Abstracts

The Property of Rapid Decay for Free Quantum Groups

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The Property of Rapid Decay (RD) was first considered by Haagerup in a famous article about the convolution algebras of the free groups [2]. A general theory was then developped and studied by Jolissaint, with applications to K-theory. We report here on the extension of this theory to discrete quantum groups : definition, examples, applications. Proofs and details can be found in [5].

Let us denote by (S, δ) the Hopf C^*-algebra of c_0-functions on a discrete quantum group [4], and by h_L, h_R its Haar weights. As a C^*-algebra, S is a direct sum $\bigoplus_{\alpha\in\mathcal{I}} B(H_\alpha)$ of matrix algebras, and we denote by $p_\alpha \in S$ the minimal central projection associated to α. We moreover identify the index set $\mathcal{I}$ with the set of classes of irreducible representations of S. It is equipped with a tensor product, a conjugation and a unit object, which are respectively induced by the coproduct δ, the antipode κ and the co-unit ε.

On the other hand we denote by $(\hat{S}_{\rm red}, \hat{\delta})$ the reduced dual Hopf C^*-algebra of (S, δ), and by $\hat{h}$ its Haar state. The unital C^*-algebra $\hat{S}_{\rm red}$ is the object of interest, and Property RD is a tool to study it. We will make use of a densely defined Fourier transform $\mathcal{F} : S \supset \mathcal{S} \to \hat{S}_{\rm red}$ which induces an identification between the GNS spaces relative to h_R and $\hat{h}$.

1. Extension of the Definition

A length on (S, δ) is an unbounded multiplier $L \in S^\eta$ such that

$$L \geq 0,\ \varepsilon(L) = 0,\ \kappa(L) = L \text{ and } \delta(L) \leq L \otimes 1 + 1 \otimes L.$$

We denote by p_n the spectral projection of L associated with $[n, n+1[$. Interesting examples of lengths are the word lengths: assume that (S, δ) is finitely generated — ie $\mathcal{I}$ is generated by a finite subset $\mathcal{D} = \bar{\mathcal{D}}$ —, we put then $L = \sum l(\alpha)p_\alpha$ with

$$l(\alpha) = \min\{k \mid \exists\ \beta_1, \dots, \beta_k \in \mathcal{D}\ \alpha \subset \beta_1 \otimes \cdots \otimes \beta_k\}.$$

We define the Sobolev norms of an element $a \in \mathcal{S}$ by the standard formulas $||a||_2 := \hat{h}_R(a^*a)$ and $||a||_{2,s} = ||(1 + L)^s a||_2$. We denote by $H \supset H_s$ the respectively associated completions, and we put $H_\infty = \bigcap_{s\geq 0} H_s$. Since H identifies via the Fourier transform with the GNS space of $\hat{h}$, the space $\hat{S}_{\rm red}$ can also be considered a subspace of H.

We are now ready to give the following Definition which, like in the classical case, is about controlling the norm of $\hat{S}_{\rm red}$ by the Sobolev norms.

Definition. *Let L be a central length on (S, δ). We say that (S, δ, L) has Property RD if the following equivalent conditions are satisfied:*

(1) $\exists C, s \in \mathbb{R}_+\ \forall a \in \mathcal{S}\ ||\mathcal{F}(a)|| \leq C||a||_{2,s}$,

(2) $H_\infty \subset \hat{S}_{red}$ *inside* H,
(3) $\exists P \in \mathbb{R}[X]\ \forall k, l, n\ \forall a \in p_n\mathcal{S}\ ||p_l\mathcal{F}(a)p_k|| \leq P(n)||a||_2$.

The centrality assumption about the length may seem too restrictive in the quantum case, and one could actually give a definition for arbitrary lengths, using for instance the first condition. However in the finitely generated case all lengths are dominated by word lengths, which are central, so that it is enough to consider central lengths in the study of Property RD.

In the case of a discrete group Γ, one recovers the classical notion of Property RD. In the (quantum) ammenable case, one can show that Property RD is still equivalent to polynomial growth, and in particular duals of connected compact Lie groups G always have Property RD. In this case we have $\hat{S}_{\mathrm{red}} = C(G)$ by definition and the embedding $H_\infty \subset \hat{S}_{\mathrm{red}}$ corresponds to the inclusion $C^\infty(G) \subset C(G)$.

2. The Free Quantum Groups

Apart from discrete groups and duals of compact groups, the first test examples for a quantum Property RD should be the free quantum groups introduced by Wang [6], which are quantum analogues of the free groups.

