Introduction to Quantum Groups
Operator algebraic aspects

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Outline

1. Constructions and tools
   [Woronowicz, Baaj–Skandalis, Banica, ...]
   - Full and reduced C^*-algebra
   - The dual C^*-algebra
   - Corepresentations
   - The boundary of $\mathbb{F}O(Q)$

2. Some operator-algebraic properties
   [Voiculescu, Ruan, Tomatsu, Brannan, DFSW, DCFY]
   - Amenability
   - Approximation properties
   - Haagerup AP for $\mathbb{F}O(Q)$

The algebraic Setting

General setting

We are given: $\mathscr{A}$ *unital* $*$*-algebra and* $\Delta : \mathscr{A} \to \mathscr{A} \odot \mathscr{A}$ $*$*-hom.*

Assume: $\mathscr{A}$ *generated by* u_{ij} *such that* $\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$,
$u = (u_{ij})$ *and* $\bar{u} = (u_{ij}^*)$ *invertible in* $M_N(\mathscr{A})$.

→ $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$ and
→ $\mathrm{Span}\, \Delta(\mathscr{A})(1 \odot \mathscr{A}) = \mathrm{Span}\, \Delta(\mathscr{A})(\mathscr{A} \odot 1) = \mathscr{A} \odot \mathscr{A}$

Examples

- $G \subset U_N$ compact group of matrices
  $\mathscr{A} = \mathrm{Pol}(G)$, $u_{ij} : G \to \mathbb{C}$ coordinate maps
  $\Delta(u_{ij})(g, h) = \sum g_{ik} h_{kj} = u_{ij}(gh)$
- $\Gamma = \langle \gamma_i \rangle$ finitely generated group
  $\mathscr{A} = \mathbb{C}[\Gamma]$, $u = \mathrm{diag}(\gamma_i)$, $\Delta(\gamma) = \gamma \otimes \gamma$

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Notation: $\mathscr{A} = \mathrm{Pol}(\mathbb{G}) = \mathbb{C}[\mathbb{\Gamma}]$.
$\mathbb{G}$, $\mathbb{\Gamma}$ are a "compact and a discrete quantum group in duality".

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Examples

- $\mathscr{A} = \mathscr{A}_o(Q)$, $Q \in GL_N(\mathbb{C})$: $A_o(Q) = \langle u_{ij} \mid u \text{ unitary}, u = Q\bar{u}Q^{-1} \rangle$
  The elements $U_{ij} = \sum u_{ik} \otimes u_{kj}$ satisfy the relations → Δ.
  Notation: $\mathbb{\Gamma} = \mathbb{F}O(Q)$, $\mathbb{G} = O^+(Q)$.
- Special case $Q = \begin{pmatrix} 0 & 1 \\ -1/q & 0 \end{pmatrix}$ → $\mathscr{A}_o(Q) = \mathrm{Pol}(SU_q(2))$

The full and reduced C^*-algebras

Definition (Full C^*-algebra)

We wish to define a norm on $\mathscr{A}$ by

$$\|x\|_{\mathrm{f}} = \sup\{\|\pi(x)\|_{B(K)} \mid \pi : \mathscr{A} \to B(K) \ast\text{-rep}\}.$$

Assumption: this is finite — "$\mathscr{A}$ has an enveloping C^*-algebra".

➜ completion $A_{\mathrm{f}} = C_{\mathrm{f}}(\mathbb{G}) = C^*_{\mathrm{f}}(\mathbb{\Gamma})$

By universality, the coproduct Δ extends to

$$\Delta : C^*_{\mathrm{f}}(\mathbb{\Gamma}) \to C^*_{\mathrm{f}}(\mathbb{\Gamma}) \otimes C^*_{\mathrm{f}}(\mathbb{\Gamma})$$

and $C^*_{\mathrm{f}}(\mathbb{\Gamma})$ is a **Woronowicz C^*-algebra** (cf Adam's talk).

The full and reduced C^*-algebras

Still assume: $\mathscr{A}$ has an envelopping C^*-algebra — e.g. (u_{ij}) unitary.
Theorem: there exists a unique state $h : \mathscr{A} \to \mathbb{C}$ such that $(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = 1 \circ h$, and it is faithful on $\mathscr{A}$.

