K-theory of the unitary free quantum groups

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joint work with

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Shanghai, 2012, July 26

Outline

The main result

Let $n \geq 2$ and $Q \in GL_n(\mathbb{C})$. Consider the following unital C^*-algebras, generated by n^2 elements u_{ij} forming a matrix u, and the relations

$$A_u(Q) = \langle u_{ij} \mid u \text{ and } Q\bar{u}Q^{-1} \text{ unitaries}\rangle,$$
$$A_o(Q) = \langle u_{ij} \mid u \text{ unitary and } u = Q\bar{u}Q^{-1}\rangle.$$

They are interpreted as maximal C^*-algebras of discrete quantum groups: $A_u(Q) = C^*(\mathbb{F}U(Q))$, $A_o(Q) = C^*(\mathbb{F}O(Q))$ [Wang, Van Daele 1995].

Theorem

The discrete quantum group $\mathbb{F}U(Q)$ satisfies the strong Baum-Connes property ("$\gamma = 1$"). We have

$$K_0(A_u(Q)) = \mathbb{Z}[1] \quad \textit{and} \quad K_1(A_u(Q)) = \mathbb{Z}[u] \oplus \mathbb{Z}[\bar{u}].$$

Strategy of proof

- If $Q\bar{Q} \in \mathbb{C}I_n$ we have $\mathbb{F}U(Q) \hookrightarrow \mathbb{Z} * \mathbb{F}O(Q)$ [Banica 1997].
- $\mathbb{F}O(Q)$ satisfies strong Baum-Connes [Voigt 2009].
- Prop.: stability of strong BC under passage to "divisible" subgroups.
- Theorem: stability of strong BC under free products.
- Case $Q\bar{Q} \notin \mathbb{C}I_n$: monoidal equivalence [Bichon-De Rijdt-Vaes 2006].
- Use strong BC to compute the K-groups.

Other possible approach: Haagerup's Property [Brannan 2011] ?

Result on free products:

- classical case: for groups acting on trees [Baum-Connes-Higson 1994], [Oyono-oyono 1998], [Tu 1998]
- quantum case: uses the quantum Bass-Serre tree and the associated Julg-Valette element [V. 2004]

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Result on free products:

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Novelties:

- C^*-algebra $\mathscr{P}$ associated to the quantum Bass-Serre tree [Julg-Valette 1989] and [Kasparov-Skandalis 1991]
- Invertibility of the associated Dirac element without "rotation trick"
- Actions of Drinfel'd double $D(\mathbb{F}U(Q))$ in order to be able to take tensor products

Outline

Discrete quantum groups

Let Γ be a discrete group and consider the C^*-algebra $C_0(\Gamma)$.
The product of Γ is reflected on $C_0(\Gamma)$ by a coproduct:

$$\begin{aligned}\Delta : C_0(\Gamma) &\to M(C_0(\Gamma)\otimes C_0(\Gamma))\\ f &\mapsto ((g,h) \to f(gh)).\end{aligned}$$

A discrete quantum group $\mathbb{\Gamma}$ can be given by:

- a C^*-algebra $C_0(\mathbb{\Gamma})$ with coproduct $\Delta : C_0(\mathbb{\Gamma}) \to M(C_0(\mathbb{\Gamma})\otimes C_0(\mathbb{\Gamma}))$,
- a C^*-algebra $C^*(\mathbb{\Gamma})$ with coproduct,
- a category of corepresentations $\mathrm{Corep}\,\mathbb{\Gamma}$ (semisimple, monoidal : $\otimes$)

Classical case: $\mathbb{\Gamma} = \Gamma$ "real" discrete group $\Longleftrightarrow$ commutative $C_0(\mathbb{\Gamma})$.
Then $\mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Gamma} = \Gamma$ with $\otimes =$ product of Γ.

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In general $C_0(\mathbb{\Gamma})$ is a sum of matrix algebras:

$$C_0(\mathbb{\Gamma}) = \bigoplus \{L(H_r) \mid r \in \mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Gamma}\}.$$

The interesting algebra is $C^*(\mathbb{\Gamma})$! E.g. $C^*(\mathbb{F}U(Q)) = A_u(Q)$.
$C_0(\mathbb{\Gamma})$, $C^*(\mathbb{\Gamma})$ are both represented on a GNS space $\ell^2(\mathbb{\Gamma})$.

