Furstenberg boundary for discrete quantum groups

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Reduced group C^*-algebras

Let Γ be a (discrete) group. Regular representation:

$$\lambda : \Gamma \to B(\ell^2\Gamma), \lambda(g)(\xi) = (h \mapsto \xi(g^{-1}h)).$$

Reduced C^*-algebra: $C^*_{\mathrm{red}}(\Gamma) = \overline{\mathrm{Span}}\, \lambda(\Gamma) \subset B(\ell^2(\Gamma))$.
Canonical trace: $\tau(x) = (\xi_0 \mid x\xi_0)$ with $\xi_0(h) = \delta_{h,e}$, $\tau(\lambda(g)) = \delta_{g,e}$.
Satisifies $\tau(1) = 1$, $\tau(x^*x) \geq 0$, $\tau(xy) = \tau(yx)$ for $x, y \in C^*_{\mathrm{red}}(\Gamma)$.

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C^***-simplicity**: $C^*_{\mathrm{red}}(\Gamma)$ has no non-trivial closed two-sided ideal.
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Examples: free group F_n [Powers 1975], $PSL_n(\mathbb{Z})$ for $n \geq 2$.

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Amenability: existence of $\varepsilon : C^*_{\mathrm{red}}(\Gamma) \to \mathbb{C}$ such that $\varepsilon(\lambda(g)) = 1\ \forall g$.
Then ϵ is a trace and $\mathrm{Ker}(\epsilon)$ is a non-trivial ideal. More generally the existence of a non trivial amenable normal subgroup $N \lhd \Gamma$ is an obstruction to C^*-simplicity and the Unique Trace Property.
Amenable radical $R_{amen} \lhd \Gamma$: largest amenable normal subgroup.

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Tool: Furstenberg, 1950s, topological dynamics

- notion of Γ-boundary $\Gamma \curvearrowright X$ compact
- universal Γ-boundary $\Gamma \curvearrowright \partial_F\Gamma$

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Theorem (Kalantar, Kennedy, Breuillard, Ozawa, 2014)

Γ is C^*-simple $\Leftrightarrow$ there exists a Γ-boundary with free action
$\Leftrightarrow$ the action of Γ on $\partial_F\Gamma$ is free.
Γ has the UTP $\Leftrightarrow$ there exists a Γ-boundary with faithful action
$\Leftrightarrow$ the action of Γ on $\partial_F\Gamma$ is faithful.
Moreover $\mathrm{Ker}(\Gamma \curvearrowright \partial_F\Gamma) = R_{amen}$.

Boundary actions

Continuous action $\Gamma \curvearrowright X$ on X compact.
We have $X \subset \mathrm{Prob}(X)$ via Dirac measures and $\Gamma \curvearrowright \mathrm{Prob}(X)$.

The action $\Gamma \curvearrowright X$ is:

- minimal if $\forall x, y \in X\ \exists g_n \in \Gamma$ s.t. $\lim g_n \cdot x = y$,
  in other words: $\forall x \in X \quad \overline{\Gamma \cdot x} = X$;
- proximal if $\forall x, y \in X\ \exists g_n \in \Gamma$ s.t. $\lim g_n \cdot x = \lim g_n \cdot y$;
- strongly proximal if $\Gamma \curvearrowright \mathrm{Prob}(X)$ proximal,
  or equivalently: $\forall \nu \in \mathrm{Prob}(X) \quad \overline{\Gamma \cdot \nu} \cap X \neq \emptyset$.

X is a Γ**-boundary** if it is minimal and strongly proximal,
or equivalently: $\forall \nu \in \mathrm{Prob}(X) \quad X \subset \overline{\Gamma \cdot \nu}$.

Classical examples:

- G connected simple Lie group, $H < G$ maximal amenable, $X = G/H$
- Γ non elementary hyperbolic, $X = \partial_G \Gamma$ Gromov boundary

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Convolution operations: $\mu \in \mathrm{Prob}(\Gamma)$, $\nu \in \mathrm{Prob}(X) \to \mu * \nu \in \mathrm{Prob}(X)$

$\nu \in \mathrm{Prob}(X)$, $f \in C(X) \to P_\nu(f) = \nu * f \in \ell^\infty(\Gamma)$

