The radial MASA in free orthogonal quantum groups

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Outline

From classical to quantum

Γ discrete group → $C^*_{\mathrm{red}}(\Gamma) \subset B(\ell^2\Gamma)$
Elements $g \in \gamma$ → unitaries $\lambda(g) \in C^*_{\mathrm{red}}(\Gamma)$
Additional structure to recover Γ from $C^*_{\mathrm{red}}(\Gamma)$: coproduct

$$\Delta : C^*_{\mathrm{red}}(\Gamma) \to C^*_{\mathrm{red}}(\Gamma) \otimes C^*_{\mathrm{red}}(\Gamma), \quad \lambda(g) \mapsto \lambda(g) \otimes \lambda(g).$$

Then $\Gamma = \{u \in \mathscr{U}(C^*_{\mathrm{red}}(\Gamma)) \mid \Delta(u) = u \otimes u\}$.

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Definition

Woronowicz C^*-algebra : unital C^*-algebra A and unital $*$-homomorphism $\Delta : A \to A \otimes A$ such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ dense in $A \otimes A$.

Example: $A = C^*_{\mathrm{red}}(\Gamma)$ but also $C^*_{\max}(\Gamma)$.
➜ general notation $A = C^*(\mathbb{\Gamma})$, $\mathbb{\Gamma}$ "discrete quantum group".

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Example: $A = C(G)$, G a (classical) compact group, with

$$\Delta : C(G) \to C(G \times G), \quad \Delta(f) = ((g,h) \mapsto f(gh)).$$

Also $A = C(\mathbb{G}_q)$, q-deformation a simple compact Lie group G.

General facts

Theorem (Woronowicz, the "Haar state")

There exists a unique state $h \in C^*(\mathbb{\Gamma})^*$ such that

$$(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = 1 \cdot h.$$

- ➜ GNS representation $\lambda : C^*(\mathbb{\Gamma}) \twoheadrightarrow C^*_{\mathrm{red}}(\mathbb{\Gamma}) \subset B(\ell^2\mathbb{\Gamma})$.
- ➜ Von Neumann algebra $\mathscr{L}(\mathbb{\Gamma}) = C^*_{\mathrm{red}}(\mathbb{\Gamma})'' \subset B(\ell^2\mathbb{\Gamma})$.

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Corepresentation: $u \in C^*(\mathbb{\Gamma}) \otimes B(H_u)$ unitary s.t. $(\Delta \otimes \mathrm{id})(u) = u_{13}u_{23}$.
Can define: direct sums, tensor products, contragredient, intertwiners, ...
➜ C^*-tensor category $\mathrm{Corep}(\mathbb{\Gamma})$, irreducible coreps $\mathrm{Irr}(\mathbb{\Gamma})$.

Theorem ("Peter-Weyl-Woronowicz")

Every corepresentation is equivalent to a direct sum of irreducibles.
The coefficients $u_{\zeta,\xi} = (\mathrm{id} \otimes \omega_{\zeta,\xi})(u)$ of irreducible corepresentations span a dense subspace $\mathbb{C}[\mathbb{\Gamma}] \subset C^*(\mathbb{\Gamma})$.

Outline

Free orthogonal quantum groups

Definition (Wang)

The orthogonal free quantum group $\mathbb{F}O_N$ is given by

$$C^*(\mathbb{F}O_N) = \langle v_{ij}, 1 \leq i,j \leq N \mid v_{ij}^* = v_{ij}, v = (v_{ij})_{ij} \textit{ unitary} \rangle$$

and the coproduct $\Delta(v_{ij}) = \sum v_{ik} \otimes v_{kl}$.

There are surjective morphisms of Woronowicz C^*-algebras

$$C^*(\mathbb{F}O_N) \twoheadrightarrow C^*(\mathbb{Z}_2^{*N}),\ v_{ij} \mapsto \delta_{ij} a_i$$

$$C^*(\mathbb{F}O_N) \twoheadrightarrow C(O_N),\ v_{ij} \mapsto (g \mapsto g_{ij})$$

Known results about $\mathscr{L}(\mathbb{F}O_N)$, $N \geq 3$:

- it is a full II_1 factor [Vaes-V.] with Property AO [V., Vaes-V.]
- it has the HAP and the CBAP [Brannan, Freslon]
- it is stronly solid [Isono, Fima-V.] hence has no regular MASA
- it embeds in R^ω [Brannan-Collins-V.]

