Hecke Algebras and the Schlichting completion for discrete quantum groups

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Outline

Introduction

Let $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ be a quantum subgroup of a discrete quantum group. The von Neumann algebra of Hecke operators is the commutant of the quasi-regular representation: $\mathscr{L}(\mathbb{\Gamma}, \mathbb{\Lambda}) = B(\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda}))^{\mathbb{\Gamma}}$.

Questions:

1. (combinatorial) description of $\mathscr{L}(\mathbb{\Gamma}, \mathbb{\Lambda})$?
2. modular properties of the canonical state $(\delta_{\mathbb{\Lambda}} \mid \cdot\, \delta_{\mathbb{\Lambda}})$?
3. analytical properties of this von Neumann algebra...

In the classical case 1 is achieved using a dense subalgebra $\mathscr{H}(\Gamma, \Lambda) \simeq c_c(\Lambda\backslash\Gamma/\Lambda)$ with convolution product. Quantum case:

4. construction and description of $\mathbb{\Gamma}/\mathbb{\Lambda}$, $\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}$?
5. boundedness of the action of $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$ on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$?

For 3, 5 we construct the Schlichting completion $(\mathbb{G}, \mathbb{H})$ of $(\mathbb{\Gamma}, \mathbb{\Lambda})$.
→ construction of new locally compact quantum groups.

Outline

Discrete quantum groups and subgroups

A **discrete quantum group** $\mathbb{\Gamma}$ is given by

- a $*$-algebra of the form $c_c(\mathbb{\Gamma}) = \bigoplus_{\alpha \in I(\mathbb{\Gamma})} B(H_\alpha)$, $\dim H_\alpha < \infty$,
- a coproduct $\Delta : c_c(\mathbb{\Gamma}) \to \mathscr{M}(c_c(\mathbb{\Gamma}) \odot c_c(\mathbb{\Gamma}))$.

The existence of integrals $\varphi := h_L$, h_R is automatic.
Denote $p_\alpha = \mathrm{id}_{H_\alpha}$, $a_\alpha = p_\alpha a \in B(H_\alpha)$, $\ell^\infty(\mathbb{\Gamma}) = M(c_c(\mathbb{\Gamma}))$.
$I(\mathbb{\Gamma})$: irreducible reps of $\mathrm{Corep}(\mathbb{\Gamma}) = \mathrm{Rep}(c_c(\mathbb{\Gamma}))$.
Tensor structure : $v \otimes w := (v \otimes w) \circ \Delta$. Duality : $\bar{v} = v \circ R$.

Classical case $\mathbb{\Gamma} = \Gamma$: $\ell^\infty(\mathbb{\Gamma})$ commutative $\Leftrightarrow \forall \alpha \dim H_\alpha = 1$.
Then $I(\mathbb{\Gamma}) = \Gamma$, $\alpha \otimes \beta = \alpha\beta$, $\bar{\alpha} = \alpha^{-1}$.

A (closed quantum) **subgroup** $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ is given by $I(\mathbb{\Lambda}) \subset I(\mathbb{\Gamma})$ stable under tensor product and duality. Then $c_c(\mathbb{\Lambda}) = p_{\mathbb{\Lambda}} c_c(\mathbb{\Gamma})$, $\Delta_{\mathbb{\Lambda}} = (p_{\mathbb{\Lambda}} \otimes p_{\mathbb{\Lambda}})\Delta$, where $p_{\mathbb{\Lambda}} = \sum_{\alpha \in I(\mathbb{\Lambda})} p_\alpha \in \ell^\infty(\mathbb{\Gamma})$. We have $\mathrm{Corep}(\mathbb{\Lambda}) \subset \mathrm{Corep}(\mathbb{\Gamma})$.

The quotient space

Quotient spaces.

