

Free entropy dimension and the orthogonal free quantum groups

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Outline

1 Introduction

- Orthogonal free quantum groups
- The von Neumann algebra $\mathcal{L}(\mathbb{F}O_n)$

2 Free entropy dimension

- Free entropy
- The case of $\mathbb{F}O_n$

3 1-boundedness

- 1-boundedness
- The case of $\mathbb{F}O_n$

4 Connection to the quantum Cayley graph

- Computation of $\partial P(u)$
- The quantum Cayley graph

Orthogonal free quantum groups

Wang's algebra defined by generators and relations:

$$A_o(n) = \langle u_{ij}, 1 \leq i, j \leq n \mid u_{ij} = u_{ij}^*, (u_{ij}) \text{ unitary} \rangle.$$

It comes with a natural “group-like” structure:

$$\Delta : A_o(n) \rightarrow A_o(n) \otimes A_o(n), u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}.$$

Why “group-like”?

We have $A_o(n) \twoheadrightarrow C(O_n)$, $u_{ij} \mapsto (g \mapsto g_{ij})$ and Δ induces

$$\Delta : C(O_n) \rightarrow C(O_n \times O_n), \Delta(f)(g, h) = f(gh).$$

One can recover the compact group O_n from $(C(O_n), \Delta)$.

We denote $A_o(n) = C(O_n^+)$, where O_n^+ is a **compact quantum group**.

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Why “group-like”?

We have $A_o(n) \twoheadrightarrow C_n = C^*((\mathbb{Z}/2\mathbb{Z})^{*n})$, $u_{ij} \mapsto \delta_{ij} b_i$ and Δ induces

$$\Delta : C_n \rightarrow C_n \otimes C_n, g \mapsto g \otimes g \text{ for } g \in C^*((\mathbb{Z}/2\mathbb{Z})^{*n}).$$

One can recover $(\mathbb{Z}/2\mathbb{Z})^{*n}$ as $\{u \in \mathcal{U}(C_n) \mid \Delta(u) = u \otimes u\}$.

We denote $A_o(n) = C^*(\mathbb{F}O_n)$, where $\mathbb{F}O_n$ is a **discrete quantum group**.

Discrete/Compact quantum groups

A Woronowicz C^* -algebra is a unital C^* -algebra A with $*$ -homomorphism $\Delta : A \rightarrow A \otimes A$ (coproduct) such that

- $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$,
- $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

Notation : $A = C^*(\Gamma) = C(\mathbb{G})$.

General theory : Haar state, Peter-Weyl, Tannaka-Krein...

Theorem (Woronowicz)

There exists a unique state $h : C^(\Gamma) \rightarrow \mathbb{C}$ such that*

$$(h \otimes \text{id})\Delta = (\text{id} \otimes h)\Delta = 1 \otimes h.$$

- regular representation $\lambda : C^*(\Gamma) \rightarrow B(H)$,
- reduced Woronowicz C^* -algebra $C_{\text{red}}^*(\Gamma) = \lambda(C^*(\Gamma))$,
- von Neumann algebra $\mathcal{L}(\Gamma) = C_{\text{red}}^*(\Gamma)'' \subset B(H)$.

Known results about $\mathcal{L}(\mathbb{F}O_n)$

We restrict to the case $n \geq 3$.

Free probability:

- the elements $(\sqrt{n} u_{ij})_{i,j \leq s}$ are asymptotically free and semi-circular with respect to h as $n \rightarrow \infty$ [Banica-Collins 2007, Brannan 2014];
- free entropy dimension of the generators : $\delta_0(u) = 1$ [Brannan-Collins-V. 2012, 2016].

Von Neumann algebra:

- $\mathcal{L}(\mathbb{F}O_n)$ is not injective [Banica 1997]
- it is a full and solid II_1 factor [Vaes-V. 2007]
- it has the HAP and the CBAP [Brannan 2012, Freslon 2013]
- it is strongly solid [Isono 2015, Fima-V. 2015]

Question: is $\mathcal{L}(\mathbb{F}O_n)$ isomorphic to a free group factor?