Quantum subgroups and quotients

Different ways of specifying $\mathbb{\Lambda} \subset \mathbb{\Gamma}$:

- bisimplifiable sub-Hopf C^*-algebra $C^*(\mathbb{\Lambda}) \subset C^*(\mathbb{\Gamma})$
  conditional expectation $E : C^*(\mathbb{\Gamma}) \twoheadrightarrow C^*(\mathbb{\Lambda})$
- full subcategory $\mathrm{Corep}\,\mathbb{\Lambda} \subset \mathrm{Corep}\,\mathbb{\Gamma}$,
  containing 1, stable under $\otimes$ and duality [V. 2004]
- surj. morphism $\pi : C_0(\mathbb{\Gamma}) \to C_0(\mathbb{\Lambda})$ compatible with coproducts
  [Vaes 2005] in the locally compact case

Quotient space:

- $C_b(\mathbb{\Gamma}/\mathbb{\Lambda}) = \{f \in M(C_0(\mathbb{\Gamma})) \mid (\mathrm{id}\otimes\pi)\Delta(f) = f\otimes 1\}$
  with coaction of $C_0(\mathbb{\Gamma})$
- $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ = GNS construction of $\varepsilon_{\mathbb{\Lambda}} \circ E : C^*(\mathbb{\Gamma}) \to \mathbb{C}$
- $\mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Gamma}/\mathbb{\Lambda} = \mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Gamma}/\sim$,
  where $r \sim s$ if $r \subset s \otimes t$ with $t \in \mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Lambda}$

Divisible subgroups

$\mathbb{\Lambda} \subset \mathbb{\Gamma}$ is "divisible" if one of the following equiv. conditions is satisfied:

- There exists a $\mathbb{\Lambda}$-equivariant isomorphism $C_0(\mathbb{\Gamma}) \simeq C_0(\mathbb{\Gamma}/\mathbb{\Lambda}) \otimes C_0(\mathbb{\Lambda})$.
- There exists a $\mathbb{\Lambda}$-equivariant isomorphism $C_0(\mathbb{\Gamma}) \simeq C_0(\mathbb{\Lambda}) \otimes C_0(\mathbb{\Lambda} \backslash \mathbb{\Gamma})$.
- For all $\alpha \in \mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Gamma}/\mathbb{\Lambda}$ there exists $r = r(\alpha) \in \alpha$ such that $r \otimes t$ is irreducible for all $t \in \mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Lambda}$.

Examples:

- Every subgroup of $\mathbb{\Gamma} = \Gamma$ is divisible.
- Proposition: $\mathbb{\Gamma}_0 \subset \mathbb{\Gamma}_0 * \mathbb{\Gamma}_1$ is divisible.
- Proposition: $\mathbb{F}U(Q) \subset \mathbb{Z} * \mathbb{F}O(Q)$ is divisible.
- $\mathbb{F}O(Q)^{ev} \subset \mathbb{F}O(Q)$ is not divisible.

In the divisible case $C_0(\mathbb{\Gamma}/\mathbb{\Lambda}) \simeq \bigoplus\{L(H_{r(\alpha)}) \mid \alpha \in \mathrm{Irr}\,\mathrm{Corep}\,\mathbb{\Gamma}/\mathbb{\Lambda}\}$.

Outline

The quantum Bass-Serre tree

$\mathbb{\Gamma}_0$, $\mathbb{\Gamma}_1$ discrete quantum groups: $C_0(\mathbb{\Gamma}_i)$, $\ell^2(\mathbb{\Gamma}_i)$, $C^*(\mathbb{\Gamma}_i)$.

Free product: $\mathbb{\Gamma} = \mathbb{\Gamma}_0 * \mathbb{\Gamma}_1$ given by $C^*(\mathbb{\Gamma}) = C^*(\mathbb{\Gamma}_0) * C^*(\mathbb{\Gamma}_1)$.
We have "$\mathrm{Irr\,Corep}\,\mathbb{\Gamma} = \mathrm{Irr\,Corep}\,\mathbb{\Gamma}_0 * \mathrm{Irr\,Corep}\,\mathbb{\Gamma}_1$" [Wang 1995].