The following assertions are equivalent:

i) $\forall \nu \in \mathrm{Prob}(X) \quad X \subset \overline{\Gamma \cdot \nu}$ (X is a Γ-boundary)

ii) $\forall \nu \in \mathrm{Prob}(X) \quad \mathrm{Prob}(X) = \overline{\mathrm{Conv}}\, \Gamma \cdot \nu$

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v) all unital positive Γ-maps $T : C(X) \to \ell^\infty(\Gamma)$ are isometric
(indeed $T = P_\nu$ for $\nu = \mathrm{ev}_e \circ T$)

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Quantum case: no sets X, Γ anymore but

- noncommutative “function” algebras $C(\mathbb{X})$, $\ell^\infty(\mathbb{\Gamma})$
- state spaces $\mathrm{Prob}(\mathbb{X}) = S(C(\mathbb{X}))$, $\mathrm{Prob}(\mathbb{\Gamma}) = S_*(\ell^\infty(\mathbb{\Gamma}))$

Outline

Discrete quantum groups

A discrete quantum group $\mathbb{\Gamma}$ is given by :

- a von Neumann algebra $\ell^\infty(\mathbb{\Gamma}) = \bigoplus_{\alpha\in I}^{\ell^\infty} B(H_\alpha)$ with $\dim H_\alpha < \infty$
- a normal $*$-homomorphism $\Delta : \ell^\infty(\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})$ such that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (coproduct)
- left and right Δ-invariant nsf weights h_L, h_R on $\ell^\infty(\mathbb{\Gamma})$

Denote $\ell^2(\mathbb{\Gamma}) = L^2(\ell^\infty(\mathbb{\Gamma}), h_L)$ the GNS space for h_L.

$\mathbb{\Gamma}$ is **unimodular** if $h_L = h_R$.

The coproduct induces a tensor product $\pi \otimes \rho := (\pi \otimes \rho)\Delta$ for representations π, ρ of $\ell^\infty(\mathbb{\Gamma})$ ➞ tensor C^*-category $\mathrm{Corep}(\mathbb{\Gamma})$.

Classical case: $\mathbb{\Gamma} = \Gamma = I$, $\ell^\infty(\mathbb{\Gamma}) = \ell^\infty(\Gamma)$, $\Delta(f) = ((r,s) \mapsto f(rs))$, $h_L(f) = h_R(f) = \sum_{r\in\Gamma} f(r)$.

Actions of quantum groups

Canonical dense subalgebra : $c_0(\mathbb{\Gamma}) \subset \ell^\infty(\mathbb{\Gamma})$ given by $\bigoplus^{c_0}_{\alpha \in I}$.
A $\mathbb{\Gamma}$-C^***-algebra** is a C^*-algebra A equipped with a $*$-homomorphism $\alpha : A \to M(c_0(\mathbb{\Gamma}) \otimes A)$ such that $(\mathrm{id} \otimes \alpha)\alpha = (\Delta \otimes \mathrm{id})\alpha$ (coaction).

For $a \in A$, $\nu \in A^*$, $\mu \in c_0(\mathbb{\Gamma})^*$ we can then define

$$L_\mu(a) = (\mu \otimes \mathrm{id})\alpha(a) \in M(A),$$
$$P_\nu(a) = (\mathrm{id} \otimes \nu)\alpha(a) \in \ell^\infty(\mathbb{\Gamma}),$$
$$\mu * \nu = (\mu \otimes \nu)\alpha \in A^*.$$

A $\mathbb{\Gamma}$**-map** $T : A \to B$ is a linear map such that $T \circ L_\mu = L_\mu \circ T$.

Classical case: $\Gamma \curvearrowright X$, $A = C_0(X)$, $\alpha(f) = ((r, x) \mapsto f(r \cdot x))$.

Example: $A = c_0(\mathbb{\Gamma})$, $\alpha = \Delta$ "translation action".
By invariance, the maps L_μ extend to bounded operators on $\ell^2(\mathbb{\Gamma})$.
➜ C^*-algebra $C^*_{\mathrm{red}}(\mathbb{\Gamma}) = \overline{\mathrm{Span}}\,\{L_\mu\}$ with state $h = (\xi_0 \mid \cdot\, \xi_0)$.
Note: h is a trace $\Leftrightarrow$ $\mathbb{\Gamma}$ unimodular.