Definition (Reduced C^*-algebra)

$(\mathscr{A}, h)$ ➜ $\Lambda : \mathscr{A} \hookrightarrow H = L^2(\mathbb{G}) = \ell^2(\mathbb{\Gamma})$ with $\|\Lambda(x)\|^2 = h(x^* x)$
➜ $\lambda : \mathscr{A} \to B(H)$, $\lambda(x)\Lambda(y) = \Lambda(xy)$
➜ completion $A_\mathrm{r} = C_\mathrm{r}(\mathbb{G}) = C^*_\mathrm{r}(\mathbb{\Gamma}) = \overline{\mathrm{Img}}\,\lambda$

By invariance, the coproduct Δ extends to

$$\Delta : C^*_\mathrm{r}(\mathbb{\Gamma}) \to C^*_\mathrm{r}(\mathbb{\Gamma}) \otimes C^*_\mathrm{r}(\mathbb{\Gamma})$$

and $C^*_\mathrm{r}(\mathbb{\Gamma})$ is a **Woronowicz C^*-algebra** (cf Adam's talk).

The full and reduced C^*-algebras

Still assume: $\mathbb{C}[\mathbb{\Gamma}]$ has an envelopping C^*-algebra — e.g. (u_{ij}) unitary.

→ potentially different **Woronowicz** C^***-algebras** $C^*_\mathrm{f}(\mathbb{\Gamma})$, $C^*_\mathrm{r}(\mathbb{\Gamma})$
→ possibly others in between

Useful maps:

- $\lambda : C^*_\mathrm{f}(\mathbb{\Gamma}) \to C^*_\mathrm{r}(\mathbb{\Gamma})$ by definition (regular representation)
- $\epsilon : C^*_\mathrm{f}(\mathbb{\Gamma}) \to \mathbb{C}$ character s.t. $\epsilon(u_{ij}) = \delta_{ij}$ (trivial repr. / co-unit)
- $\Delta' : C^*_\mathrm{r}(\mathbb{\Gamma}) \to C^*_\mathrm{r}(\mathbb{\Gamma}) \otimes C^*_\mathrm{f}(\mathbb{\Gamma})$ (Fell's absorption)

The dual C^*-algebra

Multiplicative Unitary

Define $V \in B(H \otimes H)$ by putting $V(\Lambda \otimes \Lambda)(x \otimes y) = (\Lambda \otimes \Lambda)(\Delta(x)(1 \otimes y))$.

➔ $C^*_{\mathrm{r}}(\mathbb{\Gamma}) = \overline{\mathrm{Span}}\{(\omega \otimes \mathrm{id})(V) \mid \omega \in B(H)_*\}$

➔ $V(x \otimes 1)V^* = \Delta(x)$ for $x \in C^*_{\mathrm{r}}(\mathbb{\Gamma})$

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Definition (Dual C^*-algebra)

We define $\hat{A} = C_0(\mathbb{\Gamma}) = C^*(\mathbb{G})$ and a coproduct $\Delta : \hat{A} \to M(\hat{A} \otimes \hat{A})$ by

$$C_0(\mathbb{\Gamma}) = \overline{\mathrm{Span}}\{(\mathrm{id} \otimes \omega)(V) \mid \omega \in B(H)_*\},$$
$$\Delta(f) = V^*(1 \otimes f)V \text{ for } f \in C_0(\mathbb{\Gamma}).$$

We put also $C_b(\mathbb{\Gamma}) = M(C_0(\mathbb{\Gamma}))$.

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Duality

We have $V \in M(C_0(\mathbb{\Gamma}) \otimes C^*_{\mathrm{r}}(\mathbb{\Gamma}))$ ➜ duality between $C^*_{\mathrm{r}}(\mathbb{\Gamma})$ and $C_0(\mathbb{\Gamma})$:

$$\omega \in C^*_{\mathrm{r}}(\mathbb{\Gamma})^* \to \tilde{\omega} = (\mathrm{id} \otimes \omega)(V) \in C_b(\mathbb{\Gamma}).$$

V lifts to $V_{\mathrm{f}} \in M(C_0(\mathbb{\Gamma}) \otimes C^*_{\mathrm{f}}(\mathbb{\Gamma}))$ such that $V = (\mathrm{id} \otimes \lambda)(V_{\mathrm{f}})$, again:

$$\omega \in C^*_{\mathrm{f}}(\mathbb{\Gamma})^* \to \tilde{\omega} = (\mathrm{id} \otimes \omega)(V_{\mathrm{f}}) \in C_b(\mathbb{\Gamma}).$$

Irreducible (co)representations

Definition

K f.-d. Hilbert space, $v \in B(K) \otimes C^*_{\mathrm{f}}(\Gamma)$ unitary, $(\mathrm{id} \otimes \Delta)(v) = v_{12} v_{13}$
→ "representation" of $\mathbb{G}$ / "corepresentation" of Γ and $\mathscr{A}$

Following the theory of the compact case:

- $f \in B(K_1, K_2)$ intertwiner if $(f \otimes 1) v_1 = v_2 (f \otimes 1)$ → $\mathrm{Hom}(v_1, v_2)$
- $v_1 \sim v_2$ if $\mathrm{Hom}(v_1, v_2)$ contains a bijection
- v irreducible if $\mathrm{Hom}(v, v) = \mathbb{C}\mathrm{id}$ → set $\mathrm{Irr}\, \Gamma$
- direct sum, tensor product, conjugate representation, ...