The classical case $\mathbb{\Gamma} = \Gamma$

X graph with oriented edges, one edge by pair of adjacent vertices

- set of vertices: $X^{(0)} = (\Gamma/\Gamma_0) \sqcup (\Gamma/\Gamma_1)$
- set of edges: $X^{(1)} = \Gamma$
- target and source maps: $\tau_i : \Gamma \to \Gamma/\mathbb{\Gamma}_i$ canonical surjections

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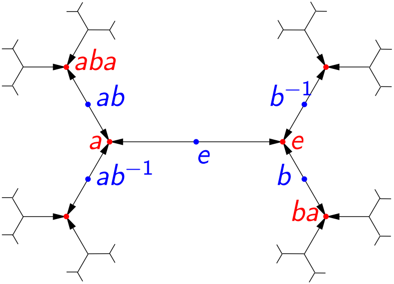

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = \langle a, b \rangle$$

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The general case

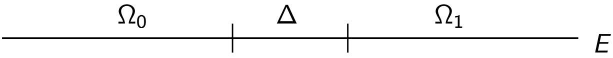
$\mathbb{X}$ "quantum graph"

- space of vertices: $\ell^2(\mathbb{X}^{(0)}) = \ell^2(\mathbb{\Gamma}/\mathbb{\Gamma}_0) \oplus \ell^2(\mathbb{\Gamma}/\mathbb{\Gamma}_1)$, $C_0(\mathbb{X}^{(0)}) = C_0(\mathbb{\Gamma}/\mathbb{\Gamma}_0) \oplus C_0(\mathbb{\Gamma}/\mathbb{\Gamma}_1)$
- space of edges: $\ell^2(\mathbb{X}^{(1)}) = \ell^2(\mathbb{\Gamma})$, $C_0(\mathbb{X}^{(1)}) = C_0(\mathbb{\Gamma})$
- target and source operators: $T_i : \ell^2(\mathbb{\Gamma}) \to \ell^2(\mathbb{\Gamma}/\mathbb{\Gamma}_i)$ unbounded $T_i f$ is bounded for all $f \in C_c(\mathbb{\Gamma}) \subset K(\ell^2(\mathbb{\Gamma}))$.

The ℓ^2 spaces are endowed with natural actions of $D(\mathbb{\Gamma})$, the operators T_i are intertwiners.

Dirac element

We put $\ell^2(\mathbb{X}) = \ell^2(\mathbb{X}^{(0)}) \oplus \ell^2(\mathbb{X}^{(1)})$ and we consider the affine line



Kasparov-Skandalis algebra $\mathscr{P} \subset C_0(E) \otimes K(\ell^2(\mathbb{X}))$
Closed subspace generated by $C_c(\mathbb{\Gamma})$, $C_c(\mathbb{\Gamma}/\mathbb{\Gamma}_0)$, $C_c(\mathbb{\Gamma}/\mathbb{\Gamma}_1)$, T_0 and T_1, with support conditions over E:

- $C_c(E) \otimes C_c(\mathbb{\Gamma})$, $C_c(\Omega_i) \otimes C_c(\mathbb{\Gamma}/\mathbb{\Gamma}_i)$,
- $C_c(\Omega_i) \otimes (T_i C_c(\mathbb{\Gamma}))$, $C_c(\Omega_i) \otimes (T_i C_c(\mathbb{\Gamma}))^*$, $C_c(\Omega_i) \otimes (T_i C_c(\mathbb{\Gamma}))(T_i C_c(\mathbb{\Gamma}))^*$.

Proposition

The natural action of $D(\mathbb{\Gamma})$ on $C_0(E) \otimes K(\ell^2(\mathbb{X}))$ restricts to $\mathscr{P}$.

Dirac element

Kasparov-Skandalis algebra $\mathscr{P} \subset C_0(E) \otimes K(\ell^2(\mathbb{X}))$

Proposition

The natural action of $D(\mathbb{\Gamma})$ on $C_0(E) \otimes K(\ell^2(\mathbb{X}))$ restricts to $\mathscr{P}$.

The inclusion $\Sigma\mathscr{P} \subset \Sigma C_0(E) \otimes K(\ell^2(\mathbb{X}))$, composed with Bott isomorphism and the equivariant Morita equivalence $K(\ell^2(\mathbb{X})) \sim_M \mathbb{C}$, defines the Dirac element $D \in KK^{D(\mathbb{\Gamma})}(\Sigma\mathscr{P}, \mathbb{C})$.

Proposition

The element D admits a left inverse $\eta \in KK^{D(\mathbb{\Gamma})}(\mathbb{C}, \Sigma\mathscr{P})$.

The dual-Dirac element η is constructed using $\mathscr{P}$ and the Julg-Valette operator $F \in B(\ell^2(\mathbb{X}))$ from [V. 2004], so that $\eta \otimes_{\Sigma\mathscr{P}} D = [F] =: \gamma$. It was already known that $\gamma = 1$ in $KK^{\mathbb{\Gamma}}$.