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Question: is $\mathcal{L}(\mathbb{F}O_n)$ isomorphic to a free group factor?

In $\mathcal{L}(F_n)$ with canonical generators a_i one has $\delta_0(a) = n$.

But it is not known whether δ_0 is an invariant of the von Neumann algebra.

However **strong 1-boundedness** is an invariant [Jung 2007]...

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Free entropy

(M, τ) : finite von Neumann algebra with fixed trace τ . $H = L^2(M, \tau)$.

Fix a tuple of self-adjoint elements $x = (x_1, \dots, x_m) \in M^m$.

$\chi(x)$: microstates free entropy / $\chi^*(x)$: non microstates free entropy.

Properties of χ and χ^* : [Voiculescu]

- $\chi(x) \in \mathbb{R} \cup \{-\infty\}$, $\chi(x_i) = \iint \ln |s - t| dx_i(s) dx_i(t) + C$.
- $\chi(x) \leq \chi(x_1) + \dots + \chi(x_m)$ with equality if x_1, \dots, x_m are freely independent.
- only for χ : assuming $\chi(x_i) > -\infty$, equality in the previous point implies free independence.

Example of free independence : $x_1 \in L^\infty(\Omega_1, p_1)$, \dots , $x_m \in L^\infty(\Omega_m, p_m)$ are free in the reduced free product $M = L^\infty(\Omega_1, p_1) * \dots * L^\infty(\Omega_m, p_m)$ with respect to the free product state τ .

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Free entropy dimension. [Voiculescu]

Assume M contains a free family $s = (s_1, \dots, s_m)$ of $(0, 1)$ -semicircular elements, also free from x . One defines:

$$\delta_0(x) = m - \liminf_{\delta \rightarrow 0} \chi(x + \delta s : s) / \ln \delta$$

$$\delta^*(x) = m - \liminf_{\delta \rightarrow 0} \chi^*(x + \delta s) / \ln \delta$$

$\delta_0(x)$ only depends on the algebra generated by x . It is not known whether it only depends on the von Neumann algebra.

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We have the following deep result:

Theorem (Biane-Capitaine-Guionnet 2003)

We have $\chi(x) \leq \chi^(x)$, hence $\delta_0(x) \leq \delta^*(x)$.*

The case of $\mathbb{F}O_n$

Theorem (Jung 2006)

If $W^*(x)$ embeds in R^ω and contains a diffuse subalgebra, then $\delta_0(x) \geq 1$.

In $\mathcal{L}(\mathbb{F}O_n)$, the subalgebra $(\sum u_{ij})'' \simeq L^\infty([-2, 2])$ is diffuse.

[Brannan-Collins-V. 2014]: $\mathcal{L}(\mathbb{F}O_n)$ embeds in R^ω .

Theorem (Connes-Shlyakhtenko 2005)

We have $\delta^*(x) \leq \beta_1^{(2)}(A, \tau) - \beta_0^{(2)}(A, \tau) + 1$ where $A = \mathbb{C}\langle x \rangle$.

By diffuseness, $\beta_0^{(2)}(A, h) = \beta_0^{(2)}(\mathbb{F}O_n) = 0$ [Kyed].

[V. 2012, Kyed-Raum-Vaes-Valvekens 2017]: $\beta_1^{(2)}(\mathbb{F}O_n) = 0$.

Corollary

For $n > 3$ we have $\delta_0(u) = \delta^*(u) = 1$ in $\mathcal{L}(\mathbb{F}O_n)$.

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1-boundedness and von Neumann isomorphisms

Recall that $\delta^*(x) = m - \liminf_{\delta \rightarrow 0} \chi^*(x + \delta s) / \ln \delta$.

Hence $\delta^*(x) \leq \alpha$ **iff** $\chi^*(x + \delta s) \leq (m - \alpha) \ln \delta + o(\ln \delta)$ as $\delta \rightarrow 0$.