The radial subalgebra

Character of a corepresentation: $\chi_u = (\mathrm{id} \otimes \mathrm{Tr})(u) \in C^*(\mathbb{\Gamma})$.

Definition

The radial subalgebra is $\mathscr{B} = \lambda(\chi_v)'' \subset \mathscr{L}(\mathbb{F}O_N)$.
One can also consider $B = C^*(\chi_v) \subset C^*(\mathbb{F}O_N)$.

Known facts:

- $\mathrm{Sp}(\chi_v) = [-N, N]$ and $B \simeq C([-N, N])$.
- Brannan: the orthogonal projection extends to a cond. expectation $E : C^*(\mathbb{F}O_N) \to B$. The positive forms $\mathrm{ev}_t \circ E$ are c_0 and converge to ε as $t \to N$ ➔ Haagerup Approximation Property.
- Banica: $\mathrm{Sp}(\lambda(\chi_v)) = [-2, 2]$ and $\mathscr{B} \simeq L^\infty([-2, 2], \mathrm{leb})$.

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Why radial? Analogy $\mathbb{F}O_N \leftrightarrow F_N = \langle a_i \rangle$, $v \leftrightarrow a = \mathrm{diag}(a_i, a_i^{-1})$.
In $\mathscr{L}(F_N)$ one has $\sum f(g)\lambda(g) \in \chi_a''$ iff $f(g) = \varphi(|g|)$. Known facts:

- χ_a'' is a singular MASA [Pytlik, Radulescu]
- $\exists\ \xi \perp \chi_a''$ such that $\mu_\xi : f \otimes g \mapsto (\xi | f\xi g)$ on $(\mathrm{Sp}\,\chi_a)^2$ is equivalent to the product measure $h\otimes h$ [Dykema-Mukherjee]

Deep result [Voiculescu]: for *any* MASA $\mathscr{C} \subset \mathscr{L}(F_N)$ there exists $\xi \perp \mathscr{C}$ st μ_ξ is not singular wrt $h\otimes h$.

The radial subalgebra

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Main result:

Theorem (Freslon-V.)

The subalgebra $\mathscr{B} \subset \mathscr{L}(\mathbb{F}O_N)$, $N \geq 3$, is a singular MASA. For any $\xi \in \mathbb{C}[\mathbb{F}O_N] \cap \mathscr{B}^\perp$, the measure μ_ξ on $[-2,2]^2$ is equivalent to $h\otimes h$.

Questions: what about the non-Kac case $\mathscr{L}(\mathbb{F}O_Q)$?
What about the "generator subalgebras" $\lambda(v_{ij})''$?

Outline

Maximal abelian subalgebras

Terminology

$\mathscr{M}$ finite von Neumann algebra with separable predual and fixed trace h
$\mathscr{B} \subset \mathscr{M}$ weakly closed abelian subalgebra, $E : \mathscr{M} \to \mathscr{B}$ cond. exp.
Normalizer: $\mathscr{N}_{\mathscr{M}}(\mathscr{B}) = \{u \in \mathscr{U}(\mathscr{M}) \mid u^* \mathscr{B} u \subset \mathscr{B}\}''$

- MASA: $\mathscr{B}' = \mathscr{B}$,
- singular MASA: $\mathscr{N}_{\mathscr{M}}(\mathscr{B}) = \mathscr{B}$,
- regular MASA: $\mathscr{B}' = \mathscr{B}$ and $\mathscr{N}_{\mathscr{M}}(\mathscr{B}) = \mathscr{M}$ (Cartan subalgebra),
- strongly mixing: $\forall\ (u_n)_n \in \mathscr{U}(\mathscr{B})^{\mathbb{N}}$ st $u_n \xrightarrow{w} 0$ and $\forall\ x, x' \in \mathscr{M}$ one has $\|E(u_n^* x u_n x') - E(x)E(x')\|_2 \to 0$ (SAHP for E)

Elementary facts:

- abelian, diffuse and strongly mixing $\Longrightarrow$ singular MASA,
- strongly mixing $\Longleftrightarrow$ $\|E(x u_n x')\|_2 \to 0$ for x (resp. x') spanning a dense subspace in $\mathscr{B}^{\perp}$ (resp. $\mathscr{M}$)
- strongly mixing $\Longleftarrow$ $\sum |(c_l x | x' c_{l'})|^2 < \infty$ for x, x' as above and $(c_l)_l$ fixed ONB of $L^2(\mathscr{B})$.