- Classical: $I(\mathbb{\Gamma})/\mathbb{\Lambda} = I(\mathbb{\Gamma})/\sim$ where $\alpha \sim \beta \Leftrightarrow \exists \lambda \in I(\mathbb{\Lambda}) \quad \beta \subset \alpha \otimes \lambda$.
- Quantum: $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty(\mathbb{\Gamma})^{\mathbb{\Lambda}} = \{a \in \ell^\infty(\mathbb{\Gamma}) \mid (1 \otimes p_{\mathbb{\Lambda}})\Delta(a) = a \otimes p_{\mathbb{\Lambda}}\}$. We have $\Delta(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})) \subset \ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.
- Categorical: $\mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda}) := \mathrm{Rep}(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}))$ is a left-$\mathrm{Corep}(\mathbb{\Gamma})$-module category with restriction functor $\mathrm{Corep}(\mathbb{\Gamma}) \to \mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$.

Some notation:

- $v_{\mathbb{\Lambda}} \in \mathrm{Corep}(\mathbb{\Lambda})$, $\mathbb{\Lambda}$-isotypical component of $v \in \mathrm{Corep}(\mathbb{\Gamma})$,
- $\kappa_\alpha = \dim_q(\bar{\alpha} \otimes \alpha)_{\mathbb{\Lambda}}$ for $\alpha \in I(\mathbb{\Gamma})$,
- $[\alpha]$ the class of α in $I(\mathbb{\Gamma})/\mathbb{\Lambda}$ (or $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$),
- $p_{[\alpha]} = \sum_{\beta \in [\alpha]} p_\beta \in \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.

Description of the quotient space

Theorem

- *We have* $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty\text{-}\bigoplus_{[\alpha]} p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.
- *We have* $p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \simeq B(H_\alpha)'_{\mathbb{\Lambda}} \cap B(H_\alpha)$ *canonically.*
- *More generally* $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}(\alpha,\beta) = B(H_\alpha, H_\beta)_{\mathbb{\Lambda}}$.
- *For* $a \in \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ *we have* $a = \sum_{[\alpha]} \kappa_\alpha^{-1}(h_R \otimes \mathrm{id})[(S^{-1}(a_\alpha) \otimes 1)\Delta(p_{\mathbb{\Lambda}})]$.

Denote $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = \bigoplus p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, $c(\mathbb{\Gamma}/\mathbb{\Lambda}) = \prod p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$.

Analogue of the counting measure on Γ/Λ :

Corollary

$c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ *admits a (unique) positive,* $\mathbb{\Gamma}$*-invariant faithful form given by*

$$\mu(a) = \sum_{[\alpha]} \kappa_\alpha^{-1} h_L(a_\alpha).$$

The Hecke algebra

Definition

For $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ *define*

$$a * b = (\mathrm{id} \otimes \mu)[\Delta(a)(1 \otimes S(b))] = (\mu S \otimes \mathrm{id})[\Delta(b)(S^{-1}(a) \otimes 1)] \in c(\mathbb{\Gamma})$$

and $a^\sharp = S(a^*) \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.

Classical case: $\forall\alpha\ \dim_q(\alpha) = 1$, $\kappa_\alpha = 1$. We recover the formula

$$(a * b)(g) = \sum_{[h]\in\Gamma/\Lambda} a(gh)b(h^{-1}).$$

Proposition-Definition

$\mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda}) := c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ *is an involutive algebra for* $*$ *and* $^\sharp$, *with unit* $p_{\mathbb{\Lambda}}$, *stable under* σ_t^R, σ_t^L *and* τ_t.

The Hecke algebra

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NB. $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ is also a (possibly degenerate) sub-$*$-algebra of $\ell^\infty(\mathbb{\Gamma})$. Denote, for $\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$:

$$L(\tau) = \#\{\alpha \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma}) \mid \alpha \subset \tau\}, \quad R(\tau) = \#\{\alpha \in I(\mathbb{\Gamma})/\mathbb{\Lambda} \mid \alpha \subset \tau\}.$$

Proposition-Definition

We say that $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a Hecke pair if $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset \ell^\infty(\mathbb{\Gamma})$ is non degenerate $\Leftrightarrow \forall\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda} \quad L(\tau) < \infty$.