Baum-Connes conjecture (1)

Category $KK^{\mathbb{\Gamma}}$: $\mathbb{\Gamma}$-C^*-algebras + morphisms $KK^{\mathbb{\Gamma}}(A, B)$

It is "triangulated":

Class of "triangles": diagrams $\Sigma Q \to K \to E \to Q$
isomorphic to cone diagrams $\Sigma B \to C_f \to A \underset{f}{\to} B$
Example: $Q = E/K$ with equivariant CP section
Motivation: yield exacts sequences via $KK(\,\cdot\,, X)$, $K(\cdot \rtimes \mathbb{\Gamma})$, ...

Two subcategories:

$$TI_{\mathbb{\Gamma}} = \{\mathrm{ind}_E^{\mathbb{\Gamma}}(A) \mid A \in KK\}, \quad TC_{\mathbb{\Gamma}} = \{A \in KK^{\mathbb{\Gamma}} \mid \mathrm{res}_E^{\mathbb{\Gamma}}(A) \simeq 0 \text{ in } KK\}.$$

$\langle TI_{\mathbb{\Gamma}} \rangle$: localizing subcategory generated by $TI_{\mathbb{\Gamma}}$, i.e. smallest stable under suspensions, K-equivalences, cones, countable direct sums.

Classical case : $\mathbb{\Gamma} = \Gamma$ torsion-free. Γ-C^*-algebras in TI_Γ are proper, all proper Γ-C^*-algebras are in $\langle TI_\Gamma \rangle$.

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Definition (Meyer-Nest)

Strong Baum-Connes property with respect to TI : $\langle TI_{\mathbb{\Gamma}} \rangle = KK^{\mathbb{\Gamma}}$.

Implies K-amenability. If $\mathbb{\Gamma} = \Gamma$ without torsion: corresponds to the existence of a γ element with $\gamma = 1$.

Stability under free products

Theorem

If Γ_0, Γ_1 satisfy the strong Baum-Connes property with respect to TI, so does $\Gamma = \Gamma_0 * \Gamma_1$.

$\mathscr{P}$ is in $\langle TI_\Gamma \rangle$ because we have the semi-split extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_0 \oplus I_1 & \longrightarrow & \mathscr{P} & \longrightarrow & C(\Delta, C_0(\Gamma)) & \longrightarrow & 0 \\ & & \wr\vert & & & & \wr\vert & & \\ & & \Sigma \operatorname{ind}_{\Gamma_1}^{\Gamma}(\mathbb{C}) \oplus \Sigma \operatorname{ind}_{\Gamma_0}^{\Gamma}(\mathbb{C}) & & & & \operatorname{ind}_E^{\Gamma}(\mathbb{C}) & & \end{array}$$

and by hypothesis $\mathbb{C} \in KK^{\Gamma_i}$ is in $\langle TI_{\Gamma_i} \rangle$.

Stability under free products

Theorem

If $\mathbb{\Gamma}_0$, $\mathbb{\Gamma}_1$ satisfy the strong Baum-Connes property with respect to TI, so does $\mathbb{\Gamma} = \mathbb{\Gamma}_0 * \mathbb{\Gamma}_1$.

Since $\mathscr{P}$ is in $\langle TI_{\mathbb{\Gamma}} \rangle$ and $KK^{\mathbb{\Gamma}}(\mathrm{ind}_E^{\mathbb{\Gamma}} A, B) \simeq KK(A, \mathrm{res}_E^{\mathbb{\Gamma}} B)$,
one can reduce the "right invertibility" of $D \in KK^{\mathbb{\Gamma}}(\Sigma\mathscr{P}, \mathbb{C})$
to its "right invertibility" in $KK(\Sigma\mathscr{P}, \mathbb{C})$.

The invertibility in KK follows from a computation: $K_*(\Sigma\mathscr{P}) = K_*(\mathbb{C})$.

Conclusion: $\Sigma\mathscr{P} \simeq \mathbb{C}$ in $KK^{\mathbb{\Gamma}}$, hence $\mathbb{C} \in \langle TI_{\mathbb{\Gamma}} \rangle$.
Taking tensor products $\Sigma\mathscr{P} \boxtimes A$ yields $\langle TI_{\mathbb{\Gamma}} \rangle = KK^{\mathbb{\Gamma}}$, but one has to consider actions of the Drinfel'd double $D\mathbb{\Gamma}$.

