A}_u(I_n) \to \mathbb{C}F_n$, $\mathscr{A}_o(I_n) \to \mathbb{C}(\mathbb{Z}/2\mathbb{Z})^{*n}$,
– any Hopf algebra $\mathscr{S}$ associated with a unimodular "discrete quantum group" (resp.: and with self-adjoint generating corepresentation) is a Hopf quotient of some $\mathscr{A}_u(I_n)$ (resp. $\mathscr{A}_o(I_n)$),
– the irreducible corepresentations of $\mathscr{A}_u(I_n)$ are naturally indexed by the elements of the free monoid on two generators $u, \bar{u}$.

Due to these remarks, the discrete quantum groups associated with $\mathscr{A}_o(I_n)$ and $\mathscr{A}_u(I_n)$ have sometimes been called "universal", or even "free".

The analogy with free groups led in subsequent works to non-trivial C^*-algebraic results showing that $A_o(I_n)_{\mathrm{red}}$ and $A_u(I_n)_{\mathrm{red}}$, $n \geqslant 2$, indeed share important global analytical properties with free groups:

– the C^*-algebra $A_u(I_n)_{\mathrm{red}}$ is simple, the C^*-algebras $A_u(I_n)_{\mathrm{red}}$ and $A_o(I_n)_{\mathrm{red}}$ are non-nuclear (if $n \geqslant 3$) [4],
– the discrete quantum groups associated with $\mathscr{A}_u(I_n)$, $\mathscr{A}_o(I_n)$ satisfy the Akemann–Ostrand Property [21],
– the discrete quantum groups associated with $\mathscr{A}_u(I_n)$, $\mathscr{A}_o(I_n)$ have the Property of Rapid Decay [22],
– the C^*-algebra $A_o(I_n)_{\mathrm{red}}$ is simple (if $n \geqslant 3$) and exact, the von Neumann algebra $A_o(I_n)''_{\mathrm{red}}$ is a full and prime II_1 factor [18],
– the von Neumann algebra $A_u(I_n)''_{\mathrm{red}}$ is a full and prime II_1 factor [17].

The starting point of most results in [21,18,17] is the construction of quantum analogues of geometrical objects associated with free groups, namely the quantum Cayley graphs and the quantum boundaries of the discrete quantum groups associated with $\mathscr{A}_o(I_n)$, $\mathscr{A}_u(I_n)$.

In the present paper, we pursue the study of the quantum Cayley graphs of $\mathscr{A}_o(I_n)$ and $\mathscr{A}_u(I_n)$, concentrating in particular on paths to the origin, which yield in good cases path cocycles, and we derive applications to the L^2-cohomology of the algebras $\mathscr{A}_o(I_n)$, $\mathscr{A}_u(I_n)$. However, for the first time the results we obtain for $\mathscr{A}_o(I_n)$ strongly contrast with the known results for free groups. More precisely:

– on the geometrical side, we find out in Theorem 4.4 that the path cocycle of $\mathscr{A}_o(I_n)$ is trivial, hence bounded, whereas the path cocyle on F_n is unbounded, and even proper,
– on the cohomological and analytical side, we prove in Theorem 5.1 that the first L^2-cohomology group $H^1(\mathscr{A}_o(I_n), M)$ vanishes, and in particular we have for the first L^2-Betti number $\beta_1^{(2)}(\mathscr{A}_o(I_n)) = 0$ for all n, whereas $\beta_1^{(2)}(F_n) = n - 1$ and $\beta_1^{(2)}((\mathbb{Z}/2\mathbb{Z})^{*n}) = \frac{n}{2} - 1$.

In the case of $\mathscr{A}_u(I_n)$, we obtain a path cocycle which is not bounded, but also not proper. We believe that these results open a new perspective in the study of universal discrete quantum groups. In particular, they put a new emphasis on the question of a-T-menability for $\mathscr{A}_o(I_n)$, which remains open.[1]

Recall that in the classical case a-T-menability, also called Haagerup's Property, has important applications, in particular in connection with K-theory, and amounts to the existence of a metrically proper π-cocycle for some unitary representation π [7]. The results in the present paper show that, in contrast with the case of free groups, such a cocycle does not exist on $\mathscr{A}_o(I_n)$ if π is contained in (a multiple of) the regular representation.

Finally, let us mention a recent work of C. Voigt [23] about the Baum–Connes conjecture and the K-theory of $A_o(I_n)$, which also presents the feature that "invariants do not depend on n": namely it is shown that $K_0(A_o(I_n)) = K_1(A_o(I_n)) = \mathbb{Z}$. It is also proved there that $A_o(I_n)$ is K-amenable, a new property shared with free groups.

Let us describe the strategy and the organization of the paper. After recalling some notation about discrete quantum groups and classical Cayley graphs, we introduce in Section 2 the notion of path cocycle for discrete quantum groups and prove some elementary facts about bounded cocycles and path cocycles. Then we continue the study of quantum Cayley trees started in [21]. We introduce in Section 3 a "rotation operator" and "quasi-classical" subspaces, and we give a new description of the space of geometrical edges.

In Section 4 we prove the existence and uniqueness of path cocycles in quantum Cayley trees and we study their triviality. We prove in particular the triviality of the path cocycle on $\mathscr{A}_o(I_n)$ for $n \geqslant 3$, and we observe that the path cocycle on $\mathscr{A}_u(I_n)$ is unbounded. Finally a "universality trick" allows us to prove in Section 5 that in fact the whole L^2-cohomology group $H^1(\mathscr{S}, M)$ vanishes in the orthogonal case.

The reader interested in the application to L^2-cohomology will probably like to skip the technical Section 3 at first, although it cannot be completely avoided from a logical point of view. Let us also note that the present article, except Section 3, relies only on Sections 3 and 4 of [21]. The results of this paper about $\mathscr{A}_o(I_n)$ have been announced at the Workshop on operator algebraic aspects of quantum groups held in Leuven in November 2008. It is a pleasure to thank the organizer, Stefaan Vaes.

1.1. Notation

Let us start from a regular multiplicative unitary $V \in B(H \otimes H)$ with a unique fixed line — i.e. V is of compact type without multiplicity [2]. Our "classical case" will be the case of the multiplicative unitary associated with a discrete group Γ: then $H = \ell^2\Gamma$ and $V = \sum_{g\in\Gamma} \mathbb{1}_g \otimes \lambda_g$, where $\mathbb{1}_g$ is the characteristic function of $\{g\}$ acting by pointwise multiplication on H, and λ_g is the left translation by g in H. Note that all tensor products between Hilbert spaces (resp. C^*-algebras, Hilbert C^*-modules) are completed with respect to the hilbertian tensor norm (resp. the minimal C^*-tensor norm).

A number of more familiar objects can be recovered from V. First of all we have the Woronowicz C^*-algebra $(S_{\mathrm{red}}, \delta)$ [25] and the completion $(\hat{S}, \hat{\delta})$ of the dual multiplier Hopf algebra [19]:

$$S_{\mathrm{red}} = \overline{\mathrm{Span}}\big\{(\omega \otimes \mathrm{id})(V) \;\big|\; \omega \in B(H)_*\big\}, \qquad \delta(x) = V(x \otimes 1)V^*;$$

[1] Note (January 2012). Since the writing of this article, a-T-menability of $\mathscr{A}_o(I_n)$ and $\mathscr{A}_u(I_n)$ has been established by Brannan [6].

$$\hat{S} = \overline{\mathrm{Span}}\{(\mathrm{id} \otimes \omega)(V) \mid \omega \in B(H)_*\}, \qquad \hat{\delta}(a) = V^*(1 \otimes a)V.$$

In the classical case we recover $S_{\mathrm{red}} = C^*_{\mathrm{red}}(\Gamma)$ and $\hat{S} = c_0(\Gamma)$ with their usual coproducts, hence the group Γ itself. In general we have $V \in M(\hat{S} \otimes S_{\mathrm{red}})$.

The theory of corepresentations of $(S_{\mathrm{red}}, \delta)$ or, equivalently, the C^*-algebra structure of $\hat{S}$, is well understood via Peter–Weyl and Tannaka–Krein type results [25]. In particular, $\hat{S}$ is a c_0-sum of matrix algebras and we denote by $\mathscr{C}$ the category of finite-dimensional representations of $\hat{S}$. It is naturally endowed with a monoidal structure for which every object α has a dual $\bar{\alpha}$. We denote by $1_{\mathscr{C}} \in \mathrm{Irr}\,\mathscr{C}$ the trivial representation.

In the classical case, the irreducible representations of $\hat{S}$ have dimension 1 and correspond naturally to the elements of Γ so that the monoidal structure (resp. the duality operation, the trivial representation) identifies to the group structure (resp. the inverse of the group, the unit $e \in \Gamma$).

For $\alpha \in \mathrm{Irr}\,\mathscr{C}$ we denote by $p_\alpha \in \hat{S}$ the central support of α, so that $p_\alpha \hat{S} \simeq L(H_\alpha)$ and $\hat{S} = c_0\text{-}\bigoplus_\alpha p_\alpha \hat{S}$. More generally to any finite subset of $\mathrm{Irr}\,\mathscr{C}$ we associate a central projection of $\hat{S}$ by summing the corresponding p_α's. In the classical case this projection is the characteristic function of the considered subset of $\mathrm{Irr}\,\mathscr{C} \simeq \Gamma$. As an exception we denote by p_0 the minimal central projection associated with the trivial representation.

The algebraic direct sum of the matrix algebras $p_\alpha \hat{S}$ is a dense subalgebra denoted $\hat{\mathscr{S}}$. Similarly the coefficients of f.-d. corepresentations span a dense subalgebra $\mathscr{S} \subset S_{\mathrm{red}}$, and for every $\alpha \in \mathrm{Irr}\,\mathscr{C}$ the product $(p_\alpha \otimes \mathrm{id})V$ lies in the algebraic tensor product $\hat{\mathscr{S}} \otimes \mathscr{S}$. In the classical case we have $\hat{\mathscr{S}} = \mathbb{C}^{(\Gamma)}$ and $\mathscr{S} = \mathbb{C}\Gamma$.

We denote by h the Haar state of S_{red}. The Hilbert space H identifies with the GNS space of h via a map $\Lambda_h : S_{\mathrm{red}} \to H$ given by the choice of a fixed unit vector of V, $\xi_0 = \Lambda_h(1)$: we have then $h = (\xi_0| \cdot \xi_0)$. We denote by $\mathscr{H}$ the dense subspace $\Lambda_h(\mathscr{S}) \subset H$. In the classical case we can take $\xi_0 = \mathbb{1}_e \in \ell^2\Gamma$.

From the rich structure of $(S_{\mathrm{red}}, \delta)$ we will also need the counit $\epsilon : \mathscr{S} \to \mathbb{C}$. In fact we will use ϵ at the hilbertian level, hence we put $\epsilon(\Lambda_h(x)) = \epsilon(x)$ for $x \in \mathscr{S}$, thus defining an unbounded linear form $\epsilon : \mathscr{H} \to \mathbb{C}$. In the classical case we have $\epsilon(\lambda_g) = 1$ and $\epsilon(\mathbb{1}_g) = 1$ for all $g \in \Gamma$.

In this paper we are primarily interested in the case of the so-called universal discrete quantum groups and their free products, see [24,20]. More precisely, if $Q \in GL_n(\mathbb{C})$ is a fixed invertible matrix, the unitary universal discrete quantum group $\mathscr{A}_u(Q)$ and the orthogonal universal discrete quantum group $\mathscr{A}_o(Q)$ are defined by twisting the relations for $\mathscr{A}_o(I_n)$, $\mathscr{A}_u(I_n)$ given in the Introduction:

$$\mathscr{A}_u(Q) = \langle u_{ij} \mid (u_{ij})_{ij} \text{ and } Q(u^*_{ij})_{ij} Q^{-1} \text{ are unitary}\rangle,$$
$$\mathscr{A}_o(Q) = \mathscr{A}_u(I_n)/\langle (u_{ij})_{ij} = Q(u^*_{ij})_{ij} Q^{-1}\rangle.$$

In the case of $\mathscr{A}_o(Q)$ one requires moreover that $Q\bar{Q}$ is a scalar multiple of I_n, and we will focus on the non-amenable cases, which correspond to $n \geqslant 3$. These discrete quantum groups, and in particular their corepresentation theory, have been studied in [4].

We will recall definitions and facts about quantum Cayley graphs in the next sections as they are needed: see Reminders 2.4, 2.6, 3.1, 3.4 and 4.3. To give the right perspective for these definitions, let us recall here some facts about the classical case [14].

A (classical) graph is given by a set of vertices $\mathfrak{V}$, a set of edges $\mathfrak{E}$, an involutive reversing map $\theta : \mathfrak{E} \to \mathfrak{E}$ without fixed points, and an endpoints map $e : \mathfrak{E} \to \mathfrak{V}^2$ such that $e \circ \theta = \sigma \circ e$, where $\sigma(\alpha, \beta) = (\alpha, \beta)$. If we put $H = \ell^2(\mathfrak{V})$, $K = \ell^2(\mathfrak{E})$ and the graph is locally finite, the graph structure yields bounded operators $\Theta : K \to K$, $\mathbb{1}_w \mapsto \mathbb{1}_{\theta(w)}$ and $E : K \to H \otimes H$, $\mathbb{1}_w \mapsto \mathbb{1}_{e(w)}$. It is useful to consider non-oriented, or "geometrical", edges obtained by identifying each edge with the reversed one. At the hilbertian level we will use the subspace of antisymmetric vectors $K_g = \mathrm{Ker}(\Theta + \mathrm{id})$.

