Free entropy dimension and the orthogonal free quantum groups

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Outline

1. Introduction
   - Orthogonal free quantum groups
   - The von Neumann algebra $\mathscr{L}(\mathbb{F}O_n)$
2. Free entropy dimension
   - Free entropy
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   - 1-boundedness
   - The case of $\mathbb{F}O_n$
   - The quantum Cayley graph

Orthogonal free quantum groups

Wang's algebra defined by generators and relations:

$$A_o(n) = \langle u_{ij}, 1 \leq i,j \leq n \mid u_{ij} = u_{ij}^*, \ \ (u_{ij}) \ \text{ unitary}\rangle.$$

It comes with a natural "group-like" structure:

$$\Delta : A_o(n) \to A_o(n) \otimes A_o(n), \ u_{ij} \mapsto \textstyle\sum_k u_{ik} \otimes u_{kj}.$$

Why "group-like"?

We have $A_o(n) \twoheadrightarrow C(O_n)$, $u_{ij} \mapsto (g \mapsto g_{ij})$ and Δ induces

$$\Delta : C(O_n) \to C(O_n) \otimes C(O_n), \ \Delta(f)(g,h) = f(gh).$$

One can recover the compact group O_n from $(C(O_n), \Delta)$.

We denote $A_o(n) = C(O_n^+)$, where O_n^+ is a **compact quantum group**.

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Why "group-like"?

We have $A_o(n) \twoheadrightarrow C_n = C^*((\mathbb{Z}/2\mathbb{Z})^{*n})$, $u_{ij} \mapsto \delta_{ij} b_i$ and Δ induces

$$\Delta : C_n \to C_n \otimes C_n, \ g \mapsto g \otimes g \text{ for } g \in C^*((\mathbb{Z}/2\mathbb{Z})^{*n}).$$

One can recover $(\mathbb{Z}/2\mathbb{Z})^{*n}$ as $\{u \in \mathscr{U}(C_n) \mid \Delta(u) = u \otimes u\}$.

We denote $A_o(n) = C^*(\mathbb{F}O_n)$, where $\mathbb{F}O_n$ is a **discrete quantum group**.

Discrete/Compact quantum groups

A Woronowicz C^*-algebra is a unital C^*-algebra A with $*$-homomorphism $\Delta : A \to A \otimes A$ (coproduct) such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

Notation : $A = C^*(\mathbb{\Gamma}) = C(\mathbb{G})$.

General theory : Haar state, Peter-Weyl, Tannaka-Krein...

Theorem (Woronowicz)

There exists a unique state $h : C^*(\mathbb{\Gamma}) \to \mathbb{C}$ such that

$$(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = 1 \otimes h.$$

→ regular representation $\lambda : C^*(\mathbb{\Gamma}) \to B(H)$,
→ reduced Woronowicz C^*-algebra $C^*_{\mathrm{red}}(\mathbb{\Gamma}) = \lambda(C^*(\mathbb{\Gamma}))$,
→ von Neumann algebra $\mathscr{L}(\mathbb{\Gamma}) = C^*_{\mathrm{red}}(\mathbb{\Gamma})'' \subset B(H)$.

Known results about $\mathscr{L}(\mathbb{F}O_n)$

We restrict to the case $n \geq 3$.

Free probability:

- the elements $(\sqrt{n}\, u_{ij})_{i,j\leq s}$ are asymptotically free and semi-circular with respect to h as $n \to \infty$ [Banica-Collins 2007, Brannan 2014];
- free entropy dimension of the generators : $\delta_0(u) = 1$ [Brannan-Collins-V. 2012, 2016].

Von Neumann algebra:

- $\mathscr{L}(\mathbb{F}O_n)$ is not injective [Banica 1997]
- it is a full and solid II_1 factor [Vaes-V. 2007]
- it has the HAP and the CBAP [Brannan 2012, Freslon 2013]
- it is strongly solid [Isono 2015, Fima-V. 2015]

Question: is $\mathscr{L}(\mathbb{F}O_n)$ isomorphic to a free group factor?

