Cocycles on free quantum groups

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Outline

1. Cocycles on discrete quantum groups
   - Definitions
   - Analytical properties
2. Path cocycles on free quantum groups
   - Path cocycles
   - Vanishing of L^2-cocycles
3. A proper cocycle on $\mathbb{F}O_n$
   - Construction of the cocycle
   - Applications

Cocycles

Discrete quantum group $\mathbb{\Gamma}$: given by a full Woronowicz C^*-algebra $(C^*(\mathbb{\Gamma}), \Delta)$. Example: $\Delta(g) = g \otimes g$ on $C^*(\Gamma)$, Γ usual discrete group.

- unitary repr: unital $*$-hom $\pi : C^*(\mathbb{\Gamma}) \to B(H)$
- regular repr: GNS $(\ell^2(\mathbb{\Gamma}), \xi_0, \lambda)$ of Haar state h
- reduced C^*-algebra: $C^*_{\rm red}(\mathbb{\Gamma}) = \lambda(C^*(\mathbb{\Gamma}))$
- trivial repr: co-unit $\epsilon : C^*(\mathbb{\Gamma}) \to \mathbb{C}$
- dense Hopf algebra: $\mathbb{C}[\mathbb{\Gamma}] \subset C^*(\mathbb{\Gamma})$ with antipode S

There is a dual Hopf C^*-algebra $C_0(\mathbb{\Gamma})$ with duality described by a multiplicative unitary $V \in M(C_0(\mathbb{\Gamma}) \otimes C^*_{\rm red}(\mathbb{\Gamma}))$.

Definition

A π-*cocycle* on $\mathbb{\Gamma}$ is a derivation $c : \mathbb{C}[\mathbb{\Gamma}] \to {}_\pi H_\epsilon$, i.e. a linear map such that $c(xy) = \pi(x)c(y) + c(x)\epsilon(y)$. It is *trivial* if there is a *fixed vector* $\xi \in H$, such that $c(x) = \pi(x)\xi - \xi\epsilon(x)$.

Connection with quantum Dirichlet forms

"Generating functional": $\psi \in \mathbb{C}[\mathbb{\Gamma}]^*$ such that

$$\psi(1) = 0,\ \psi(x^*) = \overline{\psi(x)} \text{ and } \psi(x^*x) \le 0 \text{ for } x \in \operatorname{Ker} \epsilon.$$

→ convolution semigroup of states, quantum Levy process, ..., and also:

→ Dirichlet form $\mathcal{E}$ under a symmetry condition [Cipriani-Franz-Kula]. In the tracial case: $\mathcal{E}(x\xi_0) = h(x^*(\psi * x))$.

Proposition (V.)

Assume $c : \mathbb{C}[\mathbb{\Gamma}] \to H$ *to be* real, *i.e.*

$$(c(x)|c(y)) \in \mathbb{R} \text{ as soon as } x = S(x)^* \text{ and } y = S(y^*).$$

Then: $\psi : x \mapsto (c(S(x_{(1)})^*)|c(x_{(2)}))$ *is a generating functional,*

$\psi(x^*y) = -2(c(x)|c(y))$ *for all* $x, y \in \operatorname{Ker} \epsilon$,

ψ *is symmetric:* $\psi \circ S = \psi$.

Note : if h is tracial, reality is not needed to get a generating functional. In the classical case, it is not needed either for symmetry.

Analytical properties

Cocycle c → "function" $C = (\mathrm{id}\otimes c)(V)$, unbdd multiplier of $C_0(\mathbb{\Gamma})\otimes H$.
We have $C_0(\mathbb{\Gamma}) \simeq \bigoplus_\alpha B(H_\alpha)$ → $C = (C_\alpha)_\alpha \in \prod B(H_\alpha, H_\alpha\otimes H)$.

Say that c is *bounded* if $(\|C_\alpha\|)_\alpha$ is bounded,
(metrically) *proper* if $\|(C_\alpha^* C_\alpha)^{-1}\| \to_\alpha 0$.

Lemma [V. 2012]: A cocycle c is bounded *iff* it is trivial.