If moreover the graph is connected and endowed with an origin we have a natural length function on vertices given by the distance to the origin and we can consider ascending edges, i.e. edges which start at some length n and end at length $n + 1$. At the hilbertian level we will use the partition of unity $\mathrm{id}_H = \sum p_n$ given by the characteristic functions of the corresponding spheres, and the orthogonal projection $p_+ : K \to K_+$ onto the subspace of functions supported on ascending edges.

Denote finally by $\mathscr{H}$, $\mathscr{K}$, and $\mathscr{K}_g = \mathscr{K} \cap K_g$ the natural dense subspaces of functions with finite support, and define $\epsilon : \mathscr{H} \to \mathbb{C}$ by putting $\epsilon(\mathbb{1}_v) = 1$ for all $v \in \mathfrak{V}$. Then one can introduce target and source maps $E_1 = (\mathrm{id} \otimes \epsilon) \circ E$, $E_2 = (\epsilon \otimes \mathrm{id}) \circ E$ which are bounded in the locally finite case. It is an exercise to show that the graph under consideration is a tree **iff** the restriction $E_2 : \mathscr{K}_g \to \mathrm{Ker}\,\epsilon \subset \mathscr{H}$ is a bijection. The corresponding issue in the quantum case will prove to be quite intricate.

Consider now a discrete group Γ and fix a finite subset $\mathscr{D} \subset \Gamma$ such that $\mathscr{D}^{-1} = \mathscr{D}$ and $e \notin \mathscr{D}$. The Cayley graph associated with $(\Gamma, \mathscr{D})$ can be described in two equivalent ways:

1. we take $\mathfrak{V} = \Gamma$ and $\mathfrak{E} = \{(g, h) \in \Gamma^2 \mid h \in g\mathscr{D}\}$, with the evident maps $\theta : (g, h) \mapsto (h, g)$ and $e : (g, h) \mapsto (g, h)$;
2. we take $\mathfrak{V} = \Gamma$ and $\mathfrak{E} = \Gamma \times \mathscr{D}$, with the reversing map $\theta : (g, h) \mapsto (gh, h^{-1})$ and the endpoints map $e : (g, h) \mapsto (g, gh)$.

The first presentation leads to the notion of classical Cayley graph for discrete quantum groups, whereas the second presentation, or rather the Hilbert spaces and operators associated with it as above, leads to the notion of quantum Cayley graph for discrete quantum groups [21].

2. Generalities on path cocycles

2.1. Trivial and bounded cocycles

Let Γ be a discrete group, and $\mathscr{S} = \mathbb{C}\Gamma$ the group algebra, endowed with its natural structure of $*$-Hopf algebra. Extending maps from Γ to $\mathbb{C}\Gamma$ by linearity, unitary representations $\pi : \Gamma \to B(K)$ correspond to non-degenerate $*$-representations of $\mathscr{S}$, and π-cocycles $c : \Gamma \to K$ yield derivations from $\mathscr{S}$ to the $\mathscr{S}$-bimodule ${}_\pi K_\epsilon$, where ϵ is the trivial character on Γ.

For the quantum case we make the following evident definition:

Definition 2.1. Let $(\mathscr{S}, \epsilon)$ be a unital algebra with fixed character $\epsilon \in \mathscr{S}^*$. Let $\pi : \mathscr{S} \to B(K)$ be a unital representation on a vector space K. A *π-cocycle* is a derivation from $\mathscr{S}$ to the bimodule ${}_\pi K_\epsilon$, i.e. a linear map $c : \mathscr{S} \to K$ such that

$$\forall x, y \in \mathscr{S} \quad c(xy) = \pi(x)c(y) + c(x)\epsilon(y).$$

We say that c is *trivial* if the corresponding derivation is inner, i.e. there exists $\xi \in K$ such that $c(x) = \pi(x)\xi - \xi\epsilon(x)$. We call ξ a *fixed vector* relatively to c.

When $(\mathscr{S}, \epsilon)$ is associated with a discrete quantum group as in Section 1.1, cocycles can also be considered from the dual point of view: to any unital $*$-representation $\pi : \mathscr{S} \to L(K)$ is associated a corepresentation $X \in M(\hat{S} \otimes K(K))$, and to any π-cocycle $c : \mathscr{S} \to K$ is associated an "unbounded multiplier" of the Hilbert C^*-module $\hat{S} \otimes K$:

$$C = (\mathrm{id} \otimes c)(V) \in (\hat{S} \otimes K)^\eta := \prod_\alpha (p_\alpha \hat{S} \otimes K),$$

where α runs over $\mathrm{Irr}\,\mathscr{C}$. Note that C is best understood as an unbounded operator from H to $H \otimes K$.

It is immediate to rewrite the cocyle relation as follows — an equality between elements of the algebraic tensor product $p_\alpha \hat{S} \otimes p_\beta \hat{S} \otimes K$ after multiplying by any $p_\alpha \otimes p_\beta \otimes \mathrm{id}$ on the left:

$$(\hat{\delta} \otimes \mathrm{id})(C) = V_{12}^* C_{23} V_{12} = X_{13} C_{23} + C_{13}.$$

Similarly, c is trivial with fixed vector ξ **iff** $C = X(1 \otimes \xi) - 1 \otimes \xi$.

In this paper we will mainly work with $c : \mathscr{S} \to K$, but C can be useful when it comes to boundedness. We say that the cocycle c is bounded if $C = (\mathrm{id} \otimes c)(V)$ is bounded in the following equivalent meanings: the family $((p_\alpha \otimes c)(V))_\alpha$ is bounded with respect to the C^*-hilbertian norms; the operator $(\mathrm{id} \otimes c)(V) : \mathscr{H} \to H \otimes K$ is bounded; left multiplication by $(\mathrm{id} \otimes c)(V)$ defines an element of the $M(\hat{S})$-Hilbert module $M(\hat{S} \otimes K)$ [1].

It is a classical feature of cocycles with values in unitary representations that triviality is equivalent to boundedness. We prove now a quantum generalization of this result, which will not be used in the rest of the article but we find it of independent interest.

The classical proof relies on the center construction for bounded subsets of Hilbert spaces. In the quantum case, it is more convenient to rephrase it in terms of metric projections. Recall that a metric projection of an element ζ of a Banach space E onto the subspace F is an element $\xi \in F$ such that $\|\zeta - \xi\| = d(\zeta, F)$. The subspace F is called Chebyshev if all elements of E have a unique metric projection onto F. All subspaces of Hilbert spaces are Chebyshev, and in the case of Hilbert C^*-modules we have the elementary:

Lemma 2.2. *Let A be a unital C^*-algebra and E a Hilbert A-module. Let $F \subset E$ be a closed subspace. Let B be a C^*-algebra.*

1. *Assume F is reflexive and $(\zeta|\zeta)$ is invertible in A for every $\zeta \in F \setminus \{0\}$. Then F is a Chebyshev subspace of E.*
2. *Assume that $\xi \in E$ has a unique metric projection $\zeta \in F$ and consider the Hilbert $M(B \otimes A)$-module $M(B \otimes E)$. Then $1 \otimes \zeta$ is the unique metric projection of $1 \otimes \xi$ onto $M(B \otimes F)$.*

Proof. 1. Existence of metric projections in F follows from reflexivity by weak compactness of closed balls in F [15, Cor. 2.1]. Assume $\xi, \xi' \in F$ are metric projections of ζ. Then we put $d = d(\zeta, F)$, $\eta = (\xi + \xi')/2 \in F$ and we have, by the parallelogram identity for Hilbert modules:

$$(\zeta - \eta|\zeta - \eta) = \frac{1}{2}(\zeta - \xi|\zeta - \xi) + \frac{1}{2}\big(\zeta - \xi'|\zeta - \xi'\big) - \frac{1}{4}\big(\xi - \xi'|\xi - \xi'\big).$$

Now we have $(\zeta - \xi|\zeta - \xi)$, $(\zeta - \xi'|\zeta - \xi') \leqslant d^2 1_A$ and if $\xi \neq \xi'$ we get $\epsilon > 0$ such that $(\xi - \xi'|\xi - \xi') \geqslant \epsilon 1_A$, by hypothesis on F. Hence we get $\|\zeta - \eta\|^2 \leqslant d^2 - \epsilon/4 < d^2$, a contradiction.

2. We clearly have $\|1 \otimes \zeta - 1 \otimes \xi\| = \|\zeta - \xi\| =: d$. Take $\eta \in M(B \otimes F)$ such that $\|\eta - 1 \otimes \xi\| \leqslant d$. For any state φ of B there is a well-defined contraction $\varphi \otimes \mathrm{id} : M(B \otimes E) \to E$. In particular $\|(\varphi \otimes \mathrm{id})(\eta) - \xi\| \leqslant d$, so the hypothesis implies $(\varphi \otimes \mathrm{id})(\eta) = \zeta$ for all φ. This clearly implies $\eta = 1 \otimes \zeta$. □

Proposition 2.3. *Let $\pi : S \to L(K)$ be a $*$-representation of the full C^*-algebra of a discrete quantum group, and $c : \mathscr{S} \to K$ a π-cocycle. If $(\mathrm{id} \otimes c)(V) \in M(\hat{S} \otimes K)$ then c is trivial.*

Proof. Put $C = (\mathrm{id} \otimes c)(V) \in M(\hat{S} \otimes K)$ and let $X \in L(H \otimes K)$ be the corepresentation corresponding to π. We apply Lemma 2.2 to the subspace $F = 1 \otimes K$ of $E = M(\hat{S} \otimes K)$. Reflexivity is immediate since F is a Hilbert space, and the second condition of the lemma is trivially satisfied since $(1 \otimes \zeta|1 \otimes \zeta) = \|\zeta\|_K^2 \, 1_{M(\hat{S})}$. Denote by $1 \otimes \xi$ the metric projection of C onto $1 \otimes K$ and put $d = \|1 \otimes \xi - C\|$.

By Lemma 2.2 the vector $1 \otimes 1 \otimes \xi$ is the metric projection of $1 \otimes C$ onto $M(\hat{S} \otimes 1 \otimes K)$. Since X_{13} is a unitary which stabilizes $M(\hat{S} \otimes 1 \otimes K)$ and C_{13} belongs to $M(\hat{S} \otimes 1 \otimes K)$, we can deduce that $V_{12}^* C_{23} V_{12} = X_{13} C_{23} + C_{13}$ lies at distance d from $M(\hat{S} \otimes 1 \otimes K)$ with unique metric projection $X_{13}(1 \otimes 1 \otimes \xi) + C_{13}$. But $1 \otimes 1 \otimes \xi$ is evidently a vector in $M(\hat{S} \otimes 1 \otimes K)$ lying at distance d from $V_{12}^* C_{23} V_{12}$. Hence we have that $X(1 \otimes \xi) + C = 1 \otimes \xi$, and the cocycle is trivial. □

2.2. Path cocycles

We investigate in this paper a geometrical method to construct particular cocycles on $\mathscr{S}$, the so-called path cocycles. Before introducing this notion we need to recall the definitions of the classical and quantum Cayley graphs associated with a discrete quantum group.

Reminder 2.4. As explained in Section 1.1, let $\mathscr{C}$ be the category of finite-dimensional representations of $\hat{S}$, and let $\mathrm{Irr}\,\mathscr{C}$ be a system of representants of all irreducible objects. We fix a finite subset $\mathscr{D} \subset \mathrm{Irr}\,\mathscr{C}$ such that $\bar{\mathscr{D}} = \mathscr{D}$ and $1_{\mathscr{C}} \notin \mathscr{D}$. Recall from [21, Def. 3.1] that the classical Cayley graph associated with $(\mathscr{S}, \delta, \mathscr{D})$ is given by:

- the set of vertices $\mathfrak{V} = \mathrm{Irr}\,\mathscr{C}$ and the set of edges

$$\mathfrak{E} = \{(\alpha, \beta) \in (\mathrm{Irr}\,\mathscr{C})^2 \mid \exists \gamma \in \mathscr{D} \ \beta \subset \alpha \otimes \gamma\},$$

- the canonical reversing map $\sigma : \mathfrak{E} \to \mathfrak{E}, (\alpha, \beta) \mapsto (\beta, \alpha)$ and endpoints map $i_{\mathrm{can}} : \mathfrak{E} \to \mathfrak{V} \times \mathfrak{V}$.

We endow this graph with the root $1_{\mathscr{C}}$ given by the trivial corepresentation. The elements of $\mathscr{D}$ are called directions of the Cayley graph. In the classical case of a discrete group Γ we recover the first description of the usual Cayley graph given in Section 1.1.

On the other hand, let (H, V, U) be the Kac triple associated with (S, δ) and denote by $\Sigma \in B(H \otimes H)$ the flip operator. We introduce the central projection $p_1 \in \hat{S}$ corresponding to $\mathscr{D} \subset$

$\operatorname{Irr}\mathscr{C}$ as explained in Section 1.1. Recall from [21, Def. 3.1] that the hilbertian quantum Cayley graph associated with $(\mathscr{S}, \delta, p_1)$ is given by:

– the ℓ^2-space of vertices H and the ℓ^2-space of edges $K = H \otimes p_1 H$,
– the reversing operator $\Theta = \Sigma(1 \otimes U)V(U \otimes U)\Sigma \in B(K)$ and the endpoints operator $E = V_{|K} : K \to H \otimes H$.

In the classical case of a discrete group Γ one recovers the hilbertian objects associated with the usual Cayley graph, as introduced in Section 1.1. A major feature of the quantum case is that Θ need not to be involutive, even in a "deformed" meaning.