Examples: $\mathbb{\Lambda}$ finite or finite index. Normal case: if $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, $\mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ is the convolution algebra of the quotient quantum group $\mathbb{\Gamma}/\mathbb{\Lambda}$.

Hecke Operators

Recall that $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ is endowed with a left $\mathbb{\Gamma}$-action.

Proposition

Let $(\mathbb{\Gamma},\mathbb{\Lambda})$ be a Hecke pair. We have an isomorphism

$$\mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda}) \to \mathrm{End}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))^{\mathbb{\Gamma}},\ a \mapsto T(a) := (\cdot * a)$$

with inverse $(T \to T(p_{\mathbb{\Lambda}}))$. Moreover $(x \mid T(a)y) = (T(a^\sharp)x \mid y)$ for the scalar product associated with μ.

Theorem

Let $(\mathbb{\Gamma},\mathbb{\Lambda})$ be a Hecke pair. $T(a)$ is bounded on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ for all $a \in \mathscr{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ **iff** *we have (RT) : $\kappa_\gamma \leq C_\beta\,\kappa_\alpha$ for all $\gamma \subset \alpha \otimes \beta$.*

Hecke Operators

Theorem

Let $(\mathbb{\Gamma}, \mathbb{\Lambda})$ be a Hecke pair. $T(a)$ is bounded on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ for all $a \in \mathscr{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ **iff** *we have (RT) : $\kappa_\gamma \leq C_\beta \kappa_\alpha$ for all $\gamma \subset \alpha \otimes \beta$.*

This property is not satisfied by all inclusions $\mathbb{\Lambda} \subset \mathbb{\Gamma}$.

By restriction to $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ we have also $\alpha \in \mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$. Denote $\tilde{\alpha}$ is the image of α in the Grothendieck ring $\mathbb{Z}[\mathbb{\Gamma}/\mathbb{\Lambda}]$. If $\mathbb{\Gamma}$ is unimodular:

$$\kappa_\alpha = \dim_q(\bar{\alpha} \otimes \alpha)_{\mathbb{\Lambda}} = \|\tilde{\alpha}\|_2^2.$$

Exercise

Prove in $\mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$ that (RT) is satisfied if $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a Hecke pair.

In the sequel we will see an analytical proof.

Modular properties

Canonical state on $\mathscr{H}(\Gamma, \Lambda)$: $\omega = \epsilon = (p_\Lambda \mid T(\cdot)p_\Lambda)$. It is faithful.

Proposition-Definition

Let $\nabla \in c(\Lambda\backslash\Gamma/\Lambda)$ unique such that $\mu S(a) = \mu(\nabla a)$ for all $a \in \mathscr{H}(\Gamma, \Lambda)$. Then $\theta_t : a \mapsto \sigma_t^R(\nabla^{it} a)$ is a group of $\sharp$-automorphisms of $\mathscr{H}(\Gamma, \Lambda)$ and ω is θ-KMS.

Theorem

Asumme Λ is unimodular. Then $\nabla_\alpha = (\tilde{L}([\![\alpha]\!])/\tilde{R}([\![\alpha]\!]))\, F_\alpha^2$ where the F_α are Woronowicz' modular matrices, and for $\tau = [\![\alpha]\!] \in \Lambda\backslash I(\Gamma)/\Lambda$:

$$\tilde{L}(\tau) = \sum_{[\delta]\in\Lambda\backslash I(\Gamma),[\delta]\subset\tau} (\dim_q \delta)^2/\kappa_{\bar{\delta}}$$

$$\tilde{R}(\tau) = \sum_{[\delta]\in I(\Gamma)/\Lambda,[\delta]\subset\tau} (\dim_q \delta)^2/\kappa_{\delta}.$$

There is also a more involved formula when Λ is not unimodular...