Quantum Furstenberg boundary

In the noncommutative framework it is better to work with **completely** positive (resp. isometric) maps $T : A \to B$, i.e. such that $T \otimes \mathrm{id} : M_n(A) \to M_n(B)$ is positive (resp. isometric) for all n.

Definition (KKSV, after Kalantar-Kennedy and Hamana)

A unital Γ-C^*-algebra A is a Γ-boundary if every UCP Γ-map $T : A \to B$ is automatically UCI.

This has good categorical properties : $\mathbb{C} \hookrightarrow A$ is an “essential extension” in the category of unital Γ-C^*-algebras with UCP Γ-maps as morphisms and UCI Γ-maps as embeddings.

Theorem (KKSV, after Hamana)

There exists a universal Γ-boundary $C(\partial_F \Gamma)$. For any Γ-boundary A there exists a unique Γ-equivariant $*$-homomorphism $T : A \to C(\partial_F \Gamma)$.

The amenable radical

Let $\alpha : A \to M(c_0(\mathbb{\Gamma}) \otimes A)$ be a coaction.

Definition

The cokernel $N_\alpha \subset \ell^\infty(\mathbb{\Gamma})$ of α is the weak closure of $\{P_\nu(a), a \in A, \nu \in A^*\}$. We say that α is faithful if $N_\alpha = \ell^\infty(\mathbb{\Gamma})$.

The subspace $M = N_\alpha$ is a *Baaj–Vaes subalgebra*: $\Delta(M) \subset M\bar{\otimes}M$.

In the classical case this implies $M = \ell^\infty(\Gamma)^\Lambda$ with $\Lambda \lhd \Gamma$, and for $M = N_\alpha$ we have $\Lambda = \mathrm{Ker}\,\alpha$. In the quantum case a Baaj-Vaes subalgebra M is not necessarily associated to a subgroup $\mathbb{\Lambda} < \mathbb{\Gamma}$ — it rather corresponds to a subgroup of the dual...

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The subspace $M = N_\alpha$ is a *Baaj–Vaes subalgebra*: $\Delta(M) \subset M \bar{\otimes} M$.
A $\mathbb{\Gamma}$-invariant subalgebra $M \subset \ell^\infty(\mathbb{\Gamma})$ is called *relatively amenable* if there exists a UCP $\mathbb{\Gamma}$-map $T : \ell^\infty(\mathbb{\Gamma}) \to M$.

Theorem (KKSV)

The cokernel N_F of $\Gamma \curvearrowright \partial_F \mathbb{\Gamma}$ is the unique minimal relatively amenable Baaj-Vaes subalgebra of $\ell^\infty(\mathbb{\Gamma})$.

Hence there exists a faithful $\mathbb{\Gamma}$-boundary **iff** $\ell^\infty(\mathbb{\Gamma})$ has no proper relatively amenable Baaj-Vaes subalgebra.

Outline

Orthogonal free quantum groups

Let $N \in \mathbb{N}$, $Q \in GL_N(\mathbb{C})$ s.t. $Q\bar{Q} = \pm I_N$.
One defines a "Woronowicz C^*-algebra" by generators and relations:

$$A_o(Q) = C^*\langle 1, u_{ij} \mid uu^* = u^*u = I_n, Q\bar{u}Q^{-1} = u\rangle,$$
$$\Delta : A_o(Q) \to A_o(Q) \otimes A_o(Q), u_{ij} \mapsto \textstyle\sum_k u_{ik} \otimes u_{kj}.$$

Here $\bar{u} = (u_{ij}^*)_{ij}$. This algebra has a unique bi-invariant state h → reduced algebra $\pi_h(A_o(Q)) = C^*_{\mathrm{red}}(\mathbb{F}O(Q))$.

One can then construct $\mathrm{Corep}(\mathbb{F}O(Q))$ and the dual algebra...
We have $\ell^\infty(\mathbb{F}O(Q)) = \bigoplus_{k\in\mathbb{N}}^{\ell^\infty} B(H_k)$, with $H_0 = \mathbb{C}$, $H_1 = \mathbb{C}^N$ and

$$H_k \otimes H_1 \simeq H_1 \otimes H_k \simeq H_{k-1} \oplus H_{k+1} \qquad (k \geq 1)$$

in the tensor category of representations of $\ell^\infty(\mathbb{F}O(Q))$.