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In $\mathscr{L}(F_n)$ with canonical generators a_i one has $\delta_0(a) = n$.
But it is not known whether δ_0 is an invariant of the von Neumann algebra.
However **strong** 1-**boundedness** is an invariant [Jung 2007]...

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Free entropy

(M,τ): finite von Neumann algebra with fixed trace τ. $H = L^2(M,\tau)$.
Fix a tuple of self-adjoint elements $x = (x_1, \dots, x_m) \in M^m$.
$\chi(x)$: microstates free entropy / $\chi^*(x)$: non microstates free entropy.

Properties of χ and χ^*: [Voiculescu]

- $\chi(x) \in \mathbb{R} \cup \{-\infty\}$, $\chi(x_i) = \iint \ln|s-t| \mathrm{d}x_i(s)\mathrm{d}x_i(t) + C$.
- $\chi(x) \leq \chi(x_1) + \cdots + \chi(x_m)$ with equality if $x_1, \dots, x_m$ are freely independant.
- only for χ: assuming $\chi(x_i) > -\infty$, equality in the previous point implies free independance.

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Free entropy dimension. [Voiculescu]
Assume M contains a free family $s = (s_1, \ldots, s_m)$ of $(0, 1)$-semicircular elements, also free from x. One defines:

$$\delta_0(x) = m - \liminf_{\delta \to 0} \chi(x + \delta s : s) / \ln \delta$$
$$\delta^*(x) = m - \liminf_{\delta \to 0} \chi^*(x + \delta s) / \ln \delta$$

$\delta_0(x)$ only depends on the algebra generated by x. It is not known whether it only depends on the von Neumann algebra.

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$\delta_0(x)$ only depends on the algebra generated by x. It is not known whether it only depends on the von Neumann algebra.
We have the following deep result:

Theorem (Biane-Capitaine-Guionnet 2003)

We have $\chi(x) \le \chi^*(x)$, hence $\delta_0(x) \le \delta^*(x)$.

The case of $\mathbb{F}O_n$

Theorem (Jung 2006)

If $W^*(x)$ embeds in R^ω and contains a diffuse subalgebra, then $\delta_0(x) \geq 1$.

In $\mathscr{L}(\mathbb{F}O_n)$, the subalgebra $(\sum u_{ii})'' \simeq L^\infty([-2,2])$ is diffuse.
[Brannan-Collins-V. 2014]: for $n \neq 3$, $\mathscr{L}(\mathbb{F}O_n)$ embeds in R^ω.

Theorem (Connes-Shlyakhtenko 2005)

We have $\delta^*(x) \leq \beta_1^{(2)}(A,\tau) - \beta_0^{(2)}(A,\tau) + 1$ where $A = \mathbb{C}\langle x\rangle$.

By diffuseness, $\beta_0^{(2)}(A,h) = \beta_0^{(2)}(\mathbb{F}O_n) = 0$ [Kyed].
[V. 2012, Kyed-Raum-Vaes-Valvekens 2017]: $\beta_1^{(2)}(\mathbb{F}O_n) = 0$.

Corollary

For $n > 3$ we have $\delta_0(u) = \delta^*(u) = 1$ in $\mathscr{L}(\mathbb{F}O_n)$.

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1-boundedness and von Neumann isomorphisms

Recall that $\delta^*(x) = m - \liminf_{\delta\to 0} \chi^*(x+\delta s)/\ln\delta$.
Hence $\delta^*(x) \leq \alpha$ **iff** $\chi^*(x+\delta s) \leq (\alpha - m)|\ln\delta| + o(\ln\delta)$ as $\delta \to 0$.

One says that x is α-**bounded** for δ^* if

$$\chi^*(x+\delta s) \leq (\alpha - m)|\ln\delta| + K$$

for small δ and some constant K.
There is a similar notion of α-boundedness for δ_0 [Jung].

