Stabilizer subgroups of universal compact quantum groups and the Connes embedding property

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Outline

A NC probability space

Underlying algebra defined by generators and relations [Wang]:

$$A_o(n) = \langle u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^*, \ \ (u_{ij}) \ \text{unitary} \rangle.$$

We have a coproduct which allows to convolve states:

$$\Delta : A_o(n) \to A_o(n) \otimes A_o(n), \quad u_{ij} \mapsto \textstyle\sum_k u_{ik} \otimes u_{kj}.$$

In fact $(A_o(n), \Delta)$ is a Woronowicz C^*-algebra ➜ unique bi-invariant state

$$h : A_o(n) \to \mathbb{C}, \quad (h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = 1h.$$

$u \in M_n(\mathbb{C}) \otimes A_o(n)$ universal $n \times n$ orthogonal matrix with NC entries
Haar distributed NC random orthogonal matrix

➜ joint moments of the entries u_{ij}? $n \to \infty$ asymptotics?

Old and new results

Some known results about $A_o(n)$ in NC probability:

- $\chi_1 = \sum u_{ii}$ is a semicircular variable with respect to h [Banica 1997];
- the elements $(\sqrt{n}\, u_{ij})_{i,j \leq N}$ are asymptotically free and semi-circular with respect to h as $n \to \infty$ [Banica-Collins 2007];
- computation of the spectral measure of u_{ij} with respect to h for n fixed [Banica-Collins-Zinn-Justin 2009];
- convergence of $(\sqrt{n}\, u_{ij})_{i,j \leq N}$ *strongly* in distribution [Brannan 2014].

Main result [Brannan-Collins-V.]:

- The generators u_{ij} admit matricial microstates (if $n \neq 3$).

One consequence:

- The (modified, microstate) free entropy dimension $\delta_0(u_{ij})$ equals 1.

Quantum groups

A Woronowicz C^*-algebra is a unital C^*-algebra A with $*$-homomorphism $\Delta : A \to A \otimes A$ (coproduct) such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

Notation : $A = C^*(\mathbb{\Gamma}) = C(\mathbb{G})$.

Examples :

- G compact group, $A = C(G)$, $\Delta(f) = ((x, y) \mapsto f(xy))$, characterized by commutativity of A ;
- Γ discrete group, $A = C^*(\Gamma)$, $\Delta(g) = g \otimes g$ — but also $A = C^*_{\mathrm{red}}(\Gamma)$, characterized by co-commutativity : $\Sigma\Delta = \Delta$.

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General theory :

- Haar state $h \in C^*(\mathbb{\Gamma})^* \twoheadrightarrow$ GNS representation $\lambda : C^*(\mathbb{\Gamma}) \to B(\ell^2\mathbb{\Gamma})$,
- $C^*_{\mathrm{red}}(\mathbb{\Gamma}) = \lambda(C^*(\mathbb{\Gamma}))$ is again a Woronowicz C^*-algebra,
- $\mathscr{L}(\mathbb{\Gamma}) = C^*_{\mathrm{red}}(\mathbb{\Gamma})''$ von Neumann algebra of $\mathbb{\Gamma}$,
- trivial representation / co-unit $\epsilon : C^*_{\mathrm{f}}(\mathbb{\Gamma}) = C_{\mathrm{f}}(\mathbb{G}) \to \mathbb{C}$,
- f.-d. corepresentations $v \in M_k(\mathbb{C}) \otimes C(\mathbb{G})$, intertwiners $T \in \mathrm{Hom}_{\mathbb{G}}(v, w) \subset M_{l,k}(\mathbb{C})$.

$\mathbb{\Gamma}$ is called unimodular if h is a trace, amenable if ϵ factors through λ.

Back to the algebra $A_o(n)$

Recall Wang's algebra:

$$A_o(n) = \langle u_{ij}, 1 \leq i,j \leq n \mid u_{ij} = u_{ij}^*, \ \ (u_{ij}) \text{ unitary}\rangle.$$

Consider the discrete group $FO_n = (\mathbb{Z}/2\mathbb{Z})^{*n}$ and the compact group O_n. We have two interesting quotient maps:

$$\begin{array}{lll} A_o(n) \to A_o(n)/I \simeq C^*(FO_n) & \text{with} & I = \langle u_{ij}, i \neq j\rangle, \\ A_o(n) \to A_o(n)/J \simeq C(O_n) & \text{with} & J = \langle [u_{ij}, u_{kl}]\rangle. \end{array}$$

We denote $A_o(n) = C^*(\mathbb{F}O_n) = C(O_n^+)$ with dual quantum groups:

$\mathbb{F}O_n$, the (discrete) "orthogonal free quantum group";

O_n^+, the (compact) "universal orthogonal quantum group".