Classical MASAs in $\mathscr{L}(F_N)$

Generator MASA $\mathscr{B} = a''$

$c_l = a^l$, $x = g \in F_N \setminus a^{\mathbb{Z}}$, $x' = g' \in F_N$ ➜ $(c_l x | x' c_{l'}) = \delta(g', a^l g a^{-l'})$.

This is non zero for at most one value of (l, l')! ➜ singular MASA

Radial MASA $\mathscr{B} = (\sum a_i + a_i^{-1})''$

$c_l = d_l / \|d_l\|_2$, $d_l = \sum_{W_l} g$, $W_l = \{g \in F_N | |g| = l\}$.

$x = g - h$ with $|g| = |h| = k$, $x' = g'$ with $|g'| = k'$. Then

$$(d_l x | x' d_{l'}) = \# W_l g \cap g' W_{l'} - \# W_l h \cap g' W_{l'}.$$

Lemma

Denote $w_s(g, g') = \#\{x = g' \cdots g \text{ reduced}, |x| = s\}$.
Then $w_s(g, g') = w_s(h, g') + 1/0/{-1}$ if $|g| = |h|$.

➜ $|(c_l x \mid x' c_{l'})| \leq \dfrac{\min(k, k') + 1}{(2N-1)^{(l+l')/2}}$ ➜ singular MASA

Outline

1. Discrete Quantum Groups
   - Woronowicz C^*-algebras
   - The Haar measure and corepresentations

2. The radial subalgebra in $C^*(\mathbb{F}O_N)$
   - Free orthogonal quantum groups
   - The radial subalgebra

3. The classical case
   - Maximal abelian subalgebras
   - The radial MASA in $\mathscr{L}(F_N)$

4. About the proof
   - The main estimate
   - The main lemma

The main estimate

Theorem (Banica)

$\mathrm{Irr\,Corep}\,\mathbb{F}O_N = \{v^k, k \in \mathbb{N}\}$ *with* $v^0 = 1$, $v^1 = v$ *and*

$$v^k \otimes v^1 \simeq v^{k-1} \oplus v^{k+1} \simeq v^1 \otimes v^k.$$

In particular $(\chi_k)_k$ is an ONB of $L^2(\mathscr{B})$, where $\chi_k = \chi_{v^k}$.
Moreover $\mathscr{B}^\perp$ is spanned by coefficients $v^k_{\zeta\xi}$ with $\zeta \perp \xi \in H_k = H_{v^k}$.

Theorem (Freslon-V.)

For all $\zeta \perp \xi \in H_n$, $\zeta', \xi' \in H_{n'}$ *we have*

$$(\chi_l v^n_{\zeta\xi} | v^{n'}_{\zeta'\xi'} \chi_{l'}) = O(q^{(l+l')/2})$$

with $q \in \,]0, 1[$, $q + q^{-1} = N$, $N \geq 3$.

→ strongly mixing MASA.

The main lemma

Partial traces of **Jones-Wenzl projections**
Fusion rule $v^{a+b+c} \subset v^a \otimes v^b \otimes v^c$ ➔ $P_{a,b,c} \in B(H_a \otimes H_b \otimes H_c)$.

Lemma

$X_{a,b,c} = (\mathrm{id}_a \otimes \mathrm{Tr}_b \otimes \mathrm{id}_c)(P_{a,b,c})$ *is asymptotically scalar as* $b \to \infty$*:*

$$X_{a,b,c} = \lambda_{a,b,c}(\mathrm{id}_a \otimes \mathrm{id}_b) + O(\lambda_{a,b,c} q^b).$$

NB: if a or $c = 0$, $X_{a,b,c}$ *is* scalar (intertwiner).

Interpretation: analogy with F_N

$v \in C^*(\mathbb{F}O_N) \otimes B(\mathbb{C}^N)$	$a = \mathrm{diag}(a_i, a_i^{-1}) \in C^*(F_N) \otimes C(W_1)$
$v^k \in C^*(\mathbb{F}O_N) \otimes B(H_k)$	$a_k = \sum_{\lvert g \rvert = k} g \otimes 1\!\!1_g \in C^*(F_N) \otimes C(W_k)$
$P_{a,b,c} \in B(H_a \otimes H_b \otimes H_c)$	$1\!\!1_{W_{a+b+c}} \in C(W_a \times W_b \times W_c)$
$X_{a,b,c} \in B(H_a \otimes H_b)$	$w_{a+b+c}(\cdot\,,\cdot) \in C(W_a \times W_b)$
scalar	constant