In fact we will rather use the source and target maps $E_1 = (\mathrm{id} \otimes \epsilon)V : K \to H$ and $E_2 = (\epsilon \otimes \mathrm{id})V : K \to H$, which are bounded if $\mathscr{D}$ is finite. We denote by $K_g = \operatorname{Ker}(\Theta + \mathrm{id})$ the space of geometrical, or antisymmetric, edges and by $p_g \in B(K)$ the orthogonal projection onto K_g. The spaces H, K, K_g are naturally endowed with representations of S_{red} which are intertwined by E_1, E_2 and Θ. □

Recall that we have a natural dense subspace $\mathscr{H} = \Lambda_h(\mathscr{S})$ of H, and define similarly $\mathscr{K} = \mathscr{H} \otimes p_1 H$ at the level of edges. The situation is more complicated for geometrical edges and we make the following definitions, which will be discussed in Remark 2.9. We call $\mathscr{K}_g$ the intersection of K_g with $\mathscr{K}$, and we call $\mathscr{K}_g'$ the orthogonal projection of $\mathscr{K}$ onto K_g.

Definition 2.5. The *trivial* λ*-cocycle* of $\mathscr{S}$ is the cocycle with fixed vector ξ_0, i.e. $c_0 : x \mapsto \Lambda_h(x) - \epsilon(x)\Lambda_h(1)$. A cocycle $c_g : \mathscr{S} \to K$ is called a *path cocycle* if we have $E_2 \circ c_g = c_0$ and $c(\mathscr{S}) \subset \mathscr{K}_g'$.

The motivating example is as follows: for any element g of the free group F_n, consider the unique path from e to g in the Cayley graph of F_n, and let $c_g(g) \in K_g$ denote the sum of (characteristic functions of) antisymmetric edges along this oriented path. Then it is easy to check that c_g extends by linearity to a path cocycle on $\mathbb{C}F_n$. The evident fact that this cocycle is proper (as a function on F_n) establishes the a-T-menability of F_n.

The problem of existence of path cocyles is a problem about the injectivity and surjectivity of appropriate restrictions of $E_2 : K_g \to H$. As indicated in Section 1.1, it is thus related to the question of whether the quantum Cayley graph under consideration can be considered a tree. Notice that we considered in [21] another approach to this question in terms of the ascending orientation.

We start by attacking the surjectivity side of the issue with the simple Lemma 2.7 below. Recall that we consider the counit ϵ as a linear map $\epsilon : \mathscr{H} \to \mathbb{C}$, and its kernel $\operatorname{Ker}\epsilon$ as a subspace of $\mathscr{H}$. We clearly have $\operatorname{Ker}\epsilon = \{\xi - \epsilon(\xi)\xi_0 \mid \xi \in \mathscr{H}\} = c_0(\mathscr{S})$. Hence for our purposes "surjectivity" of $E_2 : K_g \to H$ corresponds to the requirement that $E_2(\mathscr{K}_g')$ contains $\operatorname{Ker}\epsilon$.

Reminder 2.6. To state and prove the lemma we need to recall the notion of ascending orientation $K_{++} \subset K$ for quantum Cayley graphs.

We begin with spheres. The distance to the origin in the classical Cayley graph yields spheres of radius n which are subsets of $\operatorname{Irr}\mathscr{C}$. To these subsets are associated central projections $p_n \in \hat{S} \subset B(H)$. These projections play the role of characteristic functions of spheres for the quantum Cayley graph. If the classical Cayley graph is connected, they form a partition of the unit of $B(H)$.

Notice that the notation p_n is consistent with p_0, p_1 introduced earlier. When the space acted upon is clear from the context, we will moreover denote $p_n = p_n \otimes \mathrm{id} \in B(K)$, which projects onto "edges starting at distance n from the origin". We will also use $p_{<n} = \sum_{k<n} p_k$, $p_{\geqslant n} = \sum_{k \geqslant n} p_k$ etc.

Next we put $p_{\bigstar +} = \sum (p_n \otimes p_1)\hat{\delta}(p_{n+1})$ and $p_{+\bigstar} = \sum (p_n \otimes p_1)\hat{\delta}'(p_{n+1})$, where $\hat{\delta}'(a) = (U \otimes U)\Sigma\delta(a)\Sigma(U \otimes U) \in B(H \otimes H)$. The projection $p_{\bigstar +}$ lives in $\hat{S} \otimes \hat{S} \subset B(H \otimes H)$, and $p_{+\bigstar}$ is the corresponding projection in $U\hat{S}U \otimes U\hat{S}U$, acting "from the right" on H. The ascending projection of the quantum Cayley graph is by definition $p_{++} = p_{\bigstar +} p_{+\bigstar} \in B(K)$. Its image is denoted $K_{++} = p_{++}K$, and we put $\mathscr{K}_{++} = \mathscr{K} \cap K_{++} = p_{++}\mathscr{K}$.

In the classical case it is easy to check that $p_{+\bigstar} = p_{\bigstar +} = p_+$ is indeed the orthogonal projection onto the subspace of functions supported on ascending edges. Similarly p_n is the orthogonal projection onto the subspace of functions supported on the sphere of radius n. □

Lemma 2.7. *Assume that the classical Cayley graph associated with $(\mathscr{S}, \mathscr{D})$ is connected, i.e. $\mathscr{D}$ generates $\mathscr{C}$. We have then the following results for the quantum Cayley graph.*

1. *We have $(E_2 - E_1)(K_g^\perp) = 0$.*
2. *We have $(E_2 - E_1)(\mathscr{K}_{++}) = \operatorname{Ker}\epsilon$.*

As a result we have $E_2(\mathscr{K}_g) \subset \operatorname{Ker}\epsilon = E_2(\mathscr{K}_g') \subset E_2(K_g)$.

Proof. 1. Using the identities $E_1 = E_2\Theta = E_2\Theta^*$ [21, Prop. 3.6] we have:

$$(\Theta + \mathrm{id})(E_2 - E_1)^* = (\Theta + \mathrm{id})\big(\mathrm{id} - \Theta^*\big)E_2^* = \big(\Theta - \Theta^*\big)E_2^* = 0.$$

Hence $\operatorname{Im}(E_2 - E_1)^* \subset K_g$ and $K_g^\perp \subset \operatorname{Ker}(E_2 - E_1)$.

Let us explain also the last statement. We have $E_2(\Lambda_h \otimes \Lambda_h)(x \otimes y) = \Lambda_h(xy)$ and $E_1(\Lambda_h \otimes \Lambda_h)(x \otimes y) = \Lambda_h(x\epsilon(y))$ for $x, y \in \mathscr{S}$ [21, Prop. 3.6]. Since $\epsilon(xy) = \epsilon(x\epsilon(y))$, this shows that $(E_2 - E_1)(\mathscr{K})$, hence $E_2(\mathscr{K}_g) = (E_2 - E_1)(\mathscr{K}_g)$ and $E_2(\mathscr{K}_g')$, are contained in $\operatorname{Ker}\epsilon$. On the other hand 1. and 2. clearly imply that $E_2(\mathscr{K}_g') = \operatorname{Ker}\epsilon$, by definition of $\mathscr{K}_g'$.

2. The previous paragraph already shows that $(E_2 - E_1)(\mathscr{K}_{++}) \subset \operatorname{Ker}\epsilon$. Hence it remains to prove that $\xi - \epsilon(\xi)\xi_0 \in (E_2 - E_1)(\mathscr{K}_{++})$ for all $\xi \in p_n H$ and all n, and we will proceed by induction over n.

We will need to know that $E_2(p_n K_{++}) = p_{n+1}H$. The proof is as follows: for every $\gamma \in \operatorname{Irr}\mathscr{C}$ of length $n+1$ there exists by definition of the length $\alpha, \beta \in \operatorname{Irr}\mathscr{C}$ of respective lengths n, 1 such that $\gamma \subset \alpha \otimes \beta$. Then the relevant arguments of [21, Prop. 4.7 and Rem. 4.8] apply in our more general setting and show that $E_2 : \hat{\delta}(p_\gamma)(p_\alpha \otimes p_\beta)K_{++} \to p_\gamma H$ is an explicit, non-zero multiple of a (surjective) isometry.

We have now $E_2(p_n K_{++}) = p_{n+1}H$ and $E_1(p_n K_{++}) \subset p_n H$. For a fixed $\xi \in p_n H$, this yields $\zeta \in p_{n-1}K_{++}$ such that $E_2\zeta = \xi$, and $\xi' = E_1\zeta \in p_{n-1}H$. Since $E_2 - E_1$ ranges in $\operatorname{Ker}\epsilon$, we have $\epsilon(\xi) = \epsilon(\xi')$. Moreover, by induction over n one can find $\zeta' \in \mathscr{K}_{++}$ such that $(E_2 - E_1)(\zeta') = \xi' - \epsilon(\xi')\xi_0$. We have then

$$\xi - \epsilon(\xi)\xi_0 = (E_2 - E_1)(\zeta) + \xi' - \epsilon\big(\xi'\big)\xi_0 = (E_2 - E_1)\big(\zeta + \zeta'\big).$$

For $n = 0$ the result is obvious since $p_0 H = \mathbb{C}\xi_0$ and $\xi_0 - \epsilon(\xi_0)\xi_0 = 0$. □

Corollary 2.8. *If c_g is a path cocyle, then $c_g(\mathscr{S})$ is an $\mathscr{S}$-stable subspace of K_g on which E_2 is injective. Conversely, let K'_g be an $\mathscr{S}$-stable subspace of K_g on which E_2 is injective, and assume moreover that $\mathscr{K}'_g \subset K'_g$. Then there exists a unique path cocyle c_g with values in K'_g, and $c_g(\mathscr{S}) = \mathscr{K}'_g$.*

Proof. The cocyle relation implies trivially the $\mathscr{S}$-stability of $c_g(\mathscr{S})$. Assume $E_2(\zeta) = 0$ with $\zeta = c_g(x)$, $x \in \mathscr{S}$. Then we have $0 = E_2(\zeta) = c_0(x)$ by definition of a path cocyle. But clearly $c_0(x) = 0$ **iff** $x \in \mathbb{C}1$, and since 1 is mapped to 0 by any cocycle (take $x = y = 1$ in the cocycle relation) we conclude that $\zeta = 0$.

If K'_g contains $\mathscr{K}'_g$, then $E_2(K'_g)$ contains $\operatorname{Ker}\epsilon = c_0(\mathscr{S})$ by Lemma 2.7. If moreover E_2 is injective on K'_g we obtain a unique map $c_g : \mathscr{S} \to K'_g$ such that $E_2 c_g = c_0$, and its image equals $\mathscr{K}'_g$. By uniqueness, $\mathscr{S}$-stability, and since E_2 intertwines the actions of $\mathscr{S}$, the cocycle relation for c_0 implies the cocycle relation for c_g. □

Remark 2.9 *(Comment on $\mathscr{K}'_g$).* In the classical case, both subspaces $\mathscr{K}_g$ and $\mathscr{K}'_g$ coincide with the space of functions with finite support on antisymmetric edges. In the quantum case, the inclusion $\mathscr{K}_g \subset \mathscr{K}'_g$ is strict in general: we will e.g. argue in Section 3 that $\mathscr{K}'_g$ is not included in $\mathscr{K}$, whereas this is by definition the case of $\mathscr{K}_g$.

Note that $\mathscr{K}_g$ and $\mathscr{K}'_g$ are both stable under the action of the dense subalgebra $\mathscr{S} \subset S$, because this is the case of $\mathscr{K}$ and K_g. The subspace $\mathscr{K}'_g$ is obviously dense in K_g, whereas this is by no mean clear for $\mathscr{K}_g$ — we give however a proof of this density at the end of Theorem 3.5.

The choice of $\mathscr{K}'_g$ instead of $\mathscr{K}_g$ in Definition 2.5 is motivated by the results of Lemma 2.7 and Corollary 2.8. Notice that these "surjectivity" results still hold if one considers $\mathscr{K}''_g = p_g(\mathscr{K}_{++})$ instead of $\mathscr{K}'_g$, and this smaller subspace is a priori more suited for injectivity results. However the $\mathscr{S}$-stability and density of $\mathscr{K}''_g$ are not clear anymore. In fact one can prove, using the results in this article, that $\mathscr{K}'_g = \mathscr{K}''_g$ for free products of universal quantum groups. □

3. Complements on quantum Cayley trees

In this section we give a new description of the closed subspace $p_{++}K_g$ already studied in [21] and we prove a density result for $\mathscr{K}_g \subset K_g$. The main new tools are a "rotation operator" $\Phi : K_{--} \to K_{++}$, the associated shift $r' : K_{++} \to K_{++}$, and a "quasi-classical" subspace $Q_0 K$.

Reminder 3.1. We assume now that the discrete quantum group under consideration is a free product of orthogonal and unitary universal discrete quantum groups, and that $\mathscr{D}$ is the collection of the corresponding fundamental corepresentations $(u_{ij})_{ij}$ — this is the setting of [21].

In this case the classical Cayley graph is a tree, and for each edge $w = (\alpha, \beta)$ one can define the direction of w as the unique $\gamma \in \mathscr{D}$ such that $\beta \subset \alpha \otimes \gamma$, or equivalently, $(p_\alpha \otimes p_\gamma)\hat{\delta}(p_\beta) \neq 0$. Notice that there can be (at most) two edges starting from α in direction γ, one of them ascending and the other one descending.

If one puts $p_{\bigstar-} = \sum (p_n \otimes p_1)\hat{\delta}(p_{n-1})$ and $p_{-\bigstar} = \sum (p_n \otimes p_1)\hat{\delta}'(p_{n-1})$, then $\{p_{\bigstar+}, p_{\bigstar-}\}$ and $\{p_{+\bigstar}, p_{-\bigstar}\}$ are commuting partitions of unity, which also commute to the partition $\{p_n\}$. We have the following related pieces of notation: $p_{+-} = p_{+\bigstar} p_{\bigstar-}$, $K_{+-} = p_{+-}K$, $\mathscr{K}_{+-} = \mathscr{K} \cap K_{+-}$ and similarly for p_{-+} and p_{--}.