Theorem (Kyed 2011)

$\mathbb{\Gamma}$ *has Property (T) [Fima 2010]* iff *every cocycle in a unitary repr. is trivial.*

Theorem (DFSW)

$\mathbb{\Gamma}$ *admits a metrically proper real cocycle* iff *it has Haagerup's approximation property, i.e. there exists a net of states* $\varphi_k \in C^*(\mathbb{\Gamma})^*_+$ *s.t.*

$$\varphi_k \overset{w*}{\to} \epsilon \text{ and } \forall k\ (\mathrm{id}\otimes\varphi_k)(V_{\mathrm{full}}) \in C_0(\mathbb{\Gamma}).$$

Free quantum groups

The *orthogonal free quantum groups* $\mathbb{F}O_n$ [Wang 1995] are given by

$$C^*(\mathbb{F}O_n) = A_o(n) = \langle u_{ij}, 1 \leq i,j \leq n \mid u_{ij} = u_{ij}^*, \ \ (u_{ij}) \ \text{unitary} \rangle$$

with $\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$. For $n \geq 3$, the quantum group $\mathbb{F}O_n$ is non amenable [Banica 1996], exact, C^*-simple [Vaes-Vergnioux 2007], ...

Theorem (Brannan 2012)

$\mathbb{F}O_n$ satisfies Haagerup's approximation property.

Proof : explicit net of states φ_k (in fact associated multipliers) arising from the "central subalgebra" generated by $\chi = \sum u_{ii}$.

Question : What can be said about proper cocycles on $\mathbb{F}O_n$?
In which representations do they live?

Outline

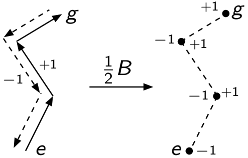
A classical path cocycle

Consider the Cayley graph of the classical free group $\Gamma = F_n = \langle S \rangle$:

- $X^{(0)} = \Gamma$, $X^{(1)} = \Gamma \times S$,
- $s(g,h) = g$, $t(g,h) = gh$, $\theta(g,h) = (gh, h^{-1})$.

Put $p(g) = \sum$ (edges along path $e \to g$) $-$ (reversed edges).
Fact. $p : \Gamma \to \ell^2(\Gamma) \otimes \ell^2(S)$ is a proper cocycle ➞ Haagerup's property.

Other fact. p is a lift of the trivial cocycle $c_0(g) = g - e$ through the boundary map $B(g \otimes h) = gh - g$ ➞ every cocycle "factors" through p.



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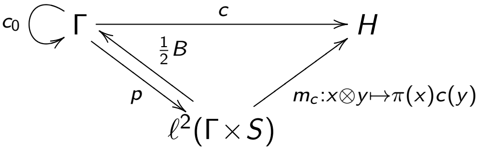

The quantum case

Denote $\ell^2(\mathbb{S}) = \mathrm{Span}\{u_{ij}\} \subset \ell^2(\Gamma) = \ell^2(\mathbb{F}O_n)$. Quantum Cayley graph:

- $\ell^2(\mathbb{X}^{(0)}) = \ell^2(\Gamma)$, $\ell^2(\mathbb{X}^{(1)}) = \ell^2(\Gamma)\otimes\ell^2(\mathbb{S})$,
- $S(x\otimes y) = x\epsilon(y)$, $T(x\otimes y) = xy$, $\Theta(x\otimes y) = xy_{(1)}\otimes S(y_{(2)})$.

Put $B = T - S$, $\ell^2_\wedge(\mathbb{X}^{(1)}) = \mathrm{Ker}(\Theta + \mathrm{id})$.
Path cocycle : $p : \mathbb{C}[\Gamma] \to \ell^2_\wedge(\mathbb{X}^{(1)})$ such that $B \circ p = c_0$.

Theorem (V. 2012)

For $\Gamma = \mathbb{F}O_n$, $n \geq 3$, the operator $B : \ell^2_\wedge(\mathbb{X}^{(1)}) \to \ell^2(\mathbb{X}^{(0)})$ is invertible. There exists a unique path cocycle, and it is bounded.

Theorem (V. 2012)

For $\Gamma = \mathbb{F}U_n$, $n \geq 3$, there exists a unique path cocycle in a suitable dense subspace of $\ell^2_\wedge(\mathbb{X}^{(1)})$. It is unbounded but not proper.

Applications

Recall the classical case: every cocycle factors through the path cocycle.

Theorem (V. 2012)

For $\Gamma = \mathbb{F}O_n$, $n \geq 3$, every λ-cocycle $c : \mathbb{C}[\Gamma] \to \ell^2(\Gamma)^k$ is trivial.