The terminology comes from the following "classical" quotients of $A_o(I_N)$:

$$A_o(I_N)/(u_{ij}, i \neq j) \simeq C^*((\mathbb{Z}/2\mathbb{Z})^{*N}), \qquad A_o(I_N)/([u_{ij}, u_{kl}]) \simeq C(O_N).$$

The Gromov boundary of $\mathbb{F}O(Q)$

Classical case: free group $\mathbb{\Gamma} = \Gamma = F_N$.
Word length: $|g|$, spheres: $S_K = \{g \in F_N; |g| = k\}$.
Gromov boundary $\partial_G F_N$: set of infinite reduced words.
It can be described as a projective limit: $\partial_G F_N = \varprojlim(S_k, \rho_k)$
where $\rho_k : S_{k+1} \to S_k$ "forgets last letter".

Quantum case: $\mathbb{\Gamma} = \mathbb{F}O(Q)$, $N \geq 3$.
We have $\ell^\infty(\mathbb{\Gamma}) = \bigoplus_{k\geq 0}^{\ell^\infty} B(H_k)$ and isometries $V_k : H_{k+1} \to H_k \otimes H_1$.
Define connecting maps $r_k : B(H_k) \to B(H_{k+1})$, $r_k(f) = V_k^*(f \otimes 1)V_k$ and

$$C(\partial_G \mathbb{F}O(Q)) = \varinjlim(B(H_k), r_k).$$

By construction we have $C(\partial_G \mathbb{F}O(Q)) \subset \ell^\infty(\mathbb{F}O(Q))/c_0(\mathbb{F}O(Q))$.

Theorem (Vaes-Vergnioux 2007)

$C(\partial_G \mathbb{F}O(Q))$ is a sub-$\mathbb{F}O(Q)$-C^*-algebra of $\ell^\infty(\mathbb{F}O(Q))/c_0(\mathbb{F}O(Q))$.

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We have "quantum traces" $\mathrm{qtr}_k : B(H_k) \to \mathbb{C}$ with $\mathrm{qtr}_{k+1} \circ r_k = \mathrm{qtr}_k$.
We get a state $\omega = \varinjlim \mathrm{qtr}_k$ on $C(\partial_G \mathbb{F}O(Q))$ and the corresponding reduced algebra $C_{\mathrm{red}}(\partial_G \mathbb{F}O(Q)) = \pi_\omega(C(\partial_G \mathbb{F}O(Q)))$.

Unique stationarity

Choose $\mu \in S(c_0(\Gamma))$. A state $\nu \in S(A)$ is μ-stationary if $\mu * \nu = \nu$.

Proposition (Kalantar)

*Assume that A admits a **unique** μ-stationary state ν and that P_ν is completely isometric. Then A is a Γ-boundary.*

One checks that ω is μ-stationary for $\mu = \mathrm{qtr}_1 \in B(H_1)^* \subset c_0(\mathbb{F}O(Q))^*$.

Theorem (Vaes-Vergnioux)

Assume $N \geq 3$. Then P_ω extends to a normal $*$-isomorphism between $C_{\mathrm{red}}(\partial\mathbb{F}O(Q))''$ and the space of harmonic functions $H^\infty_\mu(\mathbb{F}O(Q))$.

Theorem (KKSV)

For $N \geq 3$, ω is the unique μ-stationary state on $C(\partial_G\mathbb{F}O(Q))$. Hence $C_{\mathrm{red}}(\partial_G\mathbb{F}O(Q))$ is an $\mathbb{F}O(Q)$-boundary.

Outline

Uniqueness of trace

Theorem (KKSV)

Assume that $\mathbb{\Gamma}$ acts faithfully on some $\mathbb{\Gamma}$-boundary A. Then:

- *if $\mathbb{\Gamma}$ is unimodular, h is the unique trace on $C^*_{\mathrm{red}}(\mathbb{\Gamma})$;*
- *else $C^*_{\mathrm{red}}(\mathbb{\Gamma})$ does not admit any KMS state wrt the scaling group.*

Theorem (KKSV)

For $N \geq 3$, $\mathbb{F}O(Q)$ acts faithfully on $\partial_G \mathbb{F}O(Q)$.

Note: uniqueness of trace was already proved in [Vaes–Vergnioux]. In the non-unimodular case, the absence of τ-KMS state is new.

Questions. In the unimodular case, does uniqueness of trace imply the existence of a faithful boundary action?
What about free actions and C^*-simplicity?