The behaviors of E_2 and Θ with respect to these partitions of unity yield "computation rules" in quantum Cayley trees, cf. [21, Prop. 4.3], [21, Prop. 5.1] and the identity $E_2\Theta = E_1 = E_2\Theta^*$

which results from [21, Prop. 3.6]. The interplay between orientation and reversing yields two important operators: $r = -p_{+-}\Theta p_{+-}$ and $s = p_{+-}\Theta p_{++}$. □

3.1. New tools

In the following lemma we introduce a right shift r' on K_{++} which will be used in the statement and the proof of Theorem 3.5. This is done via the rotation operator Φ which is of independent interest. In the case of a classical tree, this operator corresponds to "rotating" descending edges around their target vertex, yielding the unique ascending edge with the same target.

Lemma 3.2. *Consider the quantum Cayley graph of a free product of universal discrete quantum groups.*

1. *There exists a unique bounded operator $\Phi : K_{--} \to K_{++}$ such that $(1 - p_0)E_2 p_{--} = E_2 p_{++}\Phi$. We have $\Phi(p_n K_{--}) \subset p_{n-2}K_{++}$ for $n \geqslant 2$ and $\Phi(p_1 K_{--}) = \{0\}$.*
2. *The operator $p_{++}\Theta p_{--}\Phi^*$ is a right shift r' on K_{++} such that $sr' = rs$. Moreover we have $\Phi p_{--}\Theta p_{+-} = s^*$.*

Proof. 1. We know that $E_{2|K_{++}} : K_{++} \to H$ is injective and that its image equals $(1 - p_0)H$ [21, Prop. 4.7], and we denote $\underline{E}_2 = E_2 : K_{++} \to (1 - p_0)H$ the corresponding invertible restriction. Moreover we have $(1 - p_0)E_2 p_{--} = E_2 p_{--}(1 - p_1)$. Therefore it suffices to put $\Phi = \underline{E}_2^{-1}(E_2 p_{--})(1 - p_1)$. The remaining statements hold because $E_2 p_n K_{--} = p_{n-1}H$.

2. Using the computation rules in quantum Cayley trees, we can write

$$\begin{aligned} E_2 p_{++}(rs)^* &= -E_2 p_{++}\Theta^* p_{+-}\Theta^* p_{+-} = -E_2 p_{+\bigstar}\Theta^* p_{+-}\Theta^* p_{+-} \\ &= -E_2\Theta^* p_{+-}\Theta^* p_{+-} = -E_2\Theta p_{+-}\Theta^* p_{+-} \\ &= -E_2 p_{--}\Theta p_{+-}\Theta^* p_{+-}. \end{aligned} \tag{1}$$

Similarly, using the definition of Φ:

$$\begin{aligned} E_2 p_{++}\big(sr'\big)^* &= E_2 p_{++}\Phi p_{--}\Theta^* p_{++}\Theta^* p_{+-} \\ &= (1 - p_0)E_2 p_{--}\Theta p_{++}\Theta^* p_{+-} = E_2 p_{--}\Theta p_{++}\Theta^* p_{+-}. \end{aligned} \tag{2}$$

Notice indeed that $p_0 E_2 p_{--}\Theta p_{++}\Theta^* p_{+-}$ vanishes, essentially because "all edges starting from the origin are ascending", i.e. $p_0 p_{++} = p_0$:

$$\begin{aligned} p_0 E_2 p_{--}\Theta p_{++}\Theta^* p_{+-} &= E_2 p_{--} p_1 \Theta p_{++}\Theta^* p_{+-} = E_2 p_{--}\Theta p_0 p_{++}\Theta^* p_{+-} \\ &= E_2 p_{--} p_1 \Theta\Theta^* p_{+-} = E_2 p_1 p_{--} p_{+-} = 0. \end{aligned}$$

Now we subtract (1) from (2):

$$\begin{aligned} E_2 p_{++}\big(sr' - rs\big)^* &= E_2 p_{--}\Theta p_{+\bigstar}\Theta^* p_{+-} \\ &= E_2 p_{--}\Theta\Theta^* p_{+-} = E_2 p_{--} p_{+-} = 0. \end{aligned}$$

Since E_2 is injective on K_{++}, this yields $sr' = rs$. The identity $\Phi p_{--}\Theta p_{+-} = s^*$ follows similarly from

$$E_2 p_{++}\Phi p_{--}\Theta p_{+-} = E_2 p_{--}\Theta p_{+-} = E_2\Theta p_{+-} = E_2\Theta^* p_{+-}$$
$$= E_2 p_{++}\Theta^* p_{+-} = E_2 p_{++}s^*. \qquad \square$$

We introduce now quasi-classical subspaces Q_0K, Q_0H, or rather the orthogonal projection Q_0 onto them. The name "quasi-classical" refers to the fact that we will have $\Theta^2 Q_0 = Q_0$ and $(p_{+-} + p_{-+})Q_0 = 0$, as in classical trees.

In the case of a single copy of an orthogonal group $A_o(Q)$, the projection Q_0 coincides in fact with q_0 introduced in [21, p. 125], but we will give here an independent definition. In general, Q_0K contains q_0K strictly and we call q_0K and q_0H the "classical subspaces" of the quantum Cayley tree, because they are really the ℓ^2-spaces of the classical Cayley tree — see the beginning of Section 4 for more details.

Note that Q_0H, Q_0K do not really define a (quantum) subtree of (H, K) because E_2Q_0K is not included in Q_0H in general. Nevertheless, we prove below that $E_2Q_0K_{++} \subset Q_0H$, and also $E_2(1 - Q_0)K \subset (1 - Q_0)H$, i.e. the "purely quantum" subspaces $(1 - Q_0)H$, $(1 - Q_0)K$ form a quantum subtree of (K, H).

Lemma 3.3. *Consider the quantum Cayley graph of a free product of universal discrete quantum groups.*

1. *Denote by $Q_0 \in B(K)$ the orthogonal projection onto*

   $$Q_0K = \{\zeta \in K_{++} \mid \Theta\zeta \in K_{--}\} \oplus \{\zeta \in K_{--} \mid \Theta\zeta \in K_{++}\}.$$

   Then we have $[Q_0, p_n] = [Q_0, p_{+\bigstar}] = [Q_0, p_{\bigstar+}] = [Q_0, \Theta] = 0$.
   Moreover $\Theta^2 Q_0 = Q_0$ and $Q_0K_{++} = p_{++}\operatorname{Ker} s$.
2. *Denote by $Q_0 \in B(H)$ the orthogonal projection onto*

   $$Q_0H = \{\zeta \in H \mid E_1^*\zeta \in K_{++} \oplus K_{--}\}.$$

   Then $[Q_0, p_n] = 0$ and we have $Q_0E_2(1 - Q_0) = (1 - Q_0)E_2Q_0p_{++} = 0$.

Proof. 1. It is clear from the definition that Q_0 commutes with $p_{\bigstar+}$ and $p_{+\bigstar}$. Since $\Theta p_{++}p_n = p_{n+1}\Theta p_{++}$, and similarly with p_{--}, it is also easy to check that Q_0 commutes with p_n.

Since $\Theta p_{++} = p_{\bigstar-}\Theta p_{++}$ and $\Theta p_{--} = p_{\bigstar+}\Theta p_{--}$, the definition of Q_0K can be rewritten

$$Q_0K = p_{++}\operatorname{Ker}(p_{+-}\Theta p_{++}) \oplus p_{--}\operatorname{Ker}(p_{-+}\Theta p_{--}).$$

In particular we see that $Q_0K_{++} = \operatorname{Ker} s$. Let us notice that the first term in the direct sum above also equals $p_{++}\operatorname{Ker}(p_{-+}\Theta^* p_{++})$. We can indeed use the operator W from [21, Lem. 5.2] to write, for any $\zeta \in K_{++}$:

$$p_{+-}\Theta p_{++}\zeta = 0 \iff Wp_{+-}\Theta p_{++}\zeta = 0 \iff p_{-+}\Theta^* p_{++}\zeta = 0.$$

Similarly, the second term in the direct sum equals $p_{--}\operatorname{Ker}(p_{+-}\Theta^* p_{--})$.

Now we can show that Q_0 commutes with Θ. Take indeed $\zeta \in Q_0 K_{++}$: then $\Theta\zeta$ lies in K_{--}, and in fact in $Q_0 K_{--}$ because $p_{+-}\Theta^*\Theta\zeta = p_{+-}\zeta = 0$. Similarly, we see that $\Theta Q_0 K_{++}$, $\Theta^* Q_0 K_{++}$ and $\Theta^* Q_0 K_{--}$ are all contained in $Q_0 K$, hence $[Q_0, \Theta] = 0$. Finally, since Θ and Θ^* send $Q_0 K_{++}$ to K_{--} we can write, using [21, Prop. 5.1]:

$$\Theta^2 p_{++} Q_0 = \Theta p_{--} \Theta p_{++} Q_0 = \Theta p_{--} \Theta^* p_{++} Q_0 = \Theta\Theta^* p_{++} Q_0 = p_{++} Q_0.$$

Similarly $\Theta^2 p_{--} Q_0 = p_{--} Q_0$, hence $\Theta^2 Q_0 = Q_0$.

2. We have $[Q_0, p_n] = 0$ because $E_1(p_n \otimes \mathrm{id}) = p_n E_1$. By definition of $Q_0 K$ and remarks in the first part of the proof, we have

$$(1 - Q_0)K = K_{+-} \oplus K_{-+} \oplus \overline{\mathrm{Im}}\, p_{++}\Theta p_{-+} \oplus \overline{\mathrm{Im}}\, p_{--}\Theta p_{+-}.$$

It is known that E_2 vanishes on K_{-+} and K_{+-} [21, Prop. 4.3]. Consider then $\zeta = (p_{++}\Theta p_{-+} + p_{--}\Theta p_{+-})\eta \in (1 - Q_0)K$. We have

$$E_2\zeta = E_2(\Theta p_{-+} + \Theta p_{+-})\eta = E_1(p_{-+} + p_{+-})\eta.$$

Observing moreover that $Q_0 H = \mathrm{Ker}(p_{+-} + p_{-+})E_1^* = (E_1(K_{+-} \oplus K_{-+}))^\perp$, we obtain that $Q_0 E_2(1 - Q_0) = 0$.

The last property requires more care. Taking $\zeta \in Q_0 K_{++}$, we have to show that $E_1^* E_2 \zeta = \Theta E_2^* E_2 \zeta$ lies in $K_{++} \oplus K_{--}$. Since $E_2^* H \subset K_{++} \oplus K_{--}$, this is equivalent to the fact that $E_2^* \underline{E}_2 \zeta$ lies in $Q_0 K$, where $\underline{E}_2$ denotes the invertible restriction $\underline{E}_2 = E_2 : K_{++} \to (1 - p_0)H$ as in the proof of Lemma 3.2. First we want to replace $\underline{E}_2$ with $(\underline{E}_2^{-1})^*$. This is possible because E_2 is a multiple of an isometry on each subspace $(p_\alpha \otimes p_\gamma)K_{++}$ [21, proof of Prop. 4.7], and Q_0 commutes not only with $(p_n \otimes \mathrm{id})p_{++}$, but more precisely with each $(p_\alpha \otimes p_\gamma)p_{++}$.

Let us prove now that $E_2^*(\underline{E}_2^{-1})^*\zeta$ lies in $Q_0 K$. This is clearly the case for the p_{++} component since $\zeta \in Q_0 K$ and

$$p_{++} E_2^* \big(\underline{E}_2^{-1}\big)^* = \underline{E}_2^* \big(\underline{E}_2^{-1}\big)^* = \mathrm{id}_{K_{++}}.$$

For the p_{--} component, notice that $p_{--} E_2^* (\underline{E}_2^{-1})^* = \Phi^*$. Hence we finally check that $\Phi^*\zeta$ lies in $Q_0 K$ using the last point of Lemma 3.2: we have $p_{+-}\Theta^* p_{--}\Phi^*\zeta = s\zeta$, and since $\zeta \in Q_0 K$ this vector vanishes. □

3.2. Geometrical edges

In the next theorem we give a new description of the space K_g basing on the shift r'. This description is the key point for the proof of the injectivity of E_2 on $(1 - Q_0)K_g$, see Proposition 4.1. Using the same techniques, we also take the opportunity to prove the density of the subspace $\mathscr{K}_g \subset K_g$.

Note that the non-involutivity of Θ in the quantum case, and more precisely the fact that $p_{+-}\Theta p_{+-}$ acts as a shift with respect to the distance to the origin, implies that the spectral projection p_g does not stabilize $\mathscr{K}$, and $\mathscr{K}_g'$ is not included in $\mathscr{K}$. In particular, although $\mathscr{K}_g' = \mathscr{K}_g$ in the classical case, this is no longer true in the quantum case, and Corollary 2.8 shows that $\mathscr{K}_g'$ is the correct dense subspace for the study of path cocycles.

Reminder 3.4. Recall the following results from [21]. The map $r = -p_{+-}\Theta p_{+-}$ acts as a right shift in the decomposition $K_{+-} = \bigoplus p_k K_{+-}$ and we denote by K_∞ the inductive limit of the contractive system $(p_k K_{+-}, r)$ of Hilbert spaces. Let $R_k : p_k K_{+-} \to K_\infty$ and $R = \sum R_k : \mathscr{K}_{+-} \to K_\infty$ be the canonical maps, and recall the notation $s = p_{+-}\Theta p_{++}$. Then the maps R_k are injective, and the operator $Rs : K_{++} \to K_\infty$ is a co-isometry [21, Prop. 6.2 and Thm. 6.5].

