Path cocycles in quantum Cayley trees and L^2-cohomology

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Outline

Universal discrete quantum groups

Consider the unital $*$-algebras defined by generators and relations:

$$\mathscr{A}_u(I_n) = \langle u_{ij} \mid (u_{ij}) \text{ and } (u_{ij}^*) \text{ unitary}\rangle,$$
$$\mathscr{A}_o(I_n) = \langle u_{ij} \mid u_{ij} = u_{ij}^*,\ (u_{ij}) \text{ unitary}\rangle,$$

with $1 \leq i, j \leq n$. They become Hopf $*$-algebras with

$$\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj},\quad S(u_{ij}) = u_{ji}^*,\quad \epsilon(u_{ij}) = \delta_{ij}.$$

Moreover there exists a unique positive Haar integral $h : \mathscr{A} \to \mathbb{C}$.
We can consider the GNS construction:

$$H = L^2(\mathscr{A}, h),\quad \lambda : \mathscr{A} \to B(H),\quad M = \lambda(\mathscr{A})'' \subset B(H).$$

Classical counterpart: $\mathscr{A} = \mathbb{C}G$, $H = \ell^2(G)$, with G a discrete group.

Analogies with free group algebras

- there are natural maps $\mathscr{A}_u(I_n) \twoheadrightarrow \mathbb{C}F_n$, $\mathscr{A}_o(I_n) \twoheadrightarrow \mathbb{C}(\mathbb{Z}/2\mathbb{Z})^{*n}$;
- we have $\mathscr{A}_u(I_n) \twoheadrightarrow \mathscr{B}$ for any $\mathscr{B}$ associated with a unimodular discrete quantum group and some n ;
- there is a natural correspondence between irreducible corepresentations of $\mathscr{A}_u(I_n)$ and words on u, $\bar{u}$;
- the C^*-algebras $A_u(I_n)_{\mathrm{red}}$, $A_o(I_n)_{\mathrm{red}}$ are simple, non-nuclear, exact ;
- the discrete quantum groups associated with $\mathscr{A}_u(I_n)$, $\mathscr{A}_o(I_n)$ have the Property of Rapid Decay ;
- $M = \lambda(\mathscr{A}_o(I_n))''$ is a full and prime II_1 factor.

The case $n = 2$ behaves differently, e.g. $\mathscr{A}_o(I_2) = \mathscr{C}(SU_{-1}(2))$ has polynomial growth, and will be excluded in this talk.

The Main result

For an ICC group G, we can take $\mathscr{A} = \mathbb{C}G$ and consider the Hochschild cohomology groups $H^1(\mathscr{A}, {}_\lambda H_\epsilon)$ and $H^1(\mathscr{A}, {}_\lambda M_\epsilon)$.
These groups are moreover right M-modules and we have

$$\beta_1^{(2)}(G) = \dim_M H^1(\mathscr{A}, H) = \dim_M H^1(\mathscr{A}, M).$$

Recall that $\beta_1^{(2)}(F_n) = n - 1$. In the case of the orthogonal universal discrete quantum groups we have the strongly contrasting result:

Theorem

For $n \geq 3$ we have $H^1(\mathscr{A}_o(I_n), H) = H^1(\mathscr{A}_o(I_n), M) = 0$.
In particular $\beta_1^{(2)}(\mathscr{A}_o(I_n)) = 0$. On the other hand $\beta_1^{(2)}(\mathscr{A}_u(I_n)) \neq 0$.

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Remarks:

- Collins-Härtel-Thom: $\beta_k^{(2)}(\mathscr{A}_o(I_n)) = 0$ for $k \geq 4$, $\beta_k^{(2)}(A_o(I_n)) = \beta_{4-k}^{(2)}(A_o(I_n))$, and Kyed: $\beta_0^{(2)}(\mathscr{A}_o(I_n)) = 0$.
- Voigt: Baum-Connes and K-amenability for $A_o(I_n)$, $K_0(A_o(I_n)) = K_1(A_o(I_n)) = \mathbb{Z}$
- History : Leuven 11/2008, ArXiv v1 05/2009, ArXiv v2 03/2010

More on the strategy

Strategy (for $\mathscr{A}_o$):

- Show that one particular cocycle vanishes: the *path cocycle* $c_g : \mathscr{A} \to K_g$ with values in the *quantum Cayley tree*
- Prove that this cocycle is "sufficiently universal" and vanishes "sufficiently strongly" (and use Property RD)

Consider a representation $\pi : \mathscr{A} \to L(X)$ on a vector space X.
A π-cocycle is a map $c : \mathscr{A} \to X$ such that

$$\forall x, y \in \mathscr{A} \qquad c(xy) = \pi(x)c(y) + c(x)\epsilon(y).$$

It is trivial if $c(x) = \pi(x)\xi - \xi\epsilon(x)$ for some $\xi \in L$ and all $x \in \mathscr{A}$.
$H^1(\mathscr{A}, X)$ is the space of π-cocycles modulo trivial cocycles.
We put $c_0(x) = \lambda(x)\xi_0 - \xi_0\epsilon(x)$, where $\xi_0 = \Lambda(1) \in X = H$.