Outline

Baum-Connes conjecture (2)

Each $A \in KK^{\mathbb{\Gamma}}$ has an “approximation” $\tilde{A} \to A$ with $\tilde{A} \in \langle TI_{\mathbb{\Gamma}} \rangle$, functorial and unique up to isomorphism, which fits in a triangle

$$\Sigma N \to \tilde{A} \to A \to N$$

with $N \in TC_{\mathbb{\Gamma}}$ [Meyer-Nest].

T-projective resolution of $A \in KK^{\mathbb{\Gamma}}$: complex

$$\cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

with C_i directs summands of elements of $TI_{\mathbb{\Gamma}}$, and such that

$$\cdots \longrightarrow KK(X, C_1) \longrightarrow KK(X, C_0) \longrightarrow KK(X, A) \longrightarrow 0$$

is exact for all X.

A T-projective resolution induces a spectral sequence which “computes” $K_*(\tilde{A} \rtimes \mathbb{\Gamma})$. If strong BC is satisfied, one can take $\tilde{A} = A$!

Baum-Connes conjecture (2)

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is exact for all X.

A T-projective resolution induces a spectral sequence which "computes" $K_*(\tilde{A} \rtimes \mathbb{\Gamma})$. If strong BC is satisfied, one can take $\tilde{A} = A$. In the length 1 case, one gets simply a cyclic exact sequence:

$$\begin{array}{ccccc} K_0(C_0 \rtimes \mathbb{\Gamma}) & \to & K_0(\tilde{A} \rtimes \mathbb{\Gamma}) & \to & K_1(C_1 \rtimes \mathbb{\Gamma}) \\ \uparrow & & & & \downarrow \\ K_0(C_1 \rtimes \mathbb{\Gamma}) & \leftarrow & K_1(\tilde{A} \rtimes \mathbb{\Gamma}) & \leftarrow & K_0(C_0 \rtimes \mathbb{\Gamma}). \end{array}$$

Computation of $K_*(A_u(Q))$

Proposition

We have $K_0(A_u(Q)) \simeq \mathbb{Z}$ and $K_1(A_u(Q)) \simeq \mathbb{Z}^2$.

One constructs in $KK^{\mathbb{\Gamma}}$ a resolution of $\mathbb{C}$ of the form

$$0 \longrightarrow C_0(\mathbb{\Gamma})^2 \longrightarrow C_0(\mathbb{\Gamma}) \longrightarrow \mathbb{C} \longrightarrow 0.$$

$C_0(\mathbb{\Gamma}) = \mathrm{ind}_E^{\mathbb{\Gamma}}(\mathbb{C})$ lies in $TI_{\mathbb{\Gamma}}$.
One has $K_*(C_0(\mathbb{\Gamma})) = \bigoplus \mathbb{Z}[r] = R(\mathbb{\Gamma})$, ring of corepresentations of $\mathbb{\Gamma}$.

Induced sequence in K-theory:

$$0 \longrightarrow R(\mathbb{\Gamma})^2 \xrightarrow{b} R(\mathbb{\Gamma}) \xrightarrow{d} \mathbb{Z} \longrightarrow 0,$$

exact for $b(v, w) = v(\bar{u} - n) + w(u - n)$ and $d(v) = \dim v$.
b and d lift to $KK^{\mathbb{\Gamma}}$ ➜ T-projective resolution.

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We obtain the following cyclic exact sequence:

$$\begin{array}{ccccc} K_0(C_0(\Gamma) \rtimes \Gamma) & \to & K_0(\tilde{\mathbb{C}} \rtimes \Gamma) & \to & K_1(C_0(\Gamma)^2 \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ K_0(C_0(\Gamma)^2 \rtimes \Gamma) & \leftarrow & K_1(\tilde{\mathbb{C}} \rtimes \Gamma) & \leftarrow & K_1(C_0(\Gamma) \rtimes \Gamma). \end{array}$$

But $C_0(\Gamma) \rtimes \Gamma \simeq K(\ell^2(\Gamma))$, and $\tilde{\mathbb{C}} \rtimes \Gamma \simeq C^*(\Gamma)$ by strong BC.

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We obtain the following cyclic exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} & \rightarrow & K_0(C^*(\Gamma)) & \rightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^2 & \leftarrow & K_1(C^*(\Gamma)) & \leftarrow & 0. \end{array}$$