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Application: "Decomposition of the regular repr. of $\mathbb{G}$"

There is an isomorphism

$$C_0(\mathbb{\Gamma}) \simeq \bigoplus_{\alpha \in \mathrm{Irr}\, \mathbb{\Gamma}} B(K_\alpha) \quad \text{s.t.} \quad V_{\mathrm{f}} \simeq \bigoplus v_\alpha .$$

Moreover $(\Delta \otimes \mathrm{id})(V_{\mathrm{f}}) = V_{\mathrm{f},13} V_{\mathrm{f},23} \simeq \bigoplus v_\alpha \otimes v_\beta$.

Remark: in fact f.d. representations v live in $B(K) \odot \mathscr{A}$.

→ allows to reconstruct $\mathscr{A}$ from a Woronowicz C^*-algebra

Corepresentations of $\mathbb{F}O(Q)$ and the boundary

In this slide $\Gamma = \mathbb{F}O(Q)$ and $Q\bar{Q} \in \mathbb{C}I_N$.

Theorem

One can write $\mathrm{Irr}\,\mathbb{F}O(Q) = \{v_k\}$ *with* $v_0 = 1$, $v_1 = u$, $\bar{v}_k \simeq v_k$ *and* $v_k \otimes v_1 \simeq v_{k-1} \oplus v_{k+1}$.

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An application

$r \in \mathrm{Hom}(v_{k+1}, v_k \otimes v_1)$ isometric → UCP map (cf Mike's talk)

$$R = \left(\Phi^{k,1}_{k+1}\right)^* : B(H_k) \to B(H_{k+1}), f \mapsto r^*(f \otimes \mathrm{id})r.$$

We put $C(\partial\mathbb{F}O(Q)) = \varinjlim(B(H_k), R)$.

Recall $B(H_k) \subset C_0(\Gamma)$ → $\partial\Gamma =$ "projective limit of spheres".

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Recall $B(H_k) \subset C_0(\mathbb{\Gamma})$ ➔ $\partial \mathbb{\Gamma} =$ "projective limit of spheres".

Theorem

$C(\partial \mathbb{\Gamma}) \subset C_b(\mathbb{\Gamma})/C_0(\mathbb{\Gamma})$ is a infinite-dimensional unital $*$-subalgebra.
It is stable under the left and right actions of $\mathbb{\Gamma}$.
The restriction of the left (resp. right) action is amenable (resp. trivial).

Application: exactness and property "AO^+", solidity of the von Neumann algebra $C_r^*(\mathbb{\Gamma})''$.

Amenability : definition(s)

Definition

$\mathbb{\Gamma}$ is called

- weakly amenable if $C_b(\mathbb{\Gamma})$ admits an invariant state m:
$$\forall \ \ \omega \in C_b(\mathbb{\Gamma})_*, \ f \in C_b(\mathbb{\Gamma}) \ m((\omega \otimes \mathrm{id})\Delta(f)) = \omega(1)m(f).$$
- strongly amenable if $\lambda : C^*_{\mathrm{f}}(\mathbb{\Gamma}) \to C^*_{\mathrm{r}}(\mathbb{\Gamma})$ is an isomorphism.

Note: strongly amenable $\Leftrightarrow$ ϵ factors through λ
$\Leftrightarrow$ almost invariant vectors in H.

For Γ classical, $m : \mathcal{P}(\Gamma) \to [0,1]$ invariant, finitely additive, $m(\Gamma) = 1$.

Theorem

For discrete quantum groups, weakly amenable $\Leftrightarrow$ strongly amenable.

"$\Rightarrow$" is harder if h is not tracial, and still open in the locally compact case.

Amenability : examples

Fusion rules: $\alpha \otimes \beta \simeq \bigoplus m_{\alpha\beta}^{\gamma} \gamma$ with α, β, $\gamma \in \mathrm{Irr}\, \mathbb{\Gamma}$

Theorem

Assume $\mathbb{\Gamma}$, $\mathbb{\Lambda}$ have the same fusion rules. If $\mathbb{\Gamma}$ is amenable, $\dim v_\alpha^{\mathbb{\Gamma}} \leq \dim v_\alpha^{\mathbb{\Lambda}}$ *for all α. $\mathbb{\Lambda}$ is amenable* **iff** *we have $=$ for all α.*

"Amenability is a property of the dimension function on the fusion ring."