More on the strategy

Algebraic version

Assume we can "lift" c_0 to a cocycle $c_g : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$, ie

$$(m - (\mathrm{id} \otimes \epsilon))(c_g(x)) = x - \epsilon(x)1$$

Observe that the cocycle relation for $c : \mathscr{A} \to X$ reads

$$\pi(x)c(y) = c((m - \mathrm{id} \otimes \epsilon)(x \otimes y))$$

Define $m_c : \mathscr{A} \otimes \mathscr{A} \to X$ by putting $m_c(x \otimes y) = \pi(x)c(y)$.
We obtain $c = m_c \circ c_g$.

Hence if c_g is trivial with fixed vector $\xi_g \in \mathscr{A} \otimes \mathscr{A}$, all cocycles c are trivial with fixed vector $\xi = m_c(\xi_g)$.

More on the strategy

Hilbertian version

Assume we can "lift" c_0 to a cocycle $c_g : \mathscr{A} \to H \otimes H_1$, ie

$$(m - (\mathrm{id} \otimes \epsilon))(c_g(x)) = x - \epsilon(x)1$$

Observe that the cocycle relation for $c : \mathscr{A} \to M$ reads

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With $H_1 \subset H$ finite-dimensional...

More on the strategy

The correct version

Assume we can "lift" c_0 to a cocycle $c_g : \mathscr{A} \to \mathscr{K}'_g$, ie

$$(m - (\mathrm{id} \otimes \epsilon))(c_g(x)) = x - \epsilon(x)1$$

Observe that the cocycle relation for $c : \mathscr{A} \to M$ reads

$$\pi(x)c(y) = c((m - \mathrm{id} \otimes \epsilon)(x \otimes y))$$

Define $m_c : H \otimes H_1 \to X$ by putting $m_c(x \otimes y) = \pi(x)c(y)$.
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Hence if c_g is trivial with fixed vector $\xi_g \in M \otimes H_1$, all cocycles c are trivial with fixed vector $\xi = m_c(\xi_g)$.

With $\mathscr{K}'_g \subset (\mathscr{A} \otimes \mathscr{A}_1) \cap \overline{\mathrm{Ker}}(m - \mathrm{id} \otimes \epsilon)$...

Outline

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Cayley graph associated with (G, S) is given by:

- the set of vertices G,
- the set of edges $G \times S$,
- the target map $t : (\alpha, \gamma) \to \alpha\gamma$,
- the reversing map $\theta(\alpha, \gamma) = (\alpha\gamma, \gamma^{-1})$.

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$C^*_{\mathrm{red}}(G)$ acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

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The quantum Cayley graph associated with $(\mathscr{C}, S)$ is given by:

- the space of vertices H,
- the space of edges $K = H \otimes H_1$, where $H_1 = p_S H$,
- the target operator $T : \delta_\alpha \otimes \delta_\beta \mapsto \delta_{\alpha\beta}$,
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The quantum Cayley graph associated with $(\mathscr{C}, S)$ is given by:

- the space of vertices H,
- the space of edges $K = H \otimes H_1$, where $H_1 = p_S H$,
- the target operator $T = m : K \to H$,
- the reversing operator $\Theta = \cdots$, such that $T\Theta = \mathrm{id} \otimes \epsilon$.

$C^*_{\mathrm{red}}(G)$ acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

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The quantum Cayley graph associated with $(\mathscr{C}, S)$ is given by:

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- the target operator $T = m : K \to H$,
- the reversing operator $\Theta = \cdots$, such that $T\Theta = \mathrm{id} \otimes \epsilon$.

A_{red} acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

Classical subgraphs

Quasi-classical subgraph $Q_0K \subset K$: maximal subspace on which $\Theta^2 = \mathrm{id}$.
Classical subgraph $q_0K \subset Q_0K$: fixed points for the adjoint repr. of $\hat{A}$.

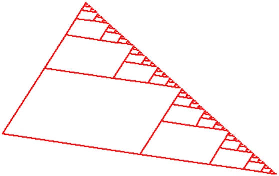
When $\mathscr{A} = \mathbb{C}G$, $q_0K = Q_0K = K$.
When $\mathscr{A} = \mathscr{A}_o(I_n)$, $q_0K = Q_0K \neq K$.
When $\mathscr{A} = \mathscr{A}_u(I_n)$, $q_0K \neq Q_0K \neq K$.

The classical and quasi-classical subgraphs are the hilbertian counterparts of "real" graphs as follows:

- vertices are elements of $\mathrm{Irr}\,\mathscr{C}$,
- edges depend on the fusions rules in $\mathrm{Irr}\,\mathscr{C}$,
- target operator with weights depending on the quantum dimensions,
- q_0H, Q_0H are not stable under the action of $\mathscr{A}$.