We start by presenting a necessary condition which proves to be usefull in that context. Replacing the "projections onto the spheres" in condition (3) by "projections onto the points of the spheres" and restricting to multiplicity-free cases we obtain the following "local version" of Property RD: there exists a polynomial $P \in \mathbb{R}[X]$ such that, for any multiplicity-free inclusion $\gamma \subset \beta \otimes \alpha$ of elements of $\mathcal{I}$

$$\forall a \in p_\alpha S\ ||p_\gamma \mathcal{F}(a) p_\beta|| \leq P(|\alpha|)||a||_2,$$

where $|\alpha|$ is the positive number such that $Lp_\alpha = |\alpha|p_\alpha$.

The interesting point about this condition is that it can be reformulated in a way that makes no reference anymore to the norm of $\hat{S}_{\mathrm{red}}$:

$$\forall a \in p_\alpha S,\ b \in p_\beta S\ ||\delta(p_\gamma)(b \otimes a)\delta(p_\gamma)||_2 \leq \sqrt{\frac{m_\gamma}{m_\beta m_\alpha}} P(|\alpha|)||b \otimes a||_2.$$

This inequality of Hilbert-Schmidt norms of matrices over $H_\beta \otimes H_\alpha$ is in fact an assertion about the relative positions in $H_\beta \otimes H_\alpha$ of the subspace equivalent to H_γ and of the cone of decomposable tensor products. Note that it is trivially verified in the case of discrete groups, since all spaces H_α are then 1-dimensional.

Using this necessary condition with the inclusions $\varepsilon \subset \bar{\alpha} \otimes \alpha$, we see that our theory is a unimodular one, although this is not apparent in the definition:

Proposition. *Non-unimodular discr. quantum groups cannot have Property RD.*

On the other hand, our necessary condition happens to be sufficient in the case of free quantum groups. In the orthogonal case this is trivial since the spheres in $\mathcal{I}$ are singletons, whereas in the unitary case this is an adaptation of the proof of Haagerup for free groups, using the freeness properties of $(\mathcal{I}, \otimes)$. By a finer study of the geometry of the fusion rules of free quantum groups, one can investigate this condition and we have finally the following quantum analogue of Haagerup's founding result:

Theorem. *The orthogonal and unitary free quantum groups have Property RD* **iff** *they are unimodular.*

3. Applications

The applications to K-theory are the first ones that come to mind to check wether the quantum theory is usable. They go back to Jolissaint [3] and rely, in the quantum case too, on a technical description of H_∞. More precisely, let L be a word length on a finitely generated discrete quantum group and denote by D the closed inner derivation by L on $B(H)$.

Proposition. *We have $\hat{S}_{red} \cap \mathrm{Dom}\, D^k \subset H_k$ and, if (S, δ, L) has Property RD with exponent s, $H_{k+s} \subset \hat{S}_{red} \cap \mathrm{Dom}\, D^k$ hence $H_\infty = \bigcap_k \mathrm{Dom}\, D^k \cap \hat{S}_{red}$.*

Standard general results about domains of closed derivations imply then that H_∞ is a full subalgebra of $\hat{S}_{\mathrm{red}}$, and in particular they have the same K-theory. Using the same techniques one can also generalize the result of V. Lafforgue stating that H_s is already a full subalgebra of $\hat{S}_{\mathrm{red}}$ for s big enough.

Finally, let us mention another application, which is part of a joint work with S. Vaes. Let $U \in M_N(\mathbb{C}) \otimes \hat{S}_{\mathrm{red}}$ be the fundamental corepresentation of a unimodular orthogonal free quantum group and consider the operator of "conjugation by the generators" $\Psi : \hat{S}_{\mathrm{red}} \to \hat{S}_{\mathrm{red}}$, $x \mapsto (\mathrm{Tr} \otimes \mathrm{id})(U^*(1 \otimes x)U)/N$.

Proposition. *If $N \geq 3$, there exists $\lambda < 1$ such that*

$$\forall x \in \hat{S}_{red}\ \hat{h}(x) = 0 \ \Rightarrow\ ||\Psi(x)||_2 \leq \lambda ||x||_2.$$

This technical result clearly implies that $\hat{S}''_{\mathrm{red}}$ is a full factor. In fact, combining the Proposition with Property RD one can transfer this "hilbertian simplicity" to the C^*-algebraic level and prove that $\hat{S}_{\mathrm{red}}$ is simple with a unique trace. The corresponding result in the unitary case was proved by Banica [1] using the freeness in $\mathcal{I}$, a method that cannot apply in the orthogonal case.

References

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