Examples

- Finite or abelian groups are amenable ; non-ab. free groups are not.
- The dual of a classical G is always amenable.
- The dual of $SU_q(2)$ is amenable.
- $\mathbb{F}O(Q)$ is amenable **iff** $N = 2$.
  Note: $\mathrm{Sp}(\sum u_{ii}) = [-2, 2]$ in $C_\mathrm{r}^*(\mathbb{\Gamma})$, and $\epsilon(\sum u_{ii}) = N$.

Approximation properties

Fix (A, h) unital separable C^*-algebra with faithful state.
Approximation property: $\exists\ T_n : A \to A$ s.t. $\forall a \in A \ \ \|T_n(a) - a\| \underset{n\,\infty}{\longrightarrow} 0$.
Some examples:

- CPAP: T_n UCP and finite rank
- CBAP: T_n uniformly CB and finite rank
- HAP: T_n UCP and compact on $L^2(A, h)$

Theorem

Γ amenable $\Rightarrow$ $C^*_{\mathrm{r}}(\Gamma)$ has the CPAP. $\Leftarrow$ holds if h is tracial.

Proof. $\Rightarrow$: $T_\varphi = (\mathrm{id} \otimes \varphi) \circ \Delta'$ for $\varphi \in C^*_{\mathrm{f}}(\Gamma)^*$, $T_\epsilon = \mathrm{id}$.

Strong amenability $\Leftrightarrow$ ϵ approximated by vector states for λ
$\Leftrightarrow$ by states φ such that $\tilde{\varphi}$ has finite rank.

$\Leftarrow$: have to reconstruct φ from T.

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Theorem

If there exist states $\varphi_n \in C_\mathrm{f}^*(\mathbb{\Gamma})^*$ s.t. $\varphi_n \to \epsilon$ $*$-weakly and $\tilde{\varphi}_n \in C_0(\mathbb{\Gamma})$, then $C_\mathrm{r}^*(\mathbb{\Gamma})$ has the HAP. $\Leftarrow$ holds if h is a trace.

Proof. $\Rightarrow$: $T_\varphi = (\mathrm{id}\otimes\varphi)\circ\Delta'$ for $\varphi \in C_\mathrm{f}^*(\mathbb{\Gamma})^*$, $T_\epsilon = \mathrm{id}$.
$\Leftarrow$: have to reconstruct φ from T.

$\mathbb{F}O(Q)$ has the HAP

Consider φ_t given by $\tilde{\varphi}_t = \sum \frac{[k+1]_t}{[k+1]_N} id_k \in \bigoplus B(H_k)$.

Clearly $\tilde{\varphi}_t \in C_0(\mathbb{\Gamma})$ and $\varphi_t \to \epsilon$ as $t \to N$. Is φ_t a state?

Approach 1 $(Q = I_N)$
Uses $B = \langle \sum u_{ii} \rangle \subset C_f^*(\mathbb{\Gamma})$.
In the unimodular case, the "orthogonal projection" extends to a positive contraction $P : C_f^*(\mathbb{\Gamma}) \twoheadrightarrow B$.
For $\mathbb{F}O(I_N)$, $B \simeq C([-N, N])$ and a computation shows that $\varphi_t = \mathrm{ev}_t \circ P$.

$\mathbb{F}O(Q)$ has the HAP

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Monoidal equivalence

"Abstract" equivalence $F : \mathrm{Corep}\,\mathbb{\Gamma}_1 \to \mathrm{Corep}\,\mathbb{\Gamma}_2$ ($\Rightarrow$ same fusion rules).
Classical cases G, Γ: implies isomorphism.
Every $O^+(Q)$ is monoidally equivalent to an $SU_q(2)$.

Approach 2

If $\tilde{\varphi} = \sum f(\alpha)\mathrm{id}_\alpha$ and $\mathbb{\Gamma} \sim_{\mathrm{mon}} \mathbb{\Gamma}'$, define φ' on $C^*_{\mathrm{f}}(\mathbb{\Gamma}')$ by $\tilde{\varphi}' = \sum f(\alpha)\mathrm{id}'_\alpha$.
Fact: $\|T_\varphi\|_{\mathrm{cb}} = \|T_{\varphi'}\|_{\mathrm{cb}}$. In particular φ state $\Leftrightarrow$ φ' state.

Proposition

On $C(SU_q(2))$, φ_t is the vaccum state in the "Podleś sphere" representations constructed by Voigt.