KK-theory for quantum groups : functorial and geometrical methods

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Crossed Products
by Quantum Groups

L.c. quantum groups

- ➜ dual Hopf C^*-algebras S, $\widehat{S}$
- ➜ regular representations $\lambda : S \twoheadrightarrow S_{\mathsf{red}} \subset L(H)$ and $\widehat{S} \to \widehat{S}_{\mathsf{red}} \subset L(H)$
- ➜ trivial representations $\varepsilon : S \to \mathbb{C}$, $\widehat{\varepsilon} : \widehat{S} \to \mathbb{C}$

Woronowicz C^*-algebras

- ➜ S unital Hopf-C^*-algebra with $\delta(S)(1 \otimes S)$ and $\delta(S)(S \otimes 1)$ dense in $S \otimes S$
- ➜ category of corepresentations $\mathcal{C}$ s.t. $\widehat{S} = \widehat{S}_{\mathsf{red}} = \bigoplus_{r \in \mathrm{Irr}\,\mathcal{C}} p_r \widehat{S}$ and $p_r \widehat{S} \simeq L(H_r)$

Ex. : $S = S_{\mathsf{red}} = C(G)$, $H = \ell^2(G)$

Ex. : $S = C^*(\Gamma)$, Irr $\mathcal{C} = \Gamma$, $p_r = 1\!\!1_{\{r\}} \in C_0(\Gamma)$

Crossed products

If S_{red} coacts on A, one can define crossed-product C^*-algebras $A \rtimes \widehat{S}$, $A \rtimes_{\mathsf{red}} \widehat{S}$.

Descent Morphisms

- ➜ S, $\widehat{S}$: full C^*-alg. of a l.c. quantum group
- ➜ A, B endowed with coactions of S_{red}

Proposition

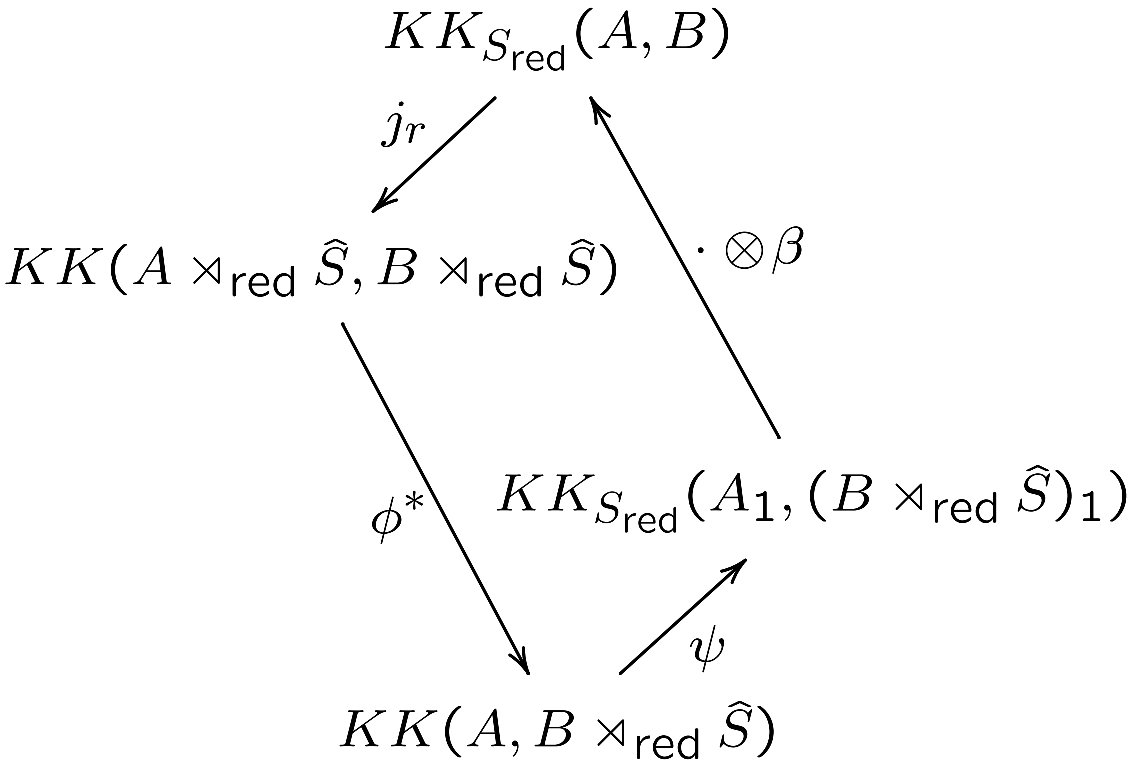
We put $E \rtimes \widehat{S} = E \otimes_B (B \rtimes \widehat{S})$

1. $K_{B \rtimes \widehat{S}}(E \rtimes \widehat{S}) \simeq K_B(E) \rtimes \widehat{S}$.
2. *If* $(E, \pi, F) \in \mathbb{E}_{S_{\mathrm{red}}}(A, B)$ *then* $(E \rtimes \widehat{S}, \pi \rtimes \mathrm{id}, F \rtimes 1) \in \mathbb{E}(A \rtimes \widehat{S}, B \rtimes \widehat{S})$.
3. *this defines a morphism* j *from* $KK_{S_{\mathrm{red}}}(A, B)$ *to* $KK(A \rtimes \widehat{S}, B \rtimes \widehat{S})$
4. $j(x \otimes_D y) = j(x) \otimes_{D \rtimes \widehat{S}} j(y)$

Green-Julg Theorem

- ➔ S_{red} : Woronowicz C^*-algebra
- ➔ A, B endowed with coactions δ_A, δ_B of S_{red}
- ➔ δ_A trivial, B S_{red}-algebra