Analogies with free group C^*-algebras

$\mathbb{F}O_n$ shares many properties with usual free groups.

- $\mathbb{F}O_n$ is non amenable for $n \geq 3$ [Banica 1997];
- $\mathscr{L}(\mathbb{F}O_n)$ is a full and strongly solid factor [Vaes-V. 2005, Isono 2012];
- Rapid Decay [V. 2007], K-amenability [Voigt 2011], ...

As far as cocycles are concerned:

- the "path cocycle" on $\mathbb{F}O_n$ is trivial and $H^1_{(2)}(\mathbb{F}O_n) = 0$ [V. 2012];
- Haagerup's Property [Brannan 2012]: existence of a proper cocyle;
- classif. of *central* generating functionals [Franz-Kula-Cipriani 2014];
- there is a proper cocyle living in the adjoint representation, which is *weakly* contained in λ [Fima-V. 2014].

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Main result of this talk, restated:

- $\mathscr{L}(\mathbb{F}O_n)$ embeds in R^ω (Connes' embedding property, $n \geq 3$).

Strategy:

- $\mathbb{F}O_2$ is amenable, hence $\mathscr{L}(\mathbb{F}O_2) \subset R^\omega$ → induction over n.
- O_n^+ is generated by two copies of O_{n-1}^+.

Outline

1. Introduction
   - The universal NC orthogonal random matrix
   - Compact and discrete quantum groups
   - Main results

2. Stabilizer subgroups
   - Generating subgroups
   - Stabilizer subgroups of O_n^+
   - Idea of the proof

3. Applications
   - Connes' embedding property
   - Free entropy dimension and microstates

Generating subgroups

$\mathbb{G}$ compact quantum group with *full* Woronowicz C^*-algebra $C_f(\mathbb{G})$.

Closed subgroup $\mathbb{H} \subset \mathbb{G}$: compact quantum group with surjective Hopf-$*$-homomorphism $\pi : C_f(\mathbb{G}) \twoheadrightarrow C_f(\mathbb{H})$.

Inner faithful $*$-homomorphism $f : C(\mathbb{G}) \to B$: for any factorization

$$f : C_f(\mathbb{G}) \xrightarrow{\pi} C_f(\mathbb{H}) \xrightarrow{g} B$$

with π surjective Hopf-$*$-homomorphism, π is an isomorphism.

Definition

Let $(\mathbb{H}_1, \pi_1)$, $(\mathbb{H}_2, \pi_2)$ be closed subgroups of $\mathbb{G}$. Then $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ if $(\pi_1 \otimes \pi_2) \circ \Delta : C_f(\mathbb{G}) \to C_f(\mathbb{H}_1) \otimes C_f(\mathbb{H}_2)$ is inner faithful.

More generally, the subgroup $\mathbb{H} \subset \mathbb{G}$ generated by $\mathbb{H}_1$, $\mathbb{H}_2$ is the *Hopf image* of $(\pi_1 \otimes \pi_2) \circ \Delta$.

Generating subgroups

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Restriction: $v \in M_n(\mathbb{C}) \otimes C_f(\mathbb{G})$ repr. of $\mathbb{G}$ + $(\mathbb{H}, \pi)$ closed subgroup
➜ $(\mathrm{id} \otimes \pi)(v)$, restricted representation of $\mathbb{H}$.

Proposition

$$\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle \iff \forall v, w \in \mathrm{Rep}(\mathbb{G})$$
$$\mathrm{Hom}_{\mathbb{G}}(v, w) = \mathrm{Hom}_{\mathbb{H}_1}(v, w) \cap \mathrm{Hom}_{\mathbb{H}_2}(v, w)$$

Examples

H_1, $H_2 \subset G$ classical compact groups → usual notions.

$\mathbb{G} = \hat{\Gamma}$ dual of discrete group Γ
→ π_i induced by surjective group morphisms $\pi_i : \Gamma \to \Gamma_i$.
→ $\hat{\Gamma} = \langle \hat{\Gamma_1}, \hat{\Gamma_2} \rangle \iff \Gamma \to \Gamma_1 \times \Gamma_2$ faithful.

Some subgroups of O_n^+:

- $\rho : C(O_n^+) \to C(O_n)$, $[u_{ij}, u_{kl}] \to 0$.
- $\pi_i : C(O_n^+) \to C(O_{n-1,i}^+) \simeq C(O_{n-1}^+)$, $u_{ii} \to 1$.

Note that $O_{n-1,i}^+ \subset O_n^+$ is the stabilizer of $e_i \in \mathbb{C}^n$.

More generally if $G_n(\mathcal{C})$ are the *easy quantum groups* associated to a category of partitions $\mathcal{C}$ stable under *block removal*, we have $G_{n-1}(\mathcal{C}) \subset G_n(\mathcal{C})$ as stabilizer subgroups.