Theorem (Jung 2007)

If x is 1-bounded for δ_0 and $\chi(x_i) > -\infty$ for at least one i, then any tuple y of generators of $W^*(x)$ is 1-bounded for δ_0 (hence $\delta_0(y) \leq 1$).

In particular if M is generated by a 1-bounded tuple of generators, it is not isomorphic to any free group factor.

Proving 1-boundedness

Consider the algebra of polynomials in m non-commuting variables $\mathbb{C}\langle X\rangle = \mathbb{C}\langle X_1, \ldots, X_m\rangle$. There are unique derivations

$$\delta_i : \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$$

such that $\delta_i(X_j) = \delta_{ij}(1 \otimes 1)$, with the bimodule structure $P \cdot (Q \otimes R) \cdot S = PQ \otimes RS$. One has e.g.

$$\partial_1(X_2X_1X_3^2X_1X_4) = X_2 \otimes X_3^2X_1X_4 + X_2X_1X_3^2 \otimes X_4.$$

Proving 1-boundedness

Consider the algebra of polynomials in m non-commuting variables $\mathbb{C}\langle X\rangle = \mathbb{C}\langle X_1,\dots,X_m\rangle$. There are unique derivations δ_i such that $\delta_i(X_j) = \delta_{ij}(1\otimes 1)$. For $P = (P_1,\dots,P_l) \in \mathbb{C}\langle X\rangle^l$, put

$$\partial P = (\partial_j P_i) \in \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle \otimes B(\mathbb{C}^m, \mathbb{C}^l).$$

Denote $H = L^2(M,\tau)$. Evaluating at $X = x$ one obtains an operator

$$\partial P(x) \in B(H\otimes H\otimes \mathbb{C}^m, H\otimes H\otimes \mathbb{C}^l),$$

which commutes to the right action $(\zeta\otimes\xi)\cdot(x\otimes y) = \zeta x\otimes y\xi$ of $M\otimes M^\circ$ on $H\otimes H$. One considers the Murray-von Neumann dimension:

$$\mathrm{rank}\,\partial P(x) = \dim_{M\otimes M^\circ} \overline{\mathrm{Im}}\ \partial P(x).$$

Theorem (Jung 2016, Shlyakhtenko 2016)

Assume that x satisfies the identities $P(x) = 0$ and that $\partial P(x)$ is of determinant class. Then x is α-bounded for δ_0 and δ^*, with

$$\alpha = m - \mathrm{rank}\,\partial P(x).$$

Relations in $\mathbb{F}O_n$

We take $m = n^2$, $X = (X_{ij})_{ij} \in \mathbb{C}\langle X_{ij}\rangle \otimes M_n(\mathbb{C})$, $x = u = (u_{ij})_{ij}$.
We consider the $l = 2n^2$ canonical relations:

$$P = (P_1, P_2) = (X^t X - 1, XX^t - 1) \in \mathbb{C}\langle X\rangle \otimes M_n(\mathbb{C})^{\oplus 2}.$$

Following [Shlyakhtenko 2016] it is easy to prove that:

Proposition

We have $n^2 - \operatorname{rank} \partial P(u) = \beta_1^{(2)}(\mathbb{F}O_n) - \beta_0^{(2)}(\mathbb{F}O_n) + 1 = 1.$

Hence if $\partial P(u)$ is of determinant class, Jung–Shlyakhtenko's result allows to conclude that u is 1-bounded.

In the case of a discrete group Γ, this would follow from Lück's determinant conjecture, which holds e.g. if Γ is sofic. In the quantum case, there is no such tool (yet?) to prove the determinant conjecture...

Computation of $\partial P(u)$

Determinant class: $(h \otimes h \otimes \mathrm{Tr})(\ln_+(\partial P(u)^* \partial P(u))) > -\infty$.