Besides it is not hard to prove that the restriction $p_{++} : K_g \to K_{++}$ is injective. When "all directions have quantum dimensions different from 2", we have the following identities [21, Thms. 5.3 and 6.5], which show in particular that $p_{++}K_g$ is closed:

$$p_{++}K_g = \{\zeta \in K_{++} \mid \exists \eta \in K_{+-}\ (1-r)\eta = s\zeta\} = \operatorname{Ker} Rs. \tag{3}$$

Moreover if $\zeta = p_{++}\xi$ and $\eta = -p_{+-}\xi$ with $\xi \in K_g$, then $s\zeta = (1-r)\eta$ and ζ, η are related by the identities $R_k p_k \eta = Rsp_{<k}\zeta = -Rsp_{\geqslant k}\zeta$. Recall that in the classical case $K_{+-} = K_\infty = 0$ so that $p_{++}K_g = K_{++}$, as expected.

Let us discuss rapidly the hypothesis that "all directions have quantum dimensions different from 2". For a free products of quantum groups $A_o(Q)$, with $Q\bar{Q} \in \mathbb{C}I_n$, and $A_u(Q)$, there is one direction $\gamma = \bar{\gamma}$ in the classical Cayley tree for each factor $A_o(Q)$, and two directions γ, $\bar{\gamma}$ for each factor $A_u(Q)$. These "fundamental corepresentations" have quantum dimension strictly greater than 2, except in the following cases and isomorphic ones: $\dim_q \gamma = 1$ for $A_o(I_1) = C^*(\mathbb{Z}/2\mathbb{Z})$ and $A_u(I_1) = C^*(\mathbb{Z})$; $\dim_q \gamma = 2$ for $A_o(I_2) = C(SU_{-1}(2))$, $A_o(Q_{-1}) := C(SU(2))$ and $A_u(I_2)$. □

Theorem 3.5. *Consider the quantum Cayley graph of a free product of universal discrete quantum groups, and assume that all directions have quantum dimensions different from* 2*.*

1. *We have* $p_{+-}K_g = s(K_{++})$*.*
2. $Q_0 p_{++}K_g = Q_0 K_{++}$ *and* $(1-Q_0)p_{++}K_g = (1-Q_0)(\mathrm{id} - r')(K_{++})$*.*
3. *The subspace* $\mathscr{K}_g$ *is dense in* K_g*.*

Proof. 1. If $\eta = s\zeta'$ with $\zeta' \in K_{++}$, put $\zeta = (1-r')\zeta'$. We have then $s\zeta = (1-r)\eta$, hence $\zeta \in p_{++}K_g$ and $\eta \in p_{+-}K_g$ by (3). This proves that $s(K_{++})$ is contained in $p_{+-}K_g$.

Conversely, let $\eta \in K_{+-}$, $\zeta \in K_{++}$ be the projections of an element of K_g. Recall that η is characterized by the identities $R_k p_k \eta = -Rsp_{\geqslant k}\zeta$. Since $\operatorname{Ker} s = Q_0 K_{++}$ and R_k is injective, we can define for each $k \in \mathbb{N}$ a vector $\zeta'_k \in (1-Q_0)p_k K_{++}$ by putting $Rs\zeta'_{k-1} = -Rsp_{\geqslant k}\zeta$. If the sum $\sum \zeta'_k$ converges to a vector $\zeta' \in K_{++}$ we clearly have $s\zeta' = \eta$ and this shows that $p_{+-}K_g \subset s(K_{++})$. The proof of this convergence is the main issue of the theorem and will be given at point 4.

2. On the other hand the vector η in (3) lies in $p_{+-}K_g$. Once we know that $p_{+-}K_g = s(K_{++})$ we can therefore write

$$\begin{aligned}(1-Q_0)p_{++}K_g &= \{\zeta \in (1-Q_0)K_{++} \mid \exists \zeta' \in (1-Q_0)K_{++}\ (1-r)s\zeta' = s\zeta\} \\ &= \{\zeta \in (1-Q_0)K_{++} \mid \exists \zeta' \in (1-Q_0)K_{++}\ s(1-r')\zeta' = s\zeta\} \\ &= (1-Q_0)(\mathrm{id} - r')(K_{++}),\end{aligned}$$

since s is injective on $(1-Q_0)K_{++}$ and vanishes on Q_0K_{++}.

Finally, since s vanishes on Q_0K_{++} there is no obstruction in (3) for a vector of Q_0K_{++} to belong to $p_{++}K_g$, hence $Q_0p_{++}K_g = Q_0K_{++}$.

3. If ζ is an element of $p_{++}K_g$ and η is the associated vector in K_{+-}, we know [21, proof of Thm. 5.3] that ζ is the image by p_{++} of the following element $\xi \in K_g$:

$$\xi = \zeta - \eta - W\eta + p_{--}\Theta(\eta - \zeta).$$

Put $r'' = (1 - Q_0)r'$. By injectivity of $p_{++} : K_g \to K$ and the previous points, the equation above shows that K_g is the image of the following continuous application:

$$K_{++} \to K, \quad \zeta' \mapsto \big(1 - r''\big)\zeta' - (1 + W)s\zeta' + p_{--}\Theta\big(s + r'' - 1\big)\zeta'.$$

Since $\mathscr{K}_{++}$ is dense in K_{++} and mapped into $\mathscr{K}$ by this application, we conclude that $\mathscr{K}_g$ is dense in K_g.

4. In the case when no direction has quantum dimension 2, we know from [21, proof of Thm. 6.5] that $K^{-1}\|\zeta\| \leqslant \|R_k\zeta\| \leqslant \|\zeta\|$ for every k, $\zeta \in p_kK_{+-}$ and a constant K depending only on the tree. Hence we obtain from the identity $Rs\zeta'_{k-1} = -Rsp_{\geqslant k}\zeta$ the following inequality:

$$\big\|s\zeta'_{k-1}\big\| \leqslant K\sum_{j=k}^{\infty}\|sp_j\zeta\|.$$

Now there are orthogonal subspaces $q_lP_\gamma p_jK_{++}$ of p_jK_{++} on which the map s is a multiple of an isometry with a norm known explicitly in terms of the sequence of quantum dimensions (m_k) associated to the direction γ [21, Lem. 6.3]:

$$\|sq_lP_\gamma p_j\| = \sqrt{\frac{m_lm_{l-1}}{m_jm_{j+1}}}.$$

In the case of A_o there is only one direction γ and $P_\gamma = \mathrm{id}$. Since q_lP_γ commutes with all our structural maps so far it commutes in particular with the construction of ζ'_k from ζ and we obtain:

$$\big\|q_lP_\gamma\zeta'_{k-1}\big\| \leqslant K\sum_{j=k}^{\infty}\sqrt{\frac{m_{k-1}m_k}{m_jm_{j+1}}}\|p_jq_lP_\gamma\zeta\|.$$

In the case when the direction γ has quantum dimension different from 2, its sequence of dimensions has the form $m_{k-1} = (a^k - a^{-k})/(a - a^{-1})$ for some $a > 1$ and it is easy to check that one has then $\frac{m_{k-1}}{m_j} \leqslant a^{-(j-k)}$ for all $j \geqslant k-1$. This allows to write:

$$\begin{aligned}\big\|q_lP_\gamma\zeta'_{k-1}\big\|^2 &\leqslant K^2\left(\sum_{j=k}^{\infty}a^{k-j}\|p_jq_lP_\gamma\zeta\|\right)^2\\ &\leqslant K^2\sum_{j=k}^{\infty}a^{k-j}\sum_{j=k}^{\infty}a^{k-j}\|p_jq_lP_\gamma\zeta\|^2\end{aligned}$$

$$= \frac{K^2}{1-a^{-1}} \sum_{j=k}^{\infty} a^{k-j} \| p_j q_l P_\gamma \zeta \|^2, \quad \text{hence}$$

$$\sum_{k=1}^{\infty} \| q_l P_\gamma \zeta'_{k-1} \|^2 \leqslant \frac{K^2}{1-a^{-1}} \sum_{1 \leqslant k \leqslant j} a^{k-j} \| p_j q_l P_\gamma \zeta \|^2$$

$$\leqslant \frac{K^2}{(1-a^{-1})^2} \sum_{j=1}^{\infty} \| p_j q_l P_\gamma \zeta \|^2 = \frac{K^2 \| q_l P_\gamma \zeta \|^2}{(1-a^{-1})^2}.$$

Finally we sum over l, take the smallest a when γ varies, sum over γ and apply the "cut-and-paste" process [21, Rem. 6.4(2)] to recover the whole of the tree, i.e. the whole of (ζ'_k) and ζ. Since the projections P_γ on the one hand, and q_l on the other hand, are mutually orthogonal, the inequality above shows that $(\sum \zeta'_k)$ converges. □

4. Path cocycles in quantum Cayley trees

In this section we go on with the search for paths in quantum Cayley graphs of universal quantum groups, by proving injectivity results for $E_2 : K_g \to H$. We will then examine the triviality of the cocycles obtained in this way, distinguishing the "orthogonal case", and the remaining "general" cases of free products of unitary and orthogonal universal discrete quantum groups.

Recall that a classical graph is a tree **iff** $E_2 : \mathscr{K}_g \to \operatorname{Ker} \epsilon \subset \mathscr{H}$ is bijective. Hence our study has a geometrical interpretation, namely the question whether the quantum Cayley graphs of universal quantum groups are trees. We have already seen in [21] that the ascending orientation of our quantum Cayley graphs behaves like in rooted trees, e.g. $E_2 : K_{++} \to H$ is injective.

One cannot expect the map $E_2 : K_g \to H$ to be injective at the ℓ^2 level in general: it is already not the case for non-abelian free groups because of exponential growth. Surprisingly, the results of the previous section yield a strong "ℓ^2-injectivity" result for the "purely quantum" part $(1 - Q_0)K_g$ of the space of antisymmetrical edges.

Proposition 4.1. *Consider the quantum Cayley graph of a free product of universal discrete quantum groups. If all directions have quantum dimensions different from* 2*, then the restriction of* E_2 *to* $Q_0\mathscr{K}_g \oplus (1 - Q_0)K_g$ *is injective. Moreover* $E_2(1 - Q_0)K_g$ *is a closed subspace of* H.

Proof. Since $E_2(1 - Q_0)K_g \subset (1 - Q_0)H$ by Lemma 3.3, it suffices to show separately that E_2 is injective on $(1 - Q_0)K_g$, and that Q_0E_2 is injective on $Q_0\mathscr{K}_g$. We start with the first assertion and prove at the same time that $E_2(1 - Q_0)K_g$ is closed.

Observe that $(1 - Q_0)K_g \subset K_g$ because $[Q_0, \Theta] = 0$. Since the restrictions of E_2 and $(E_2 - E_1)$ coincide on K_g up to a constant, and by taking the adjoint, it is sufficient to show that the inclusion

$$(1 - Q_0)(E_2 - E_1)^*(H) \subset (1 - Q_0)K_g$$

given by Lemma 2.7 is in fact an equality. It is moreover equivalent to show that $(1 - Q_0)p_{++}(E_2 - E_1)^*(H) = (1 - Q_0)p_{++}K_g$ because $p_{++} : K_g \to K_{++}$ is injective with closed range.

Therefore we consider the following operator:

$$p_{++}(E_2 - E_1)^* = p_{++}(1-\Theta)E_2^* = p_{++}E_2^* - p_{++}\Theta p_{--}E_2^*$$
$$= \big(1 - p_{++}\Theta p_{--}\Phi^*\big)p_{++}E_2^* + p_{++}\Theta p_{--}E_2^* p_0.$$

According to this expression, and because $E_2(K_{++}) = (1-p_0)H$ by [21, Prop. 4.7], we have $p_{++}(E_2 - E_1)^*(H) \supset (1-r')(K_{++})$, and we obtain by Theorem 3.5 the inclusion we are looking for:

$$(1-Q_0)p_{++}(E_2 - E_1)^*(H) \supset (1-Q_0)p_{++}(K_g).$$

Now let us prove that Q_0E_2 is injective on $Q_0\mathscr{K}_g$, by contradiction. Fix $\zeta \in Q_0\mathscr{K}_g$ such that $\zeta \neq 0$ and $Q_0E_2\zeta = 0$. For each edge w of the classical Cayley tree, $w = (\alpha, \beta)$ with direction γ, we define a projection $p_w \in B(K)$ by putting $p_w = (p_\alpha \otimes p_\gamma)p_{++}$ if w is ascending, $p_w = (p_\alpha \otimes p_\gamma)p_{--}$ otherwise. In our situation it is known that $\{p_w\}$ is a partition of unity in $B(K)$ [21, Prop. 4.5], which clearly stabilizes $\mathscr{K}$.

In particular the support $\operatorname{Supp}\zeta = \{w \mid p_w\zeta \neq 0\}$ of ζ consists of edges of a disjoint union of finite subtrees of the classical Cayley tree. Note also that $\operatorname{Supp}\zeta$ is stable under the classical reversing map, since $\Theta\zeta = -\zeta$ and $\zeta \in K_{++} \oplus K_{--}$: for $w = (\alpha, \beta)$ in the support of ζ, e.g. ascending with direction γ, we have by [21, Prop. 3.7]:

$$0 \neq \Theta(p_\alpha \otimes p_\gamma)p_{+\bigstar}\zeta = \hat\delta(p_\alpha)(\mathrm{id}\otimes p_{\bar\gamma})p_{\bigstar -}\Theta\zeta = \hat\delta(p_\alpha)(\mathrm{id}\otimes p_{\bar\gamma})p_{\bigstar -}\zeta$$
$$= \hat\delta(p_\alpha)(\mathrm{id}\otimes p_{\bar\gamma})p_{--}\zeta = (p_\beta \otimes p_{\bar\gamma})p_{--}\zeta,$$

hence $\bar w = (\beta, \alpha)$ belongs to $\operatorname{Supp}\zeta$.