Classical subgraphs

In the case of $A_u(I_n)$ we have a "classical" binary tree and a "quasiclassical" union of half lines:



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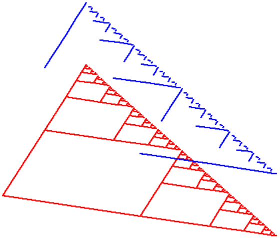

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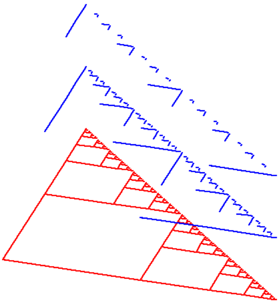

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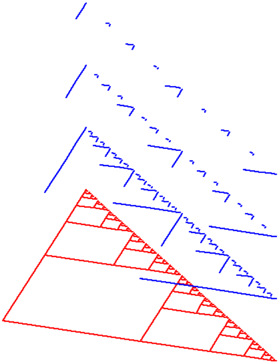

Path cocycles

We look for cocycles with values in the space of *geometric*, or antisymmetric, edges $K_g = \mathrm{Ker}(\Theta + \mathrm{id})$.
Recall that $T = m = (\mathrm{id} \otimes \epsilon)\Theta$, so that $m - \mathrm{id} \otimes \epsilon = 2T$ on K_g.

Definition

A *path cocycle* is a cocycle $c_g : \mathscr{A} \to K_g$ such that $T \circ c_g = c_0$.

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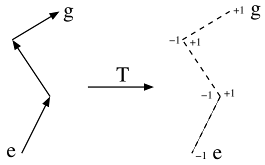
Example: in the Cayley tree of F_n, denote by $c_g(g) \in K_g$ the sum of the antisymmetric edges on the path from the origin to g.



Some general results

We consider a free product of A_o's and A_u's with $n \geq 3$.
We denote $\mathscr{K}_g'$ the orthogonal projection of $\mathscr{A} \otimes \mathscr{A}$ onto K_g.

Proposition

If T is injective on $\mathscr{K}_g'$, then there exists a unique path cocycle $c_g : \mathscr{A} \to \mathscr{K}_g'$.

In the case of F_n we have $\mathscr{K}_g' = K_g \cap (\mathscr{A} \otimes \mathscr{A})$ and T is injective only on this dense subspace. On the "purely quantum part" of our quantum trees we have the much stronger property:

Theorem

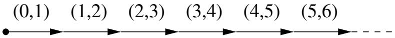
T is injective with closed range on $(1 - Q_0)K_g$.

The orthogonal case

Proposition

In the case of $A_o(Q)$, with $Q \in GL(n,\mathbb{C})$, $Q\bar{Q} \in \mathbb{C}I_n$, $n \geq 3$, the target operator $T : K_g \to H$ is invertible. As a result there exists a unique path cocycle $c_g : \mathscr{A} \to K_g$, and it is trivial.

The main reason is that $q_0 K_g = Q_0 K_g$ comes from the half-line:



We can even compute the fixed vector $\xi_g = T^{-1}\xi_0$ for c_g:

$$\xi_g = \sum_{n\geq 0} \frac{\xi_{(\alpha_n,\alpha_{n+1})} - \xi_{(\alpha_{n+1},\alpha_n)}}{\sqrt{\dim_q \alpha_n \dim_q \alpha_{n+1} \dim_q \alpha_1}}.$$

By property RD it lies in $M \otimes H_1$. $\implies \beta_1^{(2)}(\mathscr{A}_o(I_n)) = 0$ ■

The unitary case

The quasiclassical subgraph is a union of trees $\Rightarrow$ T injective on $\mathscr{K}_g'$. Hence we have a unique path cocycle $c_g : \mathscr{A} \to \mathscr{K}_g'$.

Let $\gamma \in M_n \otimes \mathscr{A}$ be the fundamental corepresentation of $\mathscr{A}_u(I_n)$. We consider $\alpha_k = \gamma^k$ and $\beta_k = \gamma\bar{\gamma}\gamma\cdots$.

Proposition

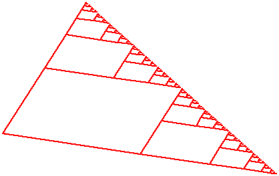
We have $\|(\mathrm{id}\otimes c_g)(\alpha_k)\| \geq C\sqrt{k}$ and $\|(\mathrm{id}\otimes c_g)(\beta_k)\| \leq D$ for all k and constants C, $D > 0$.

As a result c_g is neither trivial (bounded) nor proper.

$$\mathscr{A}_u(I_n) \text{ non-amenable} \implies \beta_1^{(2)}(\mathscr{A}_u(I_n)) \neq 0 \quad \blacksquare$$

The unitary case

Heuristically, the Proposition holds because there is no multiplicity above the zigzag path (β_k), and a lot of multiplicity above the straight line (α_k):



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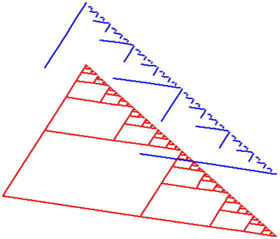

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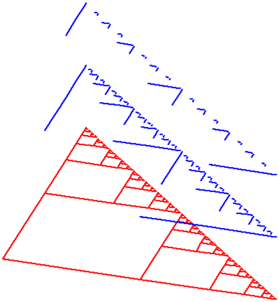

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