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Theorem

For $n \geq 4$ and $i \neq j$ we have $O_n^+ = \langle O_{n-1,i}^+, O_{n-1,j}^+ \rangle = \langle O_{n-1,i}^+, O_n \rangle$.

Brauer diagrams

$P_2(k,l)$: set of partitions of k upper points and l lower points into pairs
$NC_2(k,l) \subset P_2(k,l)$: pair partitions that can be represented by a planar diagram with noncrossing strings

Let $H = \mathbb{C}^n$ and associate to $p \in P(k,l)$ the linear map $T_p : H^{\otimes k} \to H^{\otimes l}$:

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_j \begin{pmatrix} i_1 \dots i_k \\ p \\ j_1 \dots j_l \end{pmatrix} e_{j_1} \otimes \cdots \otimes e_{j_l},$$

where the middle symbol is 1 if all blocs in p join pairs of equal indices, and 0 if not.

Then:

- $\mathrm{Hom}_{O_n}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Span}\{T_p \mid p \in P_2(k,l)\}$ [Brauer],
- $\mathrm{Hom}_{O_n^+}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Span}\{T_p \mid p \in NC_2(k,l)\}$ [Banica].

A lemma of linear algebra

Denote $TC_2(k,l) \subset P_2(k,l)$ the subset of diagrams where crossings are allowed only with lines that are connected to an upper point. Then:

Lemma

$$\mathrm{Hom}_{O^+_{n-1,i}}(1, u^{\otimes k}) = \mathrm{Span}\{T_p(e_i \otimes \cdots \otimes e_i) \mid s \leq k, p \in TC_2(s,k)\}$$

Put $\xi_s = e_1 \otimes \cdots \otimes e_1 \otimes e_2 + e_1 \otimes \cdots \otimes e_2 \otimes e_1 + \cdots + e_2 \otimes e_1 \otimes \cdots \otimes e_1 \in H^{\otimes s}$.

Lemma

We have $\mathrm{Hom}_{O^+_{n-1,i}}(1, u^{\otimes k}) \cap \mathrm{Hom}_{O^+_{n-1,j}}(1, u^{\otimes k}) = \mathrm{Hom}_{O^+_n}(1, u^{\otimes k})$ *iff the family of vectors* $\{T_p(\xi_s) \mid 1 \leq s \leq k, p \in TC_2(s,k)\}$ *is linearly independant.*

A lemma of linear algebra

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Lemma

We have $\mathrm{Hom}_{O^+_{n-1,i}}(1, u^{\otimes k}) \cap \mathrm{Hom}_{O^+_{n-1,j}}(1, u^{\otimes k}) = \mathrm{Hom}_{O^+_n}(1, u^{\otimes k})$ *iff the family of vectors* $\{T_p(\xi_s) \mid 1 \leq s \leq k, p \in TC_2(s,k)\}$ *is linearly independant.*

Lemma

If $n \geq 4$*, the linear independance property of the previous lemma is true for any* k*. As a result* $O^+_n = \langle O^+_{n-1,i}, O^+_{n-1,j} \rangle$.

Moreover we have strong numerical evidence of:

Conjecture

The same is true for $n = 3$.

Outline

Connes' embedding property

Let R^ω be an ultrapower of the hyperfinite II_1 factor.
For A unital C^*-algebra, define

$$CEP(A) = \{\tau : A \to \mathbb{C} \text{ tracial state} \mid \pi_\tau(A)'' \hookrightarrow R^\omega \text{ tracially}\},$$

where π_τ is the GNS representation.

For $\mathbb{\Gamma}$ *unimodular* discrete quantum group: $CEP(\mathbb{\Gamma}) = CEP(C^*_{\mathrm{f}}(\mathbb{\Gamma}))$.
We say that $\mathbb{\Gamma}$ is **hyperlinear** if $h \in CEP(\mathbb{\Gamma})$, i.e. if its von Neumann algebra $\mathscr{L}(\mathbb{\Gamma})$ embeds tracially in R^ω.

Proposition

- *If* $\tau_1, \tau_2 \in CEP(\mathbb{\Gamma})$ *then* $\tau_1 * \tau_2 = (\tau_1 \otimes \tau_2) \circ \Delta \in CEP(\mathbb{\Gamma})$.
- *If* $\tau_n \to \tau$ *pointwise and* $\tau_n \in CEP(\mathbb{\Gamma})$ *then* $\tau \in CEP(\mathbb{\Gamma})$.