We fix one of the subtrees in $\operatorname{Supp}\zeta$, and we choose one of its terminal edges w_1 which is not the closest one to the origin. In particular, w_1 must be ascending and the target of any other $w \in \operatorname{Supp}\zeta$ is different from the target β_1 of w_1. Now $E_2p_{w_1}\zeta$ is a non-zero element of $p_{\beta_1}H$, and it equals $Q_0E_2p_{w_1}\zeta$ by the last point of Lemma 3.3. For any other edge w with target β the vector $Q_0E_2p_w\zeta$ lies in $Q_0p_\beta H$, which is orthogonal to $p_{\beta_1}H$. Hence we obtain $Q_0E_2\zeta \neq 0$, a contradiction. □

Corollary 4.2. *Consider a quantum Cayley graph as above where all directions have quantum dimension different from* 2. *Then there exists a unique path cocycle* $c_g : \mathscr{S} \to \mathscr{K}_g'$.

Proof. Since $\Theta^2 = \mathrm{id}$ on Q_0K, we have $p_g = (\mathrm{id} - \Theta)/2$ on Q_0K, hence $Q_0\mathscr{K}_g' = Q_0\mathscr{K}_g$. As a result, we see that $\mathscr{K}_g' \subset Q_0\mathscr{K}_g \oplus (1-Q_0)K_g$, and we can apply Corollary 2.8 with $K_g' = \mathscr{K}_g'$. □

4.1. The orthogonal case

In this section we recall the definition of the classical projections q_0 and we describe the "classical subgraph" of the quantum Cayley graph obtained in this way. This description holds generally for free products of orthogonal and unitary universal discrete quantum groups, and it is strongly related to the structure of the underlying classical Cayley graph.

Only then we will restrict to the case where there is a unique factor $A_o(Q)$ in the free product quantum group under consideration. In this case the projections q_0, Q_0 coincide and, using

Proposition 4.1, we will be able to deduce that the target operator is in fact invertible from K_g to H. In particular, the path cocycle obtained previously is trivial.

Reminder 4.3. The classical subspaces $q_0 H = H_0$, $q_0 K = K_0$ can be introduced as subspaces of fixed vectors for the left–right representations of $\hat{S}$ on H, K. More precisely, we have two (resp. four) commuting representations of $\hat{S}$ on H (resp. K) given by $\hat{\pi}_2(x \otimes x') = x(Ux'U) \in B(H)$ and $\hat{\pi}_4(x \otimes y \otimes y' \otimes x') = (x \otimes y)(Ux'U \otimes Uy'U)$, and the left–right representations above are $\hat{\pi}_2 \circ \hat{\delta}$ and $\hat{\pi}_4 \circ \hat{\delta}^3$. Hence we have $q_0 = \hat{\pi}_2 \hat{\delta}(p_0)$ on H and $q_0 = \hat{\pi}_4 \hat{\delta}^3(p_0)$ on K — more generally one defines in fact a new partition of unity $\{q_l\}$ by replacing p_0 with p_{2l} in these formulae.

With this definition it is easy to check that the projections q_l commute to the structural maps E_2, Θ [21, Prop. 3.7] and, directly from the definitions, to the partitions of unity $\{p_n\}$, $\{p_{\star+}, p_{\star-}\}$, $\{p_{+\star}, p_{-\star}\}$. In particular, $q_0 \in B(H)$ and $q_0 \in B(K)$ can be viewed as projections onto a "classical subtree" (H_0, K_0). Notice however that the subspaces H_0, K_0 are not stable under the action of $\mathscr{S}$ in the quantum case. In the classical case $\hat{S}$ is commutative and we have thus $H_0 = H$, $K_0 = K$.

Moreover we have in the general case $p_{+-}K_0 = p_{-+}K_0 = 0$ hence Θ restricts to an involution on K_0 [21, Prop. 5.1], and $q_0 K \subset Q_0 K$. In the case of $A_o(I_n)$, this inclusion is in fact an equality. As a matter of fact by [21, Lemma 6.3] we have $q_0 K_{++} = \operatorname{Ker} s = Q_0 K_{++}$. By applying Θ we also get $q_0 K_{--} = Q_0 K_{--}$, hence finally $q_0 = Q_0$. □

Let us now describe this classical subtree (H_0, K_0). We identify H not only with the GNS space of h, but also with the GNS space of the left Haar weight $\hat{h}_L$ of $\hat{S}$. For each vertex $\alpha \in \operatorname{Irr} \mathscr{C}$ of the classical Cayley graph, the subspace $p_\alpha H_0 \subset p_\alpha H$ corresponds to the inclusion $1_{\mathscr{C}} \subset \bar{\alpha} \otimes \alpha$ of representations of $\hat{S}$, hence it is 1-dimensional and spanned by the vector $\xi_\alpha \in p_\alpha H$ which is the GNS image of $p_\alpha \in \hat{S}$. Note that the vector ξ_α associated to the trivial representation is $\xi_0 = \Lambda_h(1)$.

Similarly, we define $\xi_{(\alpha,\beta)}$ for any edge (α, β) of the classical Cayley graph as the GNS image in $K = H \otimes p_1 H$ of $\hat{\delta}(p_\beta)(p_\alpha \otimes p_1)$, which spans the subspace $\hat{\delta}(p_\beta)(p_\alpha \otimes p_1)K_0$ and is mapped into $p_\alpha H_0$ (resp. $p_\beta H_0$) by E_1 (resp. E_2). Denote by γ the direction of the edge from α to β in the classical Cayley graph, so that $\hat{\delta}(p_\beta)(p_\alpha \otimes p_1) = \hat{\delta}(p_\beta)(p_\alpha \otimes p_\gamma)$. The norms of the vectors ξ_α and $\xi_{(\alpha,\beta)}$ are easy to compute from the known formulae for $\hat{h}_L$, see e.g. [13, (2.13)]:

$$\begin{aligned}\|\xi_\alpha\|^2 &= m_\alpha \operatorname{Tr}_\alpha(F p_\alpha) = m_\alpha^2 \quad \text{and}\\ \|\xi_{(\alpha,\beta)}\|^2 &= m_\alpha m_\gamma (\operatorname{Tr}_\alpha \otimes \operatorname{Tr}_\gamma)\big((F \otimes F)\hat{\delta}(p_\beta)\big)\\ &= m_\alpha m_\gamma \operatorname{Tr}_\beta(F p_\beta) = m_\alpha m_\beta m_\gamma.\end{aligned}$$

Here $m_\alpha = \operatorname{Tr}_\alpha(F p_\alpha)$ is the quantum dimension of α.

Since $q_0 K_{+-} = q_0 K_{-+} = \{0\}$, the operator Θ maps $\hat{\delta}(p_\beta)(p_\alpha \otimes p_1)K_0$ isometrically to $\hat{\delta}(p_\alpha)(p_\beta \otimes p_1)K_0$, hence $\Theta(\xi_{(\alpha,\beta)}) = \xi_{(\beta,\alpha)}$ — recall that the direction of (β, α) is $\bar{\gamma}$, and $m_{\bar{\gamma}} = m_\gamma$. In particular, the vectors

$$\tilde{\xi}_{\alpha\wedge\beta} = \frac{\xi_{(\alpha,\beta)} - \xi_{(\beta,\alpha)}}{\sqrt{2 m_\alpha m_\beta m_\gamma}} \quad \text{and} \quad \tilde{\xi}_\alpha = \frac{1}{m_\alpha}\, \xi_\alpha,$$

where (α, β) runs over the classical ascending edges, and α, over the classical vertices, form hilbertian bases of $q_0 K_g$, $q_0 H$ respectively.

On the other hand, the norm of E_2 on the subspaces $\hat{\delta}(p_\beta)(p_\alpha \otimes p_\gamma)K$ is known from [21, Rem. 4.8] and we deduce

$$E_2(\xi_{(\alpha,\beta)}) = \frac{m_\alpha m_\gamma}{m_\beta}\, \xi_\beta, \qquad E_2(\tilde{\xi}_{\alpha\wedge\beta}) = \sqrt{\frac{m_\gamma}{2}} \left(\sqrt{\frac{m_\alpha}{m_\beta}}\, \tilde{\xi}_\beta - \sqrt{\frac{m_\beta}{m_\alpha}}\, \tilde{\xi}_\alpha\right).$$

We see that (H_0, K_0, Θ, E_2) is a hilbertian version of the classical Cayley tree where the relation "α is an endpoint of w" comes with weights depending on the quantum dimensions.

As a result we can find unique paths in $q_0\mathscr{K}_g$ in the following sense: for each vertex α of the classical Cayley tree, the vector $\tilde{\xi}_\alpha/m_\alpha - \xi_0$ has a unique preimage by E_2 in $\mathscr{K}_g$. Denoting by $(1 = \alpha_0, \alpha_1, \dots, \alpha_{n-1}, \alpha_n = \alpha)$ the geodesic from $1_{\mathscr{C}}$ to α in the classical Cayley tree, and by γ_i the direction from α_i to α_{i+1}, this preimage is given by

$$\zeta_\alpha = \sum_{i=0}^{n-1} \sqrt{\frac{2}{m_{\gamma_i}}}\, \frac{\tilde{\xi}_{\alpha_i\wedge\alpha_{i+1}}}{\sqrt{m_{\alpha_i} m_{\alpha_{i+1}}}}.$$

Moreover, for each infinite geodesic $\infty = (1, \alpha_1, \dots, \alpha_n, \dots)$ in the classical Cayley tree such that $(\sum m_{\alpha_i}^{-2})$ converges we get a preimage of ξ_0:

$$\zeta_\infty = \sum_{i=0}^{\infty} \sqrt{\frac{2}{m_{\gamma_i}}}\, \frac{\tilde{\xi}_{\alpha_i\wedge\alpha_{i+1}}}{\sqrt{m_{\alpha_i} m_{\alpha_{i+1}}}}. \tag{4}$$

In the case of $A_o(Q)$ there is a unique geodesic to ∞ and the previous vector is the fixed vector ξ_g for the path cocyle c_g of the next theorem:

Theorem 4.4. *Consider the quantum Cayley graph of $A_o(Q)$, with $Q^*Q \notin \mathbb{R}I_2$. Then the operator $E_2 : K_g \to H$ is invertible. As a result there exists a unique path cocycle $c_g : \mathscr{S} \to \mathscr{K}_g'$, and it is trivial.*

Proof. The classical Cayley graph of $A_o(Q)$ is the half-line with vertices α_k at integers $k \in \mathbb{N}$. We put $\tilde{\xi}_k = \tilde{\xi}_{\alpha_k}$ and $\tilde{\xi}_{k\wedge k+1} = \tilde{\xi}_{\alpha_k\wedge\alpha_{k+1}}$. If $E_2\zeta = 0$ with $\zeta = \sum \lambda_k \tilde{\xi}_k$, the equality $p_0 E_2 \zeta = 0$ reads $\lambda_0 = 0$ and an easy induction yields $\lambda_k = 0$ for all k: E_2 is injective on $q_0 K_g$. Define conversely a linear map $E_2^{-1} : q_0\mathscr{H} \to q_0 K_g$ by

$$E_2^{-1}(\tilde{\xi}_k) = -m_k \sqrt{\frac{2}{m_1}} \sum_{i=k}^{\infty} \frac{\tilde{\xi}_{i\wedge i+1}}{\sqrt{m_i m_{i+1}}}.$$

Clearly $E_2 E_2^{-1} = \mathrm{id}$, $E_2^{-1} E_{2|q_0\mathscr{K}_g} = \mathrm{id}$ and it remains to check that E_2^{-1} extends to a bounded operator on $q_0 H$. Using the asymptotics $m_k \sim C a^k$ with $a > 1$ we get

$$0 \leqslant \big(E_2^{-1}(\tilde{\xi}_k) \big| E_2^{-1}(\tilde{\xi}_l)\big) = \frac{2}{m_1} \sum_{i=\max(k,l)}^{\infty} \frac{m_k m_l}{m_i m_{i+1}} \leqslant D a^{-|k-l|}$$

for some constant D, and the matrix $(a^{-|k-l|})_{k,l}$ is easily seen to be bounded.

This proves that $E_2 : q_0K_g \to q_0H$ is invertible. Since $q_0 = Q_0$, Proposition 4.1 shows that $E_2 : (1 - q_0)K_g \to (1 - q_0)H$ is injective with closed range. Moreover, by Lemma 2.7 the subspace $E_2(K_g)$ contains $\operatorname{Ker}\epsilon$, which is dense in H. Hence $E_2 : K_g \to H$ is invertible. In particular Corollary 2.8 applies with $K'_g = K_g$. Finally $\xi_0 = \Lambda_h(1)$ is tautologically a fixed vector for c_0, and the unique vector $\xi_g \in K_g$ such that $E_2\xi_g = \xi_0$ is a fixed vector for c_g. □

4.2. The general case

We come back to the general case of free products of orthogonal and unitary universal quantum groups. We will prove that the path cocycle obtained at Corollary 4.2 is not trivial — except in the cases considered in the previous section. The strategy is as follows: we will exhibit a particularly simple infinite geodesic in the quasi-classical subtree, compute the values of the path cocycle along this geodesic, and realize that they are unbounded.

Notice that the computations of the previous section easily yield the existence in general of "weak" path cocycles which are trivial, i.e. of trivial cocycles $c : \mathscr{S} \to K_g$ such that $E_2 \circ c_g = c_0$: for each infinite geodesic starting from 1 in the classical Cayley graph, we get a vector $\zeta_\infty \in K_g$ such that $E_2\zeta_\infty = \xi_0$, and the trivial cocycle with fixed vector ζ_∞ is clearly a "weak" path cocycle. However such a path cocycle does not take its values in $\mathscr{K}_g$ nor $\mathscr{K}'_g$ in the non-orthogonal case. Notice also in the non-orthogonal case that we can choose two different geodesics ∞, ∞', and obtain in this way a non-zero vector $\zeta = \zeta_\infty - \zeta_{\infty'} \in K_g$ such that $E_2\zeta = 0$.