Identify $M_n(\mathbb{C}) \simeq p_1 H = \mathrm{Span}\{u_{ij}\xi_0\} \subset H$.
Then $u \in C^*_{\mathrm{red}}(\mathbb{F}O_n) \otimes M_n(\mathbb{C})$ acts by left mult. on $H \otimes p_1 H$.
If $S : H \to H$ is the antipode, we have in $B(H \otimes H \otimes p_1 H)$:

$$\partial P_1(u) = (1 \otimes S \otimes S) u_{23} (1 \otimes S \otimes 1) + u_{13}^*$$
$$\partial P_2(u) = (1 \otimes S \otimes S) u_{23}^* (1 \otimes S \otimes S) + u_{13} (1 \otimes 1 \otimes S)$$

Proposition

We have $\partial P_1(u)^* \partial P_1(u) = \partial P_2(u)^* \partial P_2(u)$ and it is unitarily conjugated to $(2 + 2\,\mathrm{Re}\,\Theta) \otimes 1 \in B(H \otimes p_1 H \otimes H)$, where

$$\Theta = (S \otimes 1) u (S \otimes S) \in B(H \otimes p_1 H).$$

Fact: Θ is the reversing operator of the quantum Cayley graph of $\mathbb{F}O_n$!

Decomposing the quantum Cayley graph

Classical case

For $\Lambda = \Lambda^{-1} \subset \Gamma$, the Cayley graph of (Γ, Λ) is given by

$$X^{(0)} = \Gamma,\ X^{(1)} = \Gamma \times \Lambda,$$
$$\partial : X^{(1)} \to X^{(0)} \times X^{(0)}, (g, h) \mapsto (g, gh),$$
$$\theta : X^{(1)} \to X^{(1)}, (g, h) \mapsto (gh, h^{-1}).$$

Consider $H = \ell^2(\Gamma)$, $p_1 H = \ell^2(\Lambda)$, $u = \mathrm{diag}(\lambda(g))_{g\in\Lambda}$, $S(g) = g^{-1}$. Then:

$$\Theta(g \otimes h) = (S \otimes 1)u(S \otimes S)(g \otimes h) = gh \otimes h^{-1}.$$

We have $\Theta^2 = 1$, $H \otimes p_1 H = \mathrm{Ker}(\Theta - 1) \oplus \mathrm{Ker}(\Theta + 1)$.

Decomposing the quantum Cayley graph

Classical case

We have $\Theta^2 = 1$, $H \otimes p_1 H = \mathrm{Ker}(\Theta - 1) \oplus \mathrm{Ker}(\Theta + 1)$.

Quantum case

We have $\Theta^2 \neq 1$, $\mathrm{Ker}(\Theta - 1) \oplus \mathrm{Ker}(\Theta + 1) \subsetneq H \otimes p_1 H$. The description of $\mathrm{Ker}(\Theta \pm 1)$ was an essential tool in the proof of $\beta_1^{(2)}(\mathbb{F}O_n) = 0$.

Theorem

On $\mathrm{Ker}(\Theta - 1)^\perp \cap \mathrm{Ker}(\Theta + 1)^\perp$, $\mathrm{Re}(\Theta) \simeq \bigoplus \mathrm{Re}(r_\alpha)$ is an infinite direct sum of real parts of weighted right shifts r_α.

Lemma

For any right shift r with weights in $]0, 1]$, $2 + 2\,\mathrm{Re}\, r$ is of determinant class with respect to the specific state coming from $h \otimes \mathrm{Tr}$.

Conclusion

Finally one can apply Jung-Shlyakhtenko's result:

Corollary

The generating matrix u is 1-bounded in $\mathscr{L}(\mathbb{F}O_n)$.
$\mathscr{L}(\mathbb{F}O_n)$ is not isomorphic to a free group factor.

Next questions...

- Is there a group Γ such that $\mathscr{L}(\mathbb{F}O_n) \simeq \mathscr{L}(\Gamma)$?
- What about $\mathscr{L}(\mathbb{F}O(Q))$ — the type *III* case?
- What about $\mathscr{L}(\mathbb{F}U_n)$? Recall that $\mathscr{L}(\mathbb{F}U_2) \simeq \mathscr{L}(F_2)$.