Proof. In the quantum case, the values of the path cocycle p do not have finite support → one needs an analytical version of the "factorization trick" above, which only works for ℓ^2-cocycles, and Property RD.

Applications:

- $\forall k\ \beta_k^{(2)}(\mathbb{F}O_n) = 0$ [Collins-Härtel-Thom]
- $\delta^*(\mathbb{C}[\mathbb{F}O_n], h) = 1$ by [Connes-Shlyakhtenko]
  $\delta(\mathbb{C}[\mathbb{F}O_n], h) = 1$ if $C^*_{\mathrm{red}}(\mathbb{F}O_n)''$ is R^ω-embeddable

Outline

A Deformation of $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ [Fima-V.]

Action of O_n
By definition there is a surjective map

$$\pi : C^*(\mathbb{F}O_n) = C(O_n^+) \to C(O_n).$$

By Fell's absorption principle, the coproduct Δ factors to

$$\Delta' : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*(\mathbb{F}O_n) \otimes C^*_{\mathrm{red}}(\mathbb{F}O_n).$$

We get an action of O_n on $C^*_{\mathrm{red}}(\mathbb{F}O_n)$ by automorphisms :

$$\alpha_g = ((ev_g \circ \pi) \otimes \mathrm{id}) \circ \Delta' : C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C^*_{\mathrm{red}}(\mathbb{F}O_n).$$

Deformation of $C = C^*_{\mathrm{red}}(\mathbb{F}O_n)$ inside $C \otimes C$
Consider the embedding $\iota = \Delta_{\mathrm{red}} : C = C^*_{\mathrm{red}}(\mathbb{F}O_n) \to C \otimes C$.
We deform ι by putting $\iota_g(x) = (\mathrm{id} \otimes \alpha_g)\iota : C \to C \otimes C$, for $g \in O_n$.

Note : using the conditional expectation $E : C \otimes C \to C$, one can recover Brannan's completely positive deformation, $T_t = E \circ \iota_g$ with $t = \mathrm{Tr}(g)$.

Constructing a cocycle

General scheme : deformation by automorphisms $\longleftrightarrow$ derivation into a $C, C-$bimodule $\longleftrightarrow$ cocycle in a representation.

Some notation.
u^α irreducible corepr. of $\mathbb{F}O_n \to v^\alpha = \pi_*(u^\alpha)$ representation of O_n.
$X = -X^t \neq 0 \in M_n(\mathbb{R})$ tangent vect. to O_n at $I \to$ differentation d_X.

Proposition (Fima-V.)

The cocycle associated to the previous deformation is

$$c_X : \mathbb{C}[\mathbb{F}O_n] \to \ell^2(\mathbb{F}O_n), \quad u^\alpha_{ij} \mapsto \sum_{kl} (d_X v^\alpha_{kl}) \times u^\alpha_{ik} u^{\alpha *}_{jl} \xi_0,$$

with respect to the adjoint representation $\mathrm{ad} : C^*(\mathbb{F}O_n) \to B(\ell^2(\mathbb{F}O_n))$. *Moreover it is proper.*

Example. c_X is determined by it value on generators. For $X = e_{12} - e_{21}$:

$$c_X(u_{ij}) = (u_{i1}u_{j2} - u_{i2}u_{j1})\xi_0.$$

The adjoint representation

Remark: in the unimodular case, ξ_0 is fixed by ad: $\epsilon \subset \mathrm{ad}$.
However $c_X : \mathbb{C}[\mathbb{F}O_n] \to \ell^2(\mathbb{F}O_n)^\circ = \xi_0^\perp$.

Theorem (Fima-V.)

The subrepresentation $\mathrm{ad}^\circ \subset \mathrm{ad}$ *on* $\ell^2(\mathbb{F}O_n)^\circ$ *factors through* λ.

$\mathbb{F}O_n$ has a proper cocycle in a weakly-ℓ^2 repr. : property "strong (HH)".

By Ozawa-Popa-Sinclair, using CBAP [Freslon], one gets another proof of:

Theorem (Isono 2012)

For $n \geq 3$, *the factor* $\mathscr{L}(\mathbb{F}O_n)$ *is strongly solid.*
In particular it is prime and has no Cartan subalgebra.