Outline

The C^*-Hopf algebra

Classical reminder.

Consider $\pi : \Gamma \to \mathrm{Bij}(\Gamma/\Lambda)$ by left translations. Define $G = \overline{\pi(\Gamma)}$, $H = \overline{\pi(\Lambda)}$. If (Γ, Λ) is a Hecke pair, G is a locally compact and H is compact open.

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Definition

We put $\mathscr{O}_c(\mathbb{G}) = \mathrm{alg}\text{–}\langle a * b, a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}), b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\rangle \subset c(\mathbb{\Gamma})$.
$C_0(\mathbb{G}) = C^*\text{–}\langle a * b, a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}), b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\rangle \subset \ell^\infty(\mathbb{\Gamma})$.

Classical case: $\Gamma \to G$ induces $C_0(G) \subset \ell^\infty(\Gamma)$.
If $a = \mathbb{1}_{[r]}$, $b = \mathbb{1}_{[s]}$ then $a * b = \mathbb{1}_{\{g \mid g[s]^{-1} = [r]\}}$.

The C^*-Hopf algebra

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Theorem

If $(\mathbb{\Gamma}, \mathbb{\Lambda})$ *is a Hecke pair we have*

$$\Delta(\mathscr{O}_c(\mathbb{G}))(1 \otimes \mathscr{O}_c(\mathbb{G})) = \mathscr{O}_c(\mathbb{G}) \odot \mathscr{O}_c(\mathbb{G}) = \Delta(\mathscr{O}_c(\mathbb{G}))(\mathscr{O}_c(\mathbb{G}) \otimes 1).$$

$\mathscr{O}_c(\mathbb{G})$ *is a multiplier Hopf algebra.*
$C_0(\mathbb{G})$ *is a bisimplifiable Hopf* C^**-algebra.*

The Haar weights

Corollary

$C(\mathbb{H}) := p_\Lambda C_0(\mathbb{G})$ is a Hopf C^*-algebra (with unit p_Λ). Hence it admits a Haar state h.

Moreover one has $\mathcal{O}_c(\mathbb{G})^{\mathbb{H}} = c_c(\Gamma/\Lambda)$ as subspaces of $\ell^\infty(\Gamma)$.

Corollary

$\varphi := \mu(\mathrm{id} \otimes hp_\Lambda)\Delta$ is an integral on $\mathcal{O}_c(\mathbb{G})$. $\mathbb{G}$ is an algebraic quantum group, hence a locally compact quantum group.

If $\Lambda \subset \Gamma$ is normal, $H = \pi(\Lambda) = \{1\}$, $G \simeq \Gamma/\Lambda$...

Proposition

If the action of Γ on Γ/Λ is faithful and Λ is infinite, $\mathbb{G}$ is non-discrete.

The Hecke Algebra

Let $(\mathbb{G}, \mathbb{H})$ be the Schlichting completion of a Hecke pair (Γ, Λ).
Recall that $\mathbb{H}$ is compact and put

- $c_c(\mathbb{G}/\mathbb{H}) := \mathscr{O}_c(\mathbb{G})^{\mathbb{H}} \subset \mathscr{O}_c(\mathbb{G})$ and $\ell^2(\mathbb{G}/\mathbb{H}) = \overline{c_c(\mathbb{G}/\mathbb{H})} \subset L^2(\mathbb{G})$,
- $\mathscr{H}(\mathbb{G}, \mathbb{H}) = {}^{\mathbb{H}}\mathscr{O}_c(\mathbb{G})^{\mathbb{H}}$ with the convolution product of $\mathscr{O}_c(\mathbb{G})$.

Proposition

We have $\mathscr{H}(\mathbb{G}, \mathbb{H}) \simeq \mathrm{End}(c_c(\mathbb{G}/\mathbb{H}))^{\mathbb{G}}$, $b \mapsto T'(b) := (\,\cdot * b)$.