Of course we have to exclude now the case when the quantum group under consideration consists of a single copy of $A_o(Q)$, already studied in Section 4.1. For the sake of simplicity we assume that our quantum group contains a copy of some $A_u(Q)$. We denote by $\gamma \in \operatorname{Irr}\mathscr{C}$ the corresponding fundamental corepresentation and we recall that the tensor powers $\gamma^{\otimes n}$ are irreducible and denoted $\gamma^n \in \operatorname{Irr}\mathscr{C}$. In the case where the quantum group does not contain any $A_u(Q)$, but instead two free copies of $A_o(Q_1)$, $A_o(Q_2)$ with fundamental corepresentations γ_1, γ_2, the corepresentations γ^n should be replaced by alternating irreducible tensor products $\gamma_1\gamma_2\gamma_1\cdots\gamma_k$.

Put $H_1 = p_\gamma H$, denote $\tilde\xi_1 = \tilde\xi_\gamma \in H_1$ the normalized GNS image of $p_\gamma \in \hat S$ already introduced in Section 4.1, and define $H_1^\circ = \tilde\xi_1^\perp \cap H_1$. One can check that $\epsilon(\tilde\xi_1) = m_1 := \dim_q \gamma$. Due to the irreducibility mentioned above, we have natural Hilbert space identifications $p_{\gamma^n}H = H_1^{\otimes n}$, $(p_{\gamma^n} \otimes p_\gamma)K = (p_{\gamma^n} \otimes p_\gamma)K_{++} = H_1^{\otimes n} \otimes H_1$. For any $0 \leqslant l \leqslant n$ we define the "length l" subspaces:

$$q_l H_1^{\otimes n} = H_1^{\otimes l-1} \otimes H_1^\circ \otimes \tilde\xi_1^{\otimes n-l},$$

with the convention that $q_0H_1^{\otimes n} = \mathbb{C}\tilde\xi_1^{\otimes n}$. This defines projections $q_l \in B(p_{\gamma^n}H)$ and $q_l \in B((p_{\gamma^{n-1}} \otimes p_\gamma)K)$ for any $n \in \mathbb{N}^*$ and $0 \leqslant l \leqslant n$.

This is of course compatible with the notation q_0 used in Section 4.1, and more generally this gives an explicit description of the projections $q_l = \hat\pi_2\hat\delta(p_{2l})$, $q_l = \hat\pi_4\hat\delta^3(p_{2l})$ of [21, p. 125] on the (particularly simple) subspaces $p_{\gamma^n}H$, $(p_{\gamma^n} \otimes p_\gamma)K$. In particular it is known that E_2, E_1, Θ intertwine the relevant projections q_l. Concretely, $E_2 : (p_{\gamma^n} \otimes p_\gamma)K \to p_{\gamma^{n+1}}H$ is just the identification $H_1^n \otimes H_1 = H_1^{\otimes n+1}$, the operator $E_1 = \mathrm{id} \otimes \epsilon$ maps $\zeta \otimes \tilde\xi_1 \in q_l(p_{\gamma^n} \otimes p_\gamma)K$ to $m_1\zeta$ if $l \leqslant n$, and $q_{n+1}(p_{\gamma^n} \otimes p_\gamma)K$ to 0.

Now we fix n, l, and $\zeta \in q_l p_{\gamma^n}H$. As in the proof of Lemma 2.7, we can easily construct an element $\eta \in \mathscr{K}_{++}$ such that $(E_2 - E_1)\eta = \zeta - \epsilon(\zeta)\xi_0$: we put $\eta = \eta_1 + \eta_2 + \cdots + \eta_n$ with

$\eta_i \in (p_{\gamma^{n-i}} \otimes p_\gamma)K_{++}$ uniquely determined by the identity $E_2\eta_i = E_1\eta_{i-1}$, $E_2\eta_1 = \zeta$. By the remarks in the preceding paragraph we see that η_i vanishes for $i > n-l+1$, and $\|\eta_i\| = m_1^{i-1}\|\zeta\|$ for $i \leqslant n-l+1$.

To obtain a "path from the origin to ζ" in $\mathscr{K}'_g$, it remains to project η onto K_g. This is easy because η lies "almost entirely" in Q_0K_{++}. More precisely, we assert that η_i belongs to Q_0K_{++} for $i \leqslant n-l$, and that η_{n-l+1} belongs to $(1-Q_0)K_{++}$.

As a matter of fact, the subspace $(p_{\gamma^{n+1}} \otimes p_{\bar\gamma})K_{+-}$ is isomorphic as a left–right representation of $\hat S$ to $\gamma^{n+1}\bar\gamma \otimes \bar\gamma^n = \gamma^{n+1}\bar\gamma^{n+1}$, and in particular it is irreducible of length $n+1$, so that $q_l(p_{\gamma^{n+1}} \otimes p_{\bar\gamma})K_{+-} = 0$ if $l \leqslant n$ [21, p. 125]. Since we have $\Theta(p_{\gamma^n} \otimes p_\gamma)K_{++} \subset (p_{\gamma^{n+1}} \otimes p_{\bar\gamma})K_{\bigstar -}$, this implies that $q_l(p_{\gamma^n} \otimes p_\gamma)K_{++} \subset Q_0K_{++}$ for $l \leqslant n$. Similarly, $q_{n+1}(p_{\gamma^{n+1}} \otimes p_{\bar\gamma})K_{--}$ vanishes, so that $q_{n+1}(p_{\gamma^n} \otimes p_\gamma)K_{++} \subset (1-Q_0)K_{++}$.

From these remarks we can conclude that the orthogonal projection of η_i onto K_g simply equals $(\eta_i - \Theta(\eta_i))/2$ and is orthogonal to the one of η_{n-l+1}, for each $i < n-l+1$. Summing over these i's we obtain

$$\left\|c_g(\zeta)\right\|^2 = \left\|p_g(\eta)\right\|^2 \geqslant \frac{1}{2}\sum_{i=1}^{n-l} m_1^{2(i-1)}\|\zeta\|^2 \geqslant C m_1^{2(n-l)}\|\zeta\|^2 \tag{5}$$

for some constant $C > 0$ depending only on m_1. This should be put in contrast with the situation in $A_o(Q)$, where we have $\|c_g(\zeta)\| \leqslant C\|\zeta\|$ for all ζ such that $\epsilon(\zeta) = 0$, by Theorem 4.4. Note however that this does not directly mean that the path cocycle is "unbounded". The precise statement is indeed the following one:

Proposition 4.5. *Let γ be the fundamental corepresentation of $A_u(I_N)$ and consider the corepresentations $\gamma^n \in B(H_{\gamma^n}) \otimes S$. Then the norm of $C_n := (\mathrm{id} \otimes c_g)(\gamma^n)$ as an element of $B(H_{\gamma^n}, H_{\gamma^n} \otimes K_g)$ is bounded below by $C\sqrt{n+1}$ for some constant $C > 0$ depending only on m_1. In particular the path cocycle $c_g : \mathscr{S} \to \mathscr{K}'_g$ is not trivial in the non-orthogonal case.*

Proof. We fix an orthonormal basis (e_i) of the space H_γ of the corepresentation γ, and we denote by u_{ij} the corresponding generators of $A_u(Q)$. If $\underline{i} = (i_1, \ldots, i_n)$, $\underline{k} = (j_n, \ldots, j_n)$ are multi-indices, we put $e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_n} \in H_{\gamma^n}$ and $u_{\underline{ik}} = u_{i_1k_1} \cdots u_{i_nk_n} \in \mathscr{S}$. Via the GNS map $\mathscr{S} \to H$ the vector $u_{\underline{ik}}$ identifies with an element of $p_{\gamma^n}H$.

We have by definition and inequality (5):

$$\begin{aligned}\left\|C_n(e_{\underline{i}})\right\|^2 &= \left\|\sum_{\underline{k}} e_{\underline{k}} \otimes c_g(u_{\underline{ik}})\right\|^2 = \sum_{\underline{k}} \left\|c_g(u_{\underline{ik}})\right\|^2 \\ &\geqslant C\sum_{\underline{k}}\sum_{l=0}^{n} m_1^{2(n-l)}\left\|q_l(u_{\underline{ik}})\right\|^2.\end{aligned}$$

Observe indeed that c_g maps the respective subspaces $q_l\mathscr{H}$ to the subspaces q_lK_g, which are mutually orthogonal.

It remains to find a lower bound for $\|q_l(u_{\underline{ik}})\|$. This is where we use the additional "unimodularity" assumption (i.e. $Q = I_N$) to simplify computations. In this case we can indeed identify $p_{\gamma^n}H$ with $B(H_{\gamma^n}) \simeq B(H_\gamma)^{\otimes n}$ equipped with the normalized Hilbert–Schmidt norm

$\|a\|_{HS}^2 = \mathrm{Tr}(a^*a)/m_1^n$, by sending $u_{\underline{i}\underline{k}}$ to $e_{\underline{i}}e_{\underline{k}}^*$. By definition, $q_l p_{\gamma^n} H$ corresponds then to the subspace of applications of the form $a_1 \otimes \cdots \otimes a_l \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$ with $a_i \in B(H_\gamma)$ and $\mathrm{Tr}(a_l) = 0$.

In particular, if $k_l \neq i_l$ and $k_p = i_p$ for $p > l$ we have in this identification

$$q_l\big(e_{\underline{i}}e_{\underline{k}}^*\big) = m_1^{-(n-l)}\big(e_{i_1}e_{k_1}^*\big) \otimes \cdots \otimes \big(e_{i_l}e_{k_l}^*\big) \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}.$$

Hence we obtain $\|q_l(u_{\underline{i}\underline{k}})\|^2 = m_1^{-2(n-l)} m_1^{n-l}/m_1^n = m_1^{-2n+l}$. Observing moreover that there are at least m_1^{l-1} multi-indices $\underline{k}$ satisfying the conditions above for a fixed $\underline{i}$, our first inequality yields

$$\big\|C_n(e_{\underline{i}})\big\|^2 \geqslant C \sum_{l=0}^{n} m_1^{2(n-l)} m_1^{l-1} m_1^{-2n+l} = (n+1)\frac{C}{m_1}.$$

Finally, if c_g was trivial with fixed vector $\xi_g \in K_g$, we would have $C_n(\zeta) = \gamma^n(\zeta \otimes \xi_g)$, hence $\|C_n(\zeta)\| \leqslant \|\zeta\| \times \|\xi_g\|$, for all n and $\zeta \in H_{\gamma^n}$. □

To conclude this section, let us notice that the path cocycle c_g considered in the previous proposition is unbounded, but at the same time it can be shown not to be proper. To see this, one considers the subspaces of H, K corresponding to the infinite geodesic $(1, \gamma, \gamma\bar{\gamma}, \gamma\bar{\gamma}\gamma, \ldots)$ in the classical Cayley graph of $\mathscr{A}_u(Q)$ — they are given by the projection P_γ already used in the proof of Theorem 3.5 and introduced in [21, p. 125]. Then the results of the orthogonal case apply to these subspaces, basically because $q_0 P_\gamma = Q_0 P_\gamma$. In particular one can show that the restriction of c_g to this particular geodesic is bounded.

5. Application to the first L^2-cohomology

We begin this section with some notation. Let $\mathscr{S}$ be the dense Hopf algebra associated with a discrete quantum group. For any representation $\pi : \mathscr{S} \to L(X)$ of $\mathscr{S}$ on a complex vector space X, we endow X with the trivial right $\mathscr{S}$-module structure given by ϵ, and we denote by $H^1(\mathscr{S}, X)$ the first Hochschild cohomology group of $\mathscr{S}$ with coefficients in X. Recall that we have $H^1(\mathscr{S}, X) = \mathrm{Der}(\mathscr{S}, X)/\mathrm{Inn}(\mathscr{S}, X)$, where $\mathrm{Der}(\mathscr{S}, X)$ coincides with the group of π-cocycles from $\mathscr{S}$ to X, and $\mathrm{Inn}(\mathscr{S}, X)$ is the subgroup of trivial cocycles.

The coefficient modules of interest for this section will be H, via the regular representation $\lambda : \mathscr{S} \to B(H)$, and the von Neumann algebra $M := \lambda(\mathscr{S})'' \subset B(H)$, via left multiplication by elements of $\mathscr{S}$.

In the previous sections we have studied one particular cocycle c on $\mathscr{S}$, namely the path cocycle c_g with values in K. In the case of the universal quantum groups $A_o(Q)$, we have seen that c_g is trivial as an element of $H^1(\mathscr{S}, K)$. In this section we use the vanishing of c_g to prove that many more cocycles are trivial: in fact the whole L^2-cohomology group $H^1(\mathscr{S}, M)$ vanishes in the case of $A_o(I_n)$. The heuristic reason is that c_g, being a push-back of the trivial cocycle through the restricted multiplication map E_2, is sufficiently universal amongst L^2-cocycles.

In fact a slight strengthening of the vanishing of c_g will be needed: we will need to know that the fixed vector $\xi_g \in K = H \otimes p_1 H$ lies in $\Lambda_h(M) \otimes p_1 H$.

Theorem 5.1. *Let $\mathscr{S}$, M, H be the Hopf algebra, the von Neumann algebra and the Hilbert space associated with a unimodular discrete quantum group. Using the terminology of Definition* 2.5 *we* *assume that there exists a path cocycle $c_g : \mathscr{S} \to K$ which is trivial, and which admits moreover a fixed vector ξ_g lying in $\Lambda_h(M) \otimes p_1 H \subset K$. Then we have $H^1(\mathscr{S}, M) = 0$.*

Proof. We proceed in 4 steps.