Theorem *There is an isomorphism* $KK_{S_{\text{red}}}(A,B) \simeq KK(A, B \rtimes_{\text{red}} \widehat{S})$ *given by*



K-amenability

- S, S_{red}, $\widehat{S}$, $\widehat{S}_{\mathrm{red}}$: C^*-algebras of a locally compact quantum group
- $\lambda : S \to S_{\mathrm{red}}$: regular representation
- $\varepsilon : S \to \mathbb{C}$: trivial representation

Ex. : $S = C^*(G)$, $\widehat{S} = \widehat{S}_{\mathrm{red}} = C_0(G)$

Theorem *We have i $\Rightarrow$ ii $\Rightarrow$ iii $\Rightarrow$ iv, and iv $\Rightarrow$ i when S_{red} is unital (discrete case) :*

i. $\mathbb{1} \in KK_{\widehat{S}_{\mathrm{red}}}(\mathbb{C}, \mathbb{C})$ *is represented by* $(E, 1, F)$ *with* $\delta_E \prec \delta_{\widehat{S}_{\mathrm{red}}}$ *(K-amenability)*

ii. $\forall\ A \ \ [\lambda_A] \in KK(A \rtimes S, A \rtimes_{\mathrm{red}} S)$ *is invertible*

iii. $[\lambda] \in KK(S, S_{\mathrm{red}})$ *is invertible*

iv. $\exists\ \alpha \in KK(S_{\mathrm{red}}, \mathbb{C}) \ \ \lambda^*(\alpha) = [\varepsilon] \in KK(S, \mathbb{C})$.

Amalgamated Free Products

- $T \subset S_1$, S_2 : amenable Wor. C^*-algebras
- $S = S_1 *_T S_2$: amalgamated free product
- $P : S \twoheadrightarrow T$, $R_i : S \twoheadrightarrow S_i$ cond. expect.
- E, F_i associated GNS constructions
- $\varepsilon : T$, $S_i \to \mathbb{C}$ trivial representation

Ex. : $S_i = C^*(\Gamma_i)$, $T = C^*(\Delta)$, $\Gamma = \Gamma_1 *_\Delta \Gamma_2$

Definition *Quantum Serre Tree*

- $H = F_1 \otimes_\varepsilon \mathbb{C} \oplus F_2 \otimes_\varepsilon \mathbb{C}$, $K_0 = E \otimes_\varepsilon \mathbb{C}$
- *GNS representations of S*

The classical « Serre » tree (V, E) associated to « Irr $\mathcal{C}_1 *_\mathcal{D}$ Irr $\mathcal{C}_2$ » induces a decomposition $H = \bigoplus p_n H$ and the J.-V. operator F :

- $E(r,i)^\circ \otimes_\varepsilon \mathbb{C} \xrightarrow{\sim} F_i^\circ \otimes_\varepsilon \mathbb{C}$, $\eta\mathbb{C} \to \eta_2 \otimes_\varepsilon 1_\mathbb{C}$

Theorem

1. *$(H \oplus K_0, \pi_{\mathrm{GNS}}, F)$ defines $\gamma \in KK(S_{\mathrm{red}}, \mathbb{C})$*
2. *$(\hat{S}, \hat{\delta})$ is K-amenable*

Quantum Cayley Graph

- S Wor. C^*-algebra, $\hat{S} = \hat{S}_{\rm red} \subset L(H)$
- $p_r \in \hat{S}$ min. central proj. ($r \in$ Irr $\mathcal{C}$)
- $p_1 = \sum_{r \in \mathcal{D}} p_r$, $1_{\mathcal{C}} \notin \mathcal{D}$, $\bar{\mathcal{D}} = \mathcal{D}$

Ex. : $S = C^*(\Gamma)$, $\hat{S} = C_0(\Gamma)$, $p_r = 1\!\!1_{\{r\}}$, $r \in \Gamma$

Definition *Quantum Cayley Graph*

- H *: space of the regular repr. of* S, $\hat{S}$
- $K = H \otimes p_1 H$, $S = \mathrm{id} \otimes \epsilon : K \to H$
- $\Theta = \Sigma(1 \otimes U) V (U \otimes U) \Sigma$
- *regular repr. on* H, *trivial repr. on* $p_1 H$

Definition *Classical Cayley Graph*

- $V =$ Irr $\mathcal{C}$, $\theta(r, r') = (r', r)$
- $E = \{(r, r') \in V^2 \mid r' \subset r \otimes \mathcal{D}\}$
- $C_0(V) \to L(H)$, $1\!\!1_{\{r\}} \mapsto p_r$

Free Quantum Groups

(E, V) is a tree **iff** S is a free product of free quantum groups $A_o(Q)$, $A_u(Q)$ (Wang, Banica). Then (E, V) induces a projection $p_{\star+} \in L(H)$ « on ascending edges ».

Problems ➔ $\Theta^2 \neq 1$
➔ $p_{+\star} := 1 - \Theta p_{\star+} \Theta^* \neq p_{\star+}$ ➔ $p_{++} = p_{\star+} p_{+\star}$
➔ $[p_{\star+}, u_{i,j}]$ *is compact but not of finite rank*
➔ $F = T p_{++} : K_g \to H$ *is not Fredholm*

Theorem *There exists a natural representation* $\pi_\infty : A_o(Q) \to L(H_\infty)$ *such that*

$$\begin{array}{ccc} K_g & & \\ & \searrow^{p_{++}} & \\ & & K_{++} \xrightarrow{\;B\;} H \\ & \nearrow_{R^*} & \\ H_\infty & & \end{array}$$

defines $\gamma \in KK_{\widehat{S}}(\mathbb{C}, \mathbb{C})$ *when* Tr $Q^*Q > 2$. *In the classical case* $H_\infty = 0$.