One says that x is **α -bounded** for δ^* if

$$\chi^*(x + \delta s) \leq (m - \alpha) \ln \delta + K$$

for small δ and some constant K .

There is a similar notion of α -boundedness for δ_0 [Jung].

Theorem (Jung 2007)

If x is 1-bounded for δ_0 and $\chi(x_i) > -\infty$ for at least one i , then any tuple y of generators of $W^(x)$ is 1-bounded for δ_0 (hence $\delta_0(y) \leq 1$).*

In particular if M is generated by a 1-bounded tuple of generators, it is not isomorphic to any free group factor.

Proving 1-boundedness

Consider the algebra of polynomials in m non-commuting variables $\mathbb{C}\langle X \rangle = \mathbb{C}\langle X_1, \dots, X_m \rangle$. There are unique derivations

$$\delta_i : \mathbb{C}\langle X \rangle \rightarrow \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle$$

such that $\delta_i(X_j) = \delta_{ij}(1 \otimes 1)$, with the bimodule structure:

$$P \cdot (Q \otimes R) \cdot S = PQ \otimes RS.$$

For $P = (P_1, \dots, P_l) \in \mathbb{C}\langle X \rangle^l$, put

$$\partial P = (\partial_j P_i) \in \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \otimes B(\mathbb{C}^m, \mathbb{C}^l).$$

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Denote $H = L^2(M, \tau)$. Evaluating at $X = x$ one obtains an operator

$$\partial P(x) \in B(H \otimes H \otimes \mathbb{C}^m, H \otimes H \otimes \mathbb{C}^l),$$

which commutes to the right action $(\zeta \otimes \xi) \cdot (x \otimes y) = \zeta x \otimes y \xi$ of $M \otimes M^\circ$ on $H \otimes H$. One considers the Murray-von Neumann dimension:

$$\text{rank } \partial P(x) = \dim_{M \otimes M^\circ} \overline{\text{Im}} \partial P(x).$$

Theorem (Jung 2016, Shlyakhtenko 2016)

Assume that x satisfies the identities $P(x) = 0$ and that $\partial P(x)$ is of determinant class. Then x is α -bounded for δ_0 and δ^ , with*

$$\alpha = m - \text{rank } \partial P(x).$$

Relations in $\mathbb{F}O_n$

We take $m = n^2$, $X = (X_{ij})_{ij} \in \mathbb{C}\langle X_{ij} \rangle \otimes M_n(\mathbb{C})$, $x = u = (u_{ij})_{ij}$.

We consider the $l = 2n^2$ canonical relations:

$$P = (P_1, P_2) = (X^t X - 1, X X^t - 1) \in \mathbb{C}\langle X \rangle \otimes M_n(\mathbb{C})^{\oplus 2}.$$

Following [Shlyakhtenko 2016] it is easy to prove that:

Proposition

We have $n^2 - \text{rank } \partial P(u) = \beta_1^{(2)}(\mathbb{F}O_n) - \beta_0^{(2)}(\mathbb{F}O_n) + 1 = 1$.

Hence if $\partial P(u)$ is of determinant class, Jung–Shlyakhtenko’s result allows to conclude that u is 1-bounded.

Determinant class: $(h \otimes h \otimes \text{Tr})(\ln_+(\partial P(u)^* \partial P(u))) > -\infty$.

$\ln_+(t) = \ln(t)$ if $t > 0$, $\ln_+(0) = 0$.

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In the case of a discrete group Γ , this would follow from Lück's determinant conjecture, which holds e.g. if Γ is sofic. In the quantum case, there is no such tool (yet?) to prove the determinant conjecture...

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Computation of $\partial P(u)$

Identify $M_n(\mathbb{C}) \simeq p_1 H = \text{Span}\{u_{ij}\xi_0\} \subset H$.

Then $u \in C_{\text{red}}^*(\mathbb{F}O_n) \otimes M_n(\mathbb{C})$ acts by left mult. on $H \otimes p_1 H$.