Connes' embedding property

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Proposition

- *If $\tau_1, \tau_2 \in CEP(\Gamma)$ then $\tau_1 * \tau_2 = (\tau_1 \otimes \tau_2) \circ \Delta \in CEP(\Gamma)$.*
- *If $\tau_n \to \tau$ pointwise and $\tau_n \in CEP(\Gamma)$ then $\tau \in CEP(\Gamma)$.*

Let $(\mathbb{H}_1, \pi_1)$, $(\mathbb{H}_2, \pi_2)$ be subgroups of $\mathbb{G}$.
Denote $h_i = h_{\mathbb{H}_i} \circ \pi_i : C_f(\mathbb{G}) \to \mathbb{C}$ and $h = h_{\mathbb{G}} : C_f(\mathbb{G}) \to \mathbb{C}$.

Proposition

We have $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ iff $h = \lim (h_1 * h_2)^{*n}$ pointwise.

Corollary

If $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ and $\hat{\mathbb{H}}_1$, $\hat{\mathbb{H}}_2$ are hyperlinear, then $\hat{\mathbb{G}}$ is hyperlinear.

Hyperlinearity of $\mathbb{F}O_n$

Corollary

If $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2 \rangle$ and $\hat{\mathbb{H}}_1$, $\hat{\mathbb{H}}_2$ are hyperlinear, then $\hat{\mathbb{G}}$ is hyperlinear.

Recall that $\mathbb{F}O_n = \hat{O}_n^+$ and $O_n^+ = \langle O_{n-1,i}^+, O_{n-1,j}^+ \rangle$ for $n \geq 4$.
Moreover $\mathbb{F}O_2$ is hyperlinear because it is amenable.
→ $\mathbb{F}O_n$ hyperlinear for all n **if** $O_3^+ = \langle O_{2,i}^+, O_{2,j}+ \rangle$.

Bypass to avoid the use of the conjecture at $n = 3$:

Lemma (after A. Chirvasitu)

We have $O_4^+ = \langle O_2^+ \hat{*} O_2^+, O_4 \rangle$.

Altogether:

Theorem

$\mathbb{F}O_n$ is hyperlinear for all $n \neq 3$.

Free entropy dimension

Denote by δ_0 Voiculescu's modified free entropy dimension.

Consequence of Connes' embedding property: we can apply Jung's "hyperfinite monotonicity" result. Since $\mathscr{L}(\mathbb{F}O_n)$ contains diffuse von Neumann subalgebras this yields:

Corollary

For the generators u_{ij} of $\mathscr{L}(\mathbb{F}O_n)$, $n \neq 3$, we have $1 \leq \delta_0(u_{ij})$.

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For the generators u_{ij} of $\mathscr{L}(\mathbb{F}O_n)$, $n \neq 3$, we have $1 \leq \delta_0(u_{ij})$.

On the other hand we have an upper bound coming from ℓ^2-Betti numbers. More precisely

$$\delta_0(u_{ij}) \leq \delta^*(u_{ij}) \leq \beta_1^{(2)}(\mathbb{F}O_n) - \beta_0^{(2)}(\mathbb{F}O_n) + 1$$

by [Biane-Capitaine-Guionnet] and [Connes-Shlyakhtenko].

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by [Biane-Capitaine-Guionnet] and [Connes-Shlyakhtenko]. Moreover

Theorem (V. 2012)

We have $\beta_1^{(2)}(\mathbb{F}O_n) = 0$ *for all* $n \geq 3$.

Since $\mathbb{F}O_n$ is infinite we have $\beta_0^{(2)}(\mathbb{F}O_n) = 0$ [Kyed] and finally

Corollary

For the generators u_{ij} *of* $\mathscr{L}(\mathbb{F}O_n)$, $n \neq 3$, *we have* $\delta_0(u_{ij}) = 1$.

Microstates

Connes' embedding property is equivalent to the existence of matricial microstates for the generators u_{ij}. More precisely, for every $p \in \mathbb{N}$ and $\epsilon > 0$ there exists $k \in \mathbb{N}$ and matrices $a_{ij} \in M_k(\mathbb{C})_{sa}$ such that

$$|\mathrm{tr}(a_{i_1 j_1} \cdots a_{i_q j_q}) - h(u_{i_1 j_1} \cdots u_{i_q j_q})| \leq \epsilon$$

for all $i, j \in \{1, \ldots, n\}^q$, $q \leq p$.

Using the stabilizer subgroups one can write down an explicit construction:

$$\begin{array}{c} \text{explicit microstate} \\ \text{for } O_2^+ = SU_{-1}(2) \end{array} \quad \Longrightarrow \quad \begin{array}{c} \text{explicit microstate} \\ \text{for } \{u_{ij}\} \subset O_n^+ \end{array}$$

Questions : construct "natural" microstates / a "natural" asymptotic random matrix model for O_n^+ ?