1. We observe that for any linear map $c : \mathscr{S} \to M$ the following formula, making use of the right M-module structure of H, defines a bounded map $m_c : K \to H$:

$$m_c\big(\zeta \otimes \Lambda(x)\big) = \zeta \cdot c(x).$$

The main reason is that in this formula only the values of c on the finite-dimensional subspace $\mathscr{S}_1 \subset \mathscr{S}$ corresponding to $p_1 H \subset \mathscr{H}$ play a role.

More precisely, recall that the Haar state h is a trace in the unimodular case, and denote by $\rho(x) \in B(H)$ the action by right multiplication of $x \in M$. Fix an orthonormal basis $(\Lambda(x_i))_{1 \leqslant i \leqslant N}$ of the space $p_1 H$. Any $\eta \in K$ can be decomposed into a sum $\sum \zeta_i \otimes \Lambda(x_i)$ and we have then

$$\big\|m_c(\eta)\big\| = \left\| \sum \rho\big(c(x_i)\big)\zeta_i \right\| \leqslant \sum \big\|c(x_i)\big\|_M \|\zeta_i\|_H \leqslant C\sqrt{N}\|\eta\|,$$

where C is the norm of c considered as an operator from $p_1 H$ to M.

2. We carry on the following computation, using now the fact that c is a cocycle. For all $x \in \mathscr{S}$ and $y \in \mathscr{S}_1$ we have

$$\begin{aligned} m_c\big(\Lambda(x) \otimes \Lambda(y)\big) &= xc(y) = c(xy) - c(x)\epsilon(y) \\ &= c\big(xy - x\epsilon(y)\big) = c(E_2 - E_1)\big(\Lambda(x) \otimes \Lambda(y)\big), \end{aligned}$$

considering c as a map from $\mathscr{S} \simeq \mathscr{H}$ to $M \subset H$.

In particular this shows that m_c vanishes on $\mathrm{Ker}(E_2 - E_1) \cap \mathscr{K}$, which contains $K_g^{\perp} \cap \mathscr{K}$ by Lemma 2.7. Since m_c is bounded and $K_g^{\perp} \cap \mathscr{K}$ is clearly dense in $K_g^{\perp} = \overline{\mathrm{Im}}(\Theta + \mathrm{id})^*$, we can conclude that m_c vanishes on $K_g^{\perp}$. As a result, the identity $m_c = c(E_2 - E_1)$ holds on $\mathscr{K} \oplus K_g^{\perp}$, and in particular, on $\mathscr{K}_g'$.

3. Now we apply this identity to the values of the path cocycle c_g, which lie in $\mathscr{K}_g'$ by definition. This yields, for any $x \in \mathscr{S}$:

$$m_c\big(c_g(x)\big) = c(E_2 - E_1)\big(c_g(x)\big) = 2c\big(x - \epsilon(x)1\big) = 2c(x).$$

Since m_c is defined on the whole of K, we can also put $\xi = \frac{1}{2} m_c(\xi_g)$, and we have for any $x \in \mathscr{S}$:

$$x\xi - \xi\epsilon(x) = \frac{1}{2} m_c\big(x\xi_g - \xi_g\epsilon(x)\big) = \frac{1}{2} m_c\big(c_g(x)\big) = c(x),$$

so that ξ is a fixed vector in H for $c : \mathscr{S} \to M$.

4. It remains to check that the fixed vector ξ lies in fact in M. But this is clear from the definition of m_c and the hypothesis $\xi_g \in \Lambda(M) \otimes p_1 H$. $\square$

The property of rapid decay gives a convenient way to check that a vector $\xi \in H$ lies in $\Lambda_h(M)$, and it is known to hold for $A_o(Q)$ in the unimodular case [22, Thm. 4.9]. Let us recall the definition of Property RD.

We denote by L be the classical length function on the discrete quantum group under consideration, i.e. the central unbounded multiplier of $\hat{S}$ given by $L = \sum l(\alpha) p_\alpha$, where $l(\alpha)$ is the distance from 1 to α in the classical Cayley graph. We consider L as an unbounded operator on H, with domain $\mathscr{H}$, and we denote by H^s the completion of $\mathscr{S}$ with respect to the norm $\|x\|_{2,s} = \|(1+L)^s \Lambda_h(x)\|_H$. Property RD states that there exist constants $C > 0$ and $s > 1$ such that $\|x\|_{S_{\mathrm{red}}} := \|\lambda(x)\|_{B(H)} \leqslant C\|x\|_{2,s}$ for all $x \in \mathscr{S}$. In other words we have then continuous inclusions $H^s \subset S_{\mathrm{red}} \subset M \subset H$.

Corollary 5.2. *Let $\mathscr{S}$, M be the Hopf algebra and the von Neumann algebra associated with $A_o(I_n)$, $n \geqslant 3$. Then we have $H^1(\mathscr{S}, M) = 0$.*

Proof. According to the theorem, it suffices to show that the fixed vector ξ_g of the path cocycle lies in $\Lambda(M) \otimes p_1 H$. Thank to Property RD we will in fact prove that ξ_g lies in $H^s \otimes p_1 H$, using formula (4).

Since $(1+L)^s \otimes \mathrm{id} = (k+1)^s \mathrm{id}$ on $p_k(K)$, the vectors $((1+L)^s \otimes \mathrm{id})(\tilde{\xi}_{i \wedge i+1})$ are pairwise orthogonal, and their respective norms are clearly dominated by $(i+2)^s$. As a result we have

$$\|\xi_g\|_{2,s}^2 \leqslant \frac{2}{m_1} \sum_{i=0}^{\infty} \frac{(i+2)^{2s}}{m_i m_{i+1}}.$$

The left-hand side is finite because the sequence of quantum dimensions (m_i) grows geometrically when $n \geqslant 3$, hence we are done. □

Let $(\mathscr{S}, \delta)$ be the dense Hopf algebra of a unimodular countable discrete quantum group and denote by M the von Neumann algebra associated with $\mathscr{S}$. The definition of L^2-Betti numbers given by W. Lück [11] extends without difficulty to the quantum case [9], as well as the arguments of [16,12], so that we have

$$\beta_k^{(2)}(\mathscr{S}, \delta) = \dim_M H_k(\mathscr{S}, M) = \dim_{M^\circ} H^k(\mathscr{S}, M).$$

Here the action of M° comes from multiplication of elements of M on the right of the coefficient module M.

The result of Corollary 5.2 yields immediately the following result, which strongly contrasts with the values $\beta_1^{(2)}(F_n) = n - 1$:

Corollary 5.3. *For any $n \geqslant 3$ we have $\beta_1^{(2)}(\mathscr{A}_o(I_n), \delta) = 0$.*

Notice that we also have $\beta_0^{(2)}(\mathscr{S}, \delta) = 0$ for $\mathscr{S} = \mathscr{A}_o(I_n)$, because $M = A_o(I_n)''$ is diffuse. Moreover a free resolution of the co-unit $\epsilon : \mathscr{S} \to \mathbb{C}$ has been constructed in [8], showing in particular that $\beta_k^{(2)}(\mathscr{S}, \delta) = 0$ for $k \geqslant 4$, and $\beta_{3-k}^{(2)}(\mathscr{S}, \delta) = \beta_k^{(2)}(\mathscr{S}, \delta)$ for $k \in \{0, 1, 2, 3\}$. Using the corollary above it results that $\beta_k^{(2)}(\mathscr{S}, \delta) = 0$ for all k. This also holds when $n = 1, 2$ for a completely different reason, namely amenability.

In the unitary case, Proposition 4.5 shows that $H^1(\mathscr{S}, K) \neq 0$ for $\mathscr{S} = \mathscr{A}_u(I_n)$, $n \geqslant 1$. Since K is a direct sum of finitely many copies of H as an $\mathscr{S}$-module, this implies $H^1(\mathscr{S}, H) \neq 0$. Recall that for non-amenable countable discrete groups, the vanishing of $\beta_1^{(2)}(\mathscr{S}, \delta)$ is equivalent to the vanishing of $H^1(\mathscr{S}, H)$ ([5], see also [12, Cor. 2.4]). This carries over to discrete quantum groups [10], hence we obtain $\beta_1^{(2)}(\mathscr{A}_u(I_n), \delta) \neq 0$ for $n \geqslant 2$. It seems reasonable to conjecture that $\beta_1^{(2)}(\mathscr{A}_u(I_n), \delta) = 1$.

Remark 5.4 *(A hilbertian variant).* We give now a direct proof of the vanishing of $H^1(\mathscr{A}_o(I_n), H)$ for $n \geqslant 3$, by adapting the methods of Theorem 5.1.

For a general cocycle $c : \mathscr{S} \to H$ one can define an unbounded map $m_c : \mathscr{K} \to H$ as in the theorem, and it turns out that m_c is closable as a densely defined operator on K. Consider indeed a sequence $\zeta_k = \sum_{ij} \Lambda(x_{ij,k}) \otimes \Lambda(u_{ij}) \in \mathscr{K}$ that converges to 0 and whose image under m_c converges to some vector $\eta \in H$. We have then for any $z \in \mathscr{S}$:

$$\big(\Lambda(z)\big|m_c(\zeta_k)\big) = \sum\big(x_{ij,k}^*\Lambda(z)\big|c(u_{ij})\big) = \sum\big(\rho(z)\Lambda\big(x_{ij,k}^*\big)\big|c(u_{ij})\big),$$

where $\rho(z)$, for $z \in \mathscr{S}$, is the bounded operator of right multiplication by z in the GNS construction of h. We observe that the left-hand side converges to $(\Lambda(z)|\eta)$, whereas the right-hand side converges to 0 since the sum is finite and the convergence $\zeta_k \to 0$ means that $\Lambda(x_{ij,k}) \to 0$ for each i, j. This shows that $\eta = 0$.

Now the vanishing of m_c on $\operatorname{Ker}(E_2 - E_1) \cap \mathscr{K}$ implies the vanishing of its closure $\overline{m}_c$ on $\operatorname{Ker}(E_2 - E_1) \cap \operatorname{Dom} \overline{m}_c$, so that the identity $\overline{m}_c = c \circ (E_2 - E_1)$ holds on $\mathscr{K}_g' \cap \operatorname{Dom} \overline{m}_c$. But it is easy to check that $\Lambda_h(M) \otimes p_1 H \subset \operatorname{Dom} \overline{m}_c$, and since the fixed vector ξ_g lies in $\Lambda_h(M) \otimes p_1 H$ we obtain the inclusion $c_g(\mathscr{S}) \subset \operatorname{Dom} \overline{m}_c$. We can then apply m_c to $c_g(x)$ and ξ_g as in the proof of the theorem and obtain the triviality of c.

Remark 5.5 *(The non-unimodular case).* L^2-Betti numbers have not been defined for non-unimodular discrete quantum groups. However we can still prove the vanishing of an L^2-cohomology group in this case, i.e. for $A_o(Q)$ with $n \geqslant 3$, $Q\bar{Q} \in \mathbb{C}I_n$, Q not unitary.

More precisely it is natural for the proof of Theorem 5.1 to consider the submodule $N \subset H$ of vectors ξ such that $\rho(\xi) : \mathscr{H} \to H$, $\Lambda(x) \mapsto x\xi$ is bounded. The left $\mathscr{S}$-module N can be identified with M endowed with the twisted left action $a \cdot x = \sigma_{i/2}^h(a)x$, via $(x \mapsto Jx^*\xi_0)$. In other words we are considering the group $H^1(\mathscr{S}, M)$ with the twisted action on the left of M, and the trivial action on the right. Let us explain how the techniques of Theorem 5.1 can be used to prove the vanishing of this group.

For any cocycle $c : \mathscr{S} \to N$ one can define a bounded map $m_c : K \to H$ by the formula $m_c(\xi \otimes \Lambda(y)) = \rho(c(y))\xi$ and follow the proof of the theorem. The only point that requires more care is the fact that ξ_g lies in $N \otimes p_1 H$ — since $\rho(\mathscr{S})N \subset N$, this will imply that the fixed vector ξ belongs to N. We cannot apply Property RD anymore, since it does not hold for non-unimodular discrete quantum groups. However, we have the following strong weakening of Property RD, already used in [18, Rem. 7.6], which will suffice for our purposes.

Denoting by H_r^s the completion of $\mathscr{H}$ with respect to the norm $\|\zeta\|_{2,s,r} = \|r^L(1+L)^s\zeta\|_H$, the methods of [22] yield, in the non-unimodular case, a continuous inclusion $H_r^s \subset \Lambda(M)$, where $s = 3$, $r = \|F\|$ and F is the usual positive "modular" element of $B(H_1)$ satisfying $\operatorname{Tr}(F) = \operatorname{Tr}(F^{-1})$. Note that we have also $H_r^s \subset N$ since $JH_r^s = H_r^s$ — indeed $r^L(1+L)^s$ is scalar on each $p_k H$ and $J : p_k H \to p_k H$ is an (anti-)isometry.

Therefore it suffices to check that ξ_g lies in $H_r^s \otimes p_1 H$. Since $L = k\,\mathrm{id}$ on $p_k H$ we get immediately from (4):

$$\|\xi_g\|_{2,s,r}^2 \leqslant \frac{2}{m_1} \sum_{i=0}^{\infty} \frac{r^{2i+2}(i+2)^{2s}}{m_i m_{i+1}} \leqslant C \sum_{i=0}^{\infty} \left(\frac{r}{a}\right)^{2i} (i+2)^{2s},$$

where $a > 1$ is the number such that $\mathrm{Tr}(F) = a + a^{-1}$ — we have then $m_k = (a^{k+1} - a^{-k-1})/(a - a^{-1})$ for all k, hence the second upper bound above. Now we have $a + a^{-1} = \mathrm{Tr}(F) > r + r^{-1}$ since r, hence also r^{-1}, is an eigenvalue of F, and the size of F is at least 3. This implies $a > r$, hence $\|\xi_g\|_{2,s,r} < \infty$.

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