If $S : H \rightarrow H$ is the antipode, we have in $B(H \otimes H \otimes p_1 H)$:

$$\partial P_1(u) = (1 \otimes S \otimes S)u_{23}(1 \otimes S \otimes 1) + u_{13}^*$$

$$\partial P_2(u) = (1 \otimes S \otimes S)u_{23}^*(1 \otimes S \otimes S) + u_{13}(1 \otimes 1 \otimes S)$$

Proposition

We have $\partial P_1(u)^* \partial P_1(u) = \partial P_2(u)^* \partial P_2(u)$ and it is unitarily conjugated to $(2 + 2 \text{Re } \Theta) \otimes 1 \in B(H \otimes p_1 H \otimes H)$, where

$$\Theta = (S \otimes 1)u(S \otimes S) \in B(H \otimes p_1 H).$$

Fact: Θ is the reversing operator of the quantum Cayley graph of $\mathbb{F}O_n$!

Decomposing the quantum Cayley graph

Classical case

One has $H = \ell^2(\Gamma) = \overline{\text{span}}\{g, g \in \Gamma\}$, $p_1 H = \ell^2(\Lambda)$ with $\Lambda = \Lambda^{-1} \subset \Gamma$.
 The “fundamental matrix” is $u = \text{diag}(\lambda(g))_{g \in \Lambda}$ and $S(g) = g^{-1}$. Then:

$$\Theta(g \otimes h) = (S \otimes 1)u(S \otimes S)(g \otimes h) = gh \otimes h^{-1}.$$

Interpretation: $(g, h) \rightarrow$ edge connecting g to gh in the Cayley graph
 $(gh, h^{-1}) \rightarrow$ reversed edge

In particular: $\Theta^2 = 1$ hence $H \otimes p_1 H = \text{Ker}(\Theta - 1) \oplus \text{Ker}(\Theta + 1)$.

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Quantum case

Quantum Cayley graph given by $\ell^2(\mathcal{X}^{(0)}) = H$, $\ell^2(\mathcal{X}^{(1)}) = H \otimes p_1 H$,
 the boundary operator $E = V : \ell^2(\mathcal{X}^{(1)}) \rightarrow \ell^2(\mathcal{X}^{(0)}) \otimes \ell^2(\mathcal{X}^{(0)})$,
 the reversing operator Θ .

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In particular: $\Theta^2 = 1$ hence $H \otimes p_1 H = \text{Ker}(\Theta - 1) \oplus \text{Ker}(\Theta + 1)$.

Quantum case

We have $\Theta^2 \neq 1$, $\text{Ker}(\Theta - 1) \oplus \text{Ker}(\Theta + 1) \subsetneq H \otimes p_1 H$. The description of $\text{Ker}(\Theta \pm 1)$ was an essential tool in the proof of $\beta_1^{(2)}(\mathbb{F}O_n) = 0$.

Theorem

On $\text{Ker}(\Theta - 1)^\perp \cap \text{Ker}(\Theta + 1)^\perp$, $\text{Re}(\Theta) \simeq \bigoplus \text{Re}(r_\alpha)$ is an infinite direct sum of real parts of weighted right shifts r_α .

Conclusion

Lemma

For any right shift r with weights in $]0, 1]$, $2 + 2 \operatorname{Re} r$ is of determinant class with respect to the specific state coming from $h \otimes \operatorname{Tr}$.

Finally one can apply Jung-Shlyakhtenko's result:

Corollary

*The generating matrix u is 1-bounded in $\mathcal{L}(\mathbb{F}O_n)$.
 $\mathcal{L}(\mathbb{F}O_n)$ is not isomorphic to a free group factor.*

Next questions...

- Is there a group Γ such that $\mathcal{L}(\mathbb{F}O_n) \simeq \mathcal{L}(\Gamma)$?
- What about $\mathcal{L}(\mathbb{F}U_n)$? Recall that $\mathcal{L}(\mathbb{F}U_2) \simeq \mathcal{L}(F_2)$.