By construction of $(\mathbb{G}, \mathbb{H})$ we have

$$\mathrm{End}(c_c(\mathbb{G}/\mathbb{H}))^{\mathbb{G}} = \mathrm{End}(c_c(\Gamma/\Lambda))^{\Gamma} \text{ and } \ell^2(\mathbb{G}/\mathbb{H}) \simeq \ell^2(\Gamma/\Lambda).$$

Since the operators $T'(b)$ arise from the right regular repr. of $\mathbb{G}$ we get:

Corollary

The Hecke operators $T(a)$, for $a \in \mathscr{H}(\Gamma, \Lambda)$, are bounded on $\ell^2(\Gamma/\Lambda)$.

HNN Extensions

Fixe $\mathbb{\Lambda}_{\pm 1} \subset \mathbb{\Gamma}_0$ with an isomorphism $\theta : \mathbb{\Lambda}_1 \to \mathbb{\Lambda}_{-1}$.
Consider $\mathbb{\Gamma} = HNN(\mathbb{\Gamma}_0, \theta)$ [Fima 2013]. $\mathrm{Corep}(\mathbb{\Gamma})$ is generated by $\mathrm{Corep}(\mathbb{\Gamma}_0)$ and a 1-dimensional w such that $w^\epsilon \otimes v \otimes w^{-\epsilon} = \theta_*^\epsilon(v)$ for $v \in \mathrm{Corep}(\mathbb{\Lambda}_\epsilon)$.

Proposition

- *If $\mathbb{\Lambda}_{\pm 1}$ have finite index in $\mathbb{\Gamma}_0$ and are different from $\mathbb{\Gamma}_0$, then $\mathbb{\Gamma}_0 \subset \mathbb{\Gamma}$ is almost normal, not normal, of infinite index.*
- *If $\bigcap_{k \in \mathbb{Z}} \mathrm{Dom}\, \theta^k = \{1\}$ the action of $\mathbb{\Gamma}$ on $\mathbb{\Gamma}/\mathbb{\Gamma}_0$ is faithful.*
- *We have $\nabla_w = (\tilde{L}([\![w]\!])/\tilde{R}([\![w]\!]))\, p_w$ with*
  $$\tilde{L}([\![w]\!]) = \textstyle\sum_{[\delta] \in \mathbb{\Lambda}_1 \backslash I(\mathbb{\Gamma}_0)} \dim_q(\delta \otimes \bar{\delta}) / \dim_q(\delta \otimes \bar{\delta})_{\mathbb{\Lambda}_1},$$
  $$\tilde{R}([\![w]\!]) = \textstyle\sum_{[\delta] \in I(\mathbb{\Gamma}_0)/\mathbb{\Lambda}_{-1}} \dim_q(\bar{\delta} \otimes \delta) / \dim_q(\bar{\delta} \otimes \delta)_{\mathbb{\Lambda}_{-1}}.$$

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Example. $\mathbb{\Sigma}_{\pm 1}$ non classical finite quantum groups.
Take $\mathbb{\Gamma}_0 = \prod'_{k\in\mathbb{Z}^*} \mathbb{\Sigma}_{\mathrm{sgn}(k)}$ and $\mathbb{\Lambda}_\epsilon = \prod'_{k\in\mathbb{Z}^*, k\neq\epsilon} \mathbb{\Sigma}_{\mathrm{sgn}(k)} \subset \mathbb{\Gamma}_0$.
Then the Proposition applies, $\tilde{L}([\![w]\!]) = \#\mathbb{\Sigma}_1$, $\tilde{R}([\![w]\!]) = \#\mathbb{\Sigma}_{-1}$.

$\mathbb{G}$ is a non-discrete, non-classical, non-co-classical locally compact quantum group, with non-trivial modular group if $\#\mathbb{\Sigma}_1 \neq \#\mathbb{\Sigma}_{-1}$.