Def A length on (S,δ) is an operator $L \in Z(S^\circ) \subset B(H)$ st $L \geq 0$, $\varepsilon(L) = 0$, $\kappa(L) = L$ and $\delta(L) \leq 1 \otimes L + L \otimes 1$
(These operations are well-defined on S°.)

[redacted]

Notice that the (classes of) irreducible reps [redacted] of S are indexed by R, we will denote by α the irreduc. rep associated to $\alpha \in R$ whose space is H_α. We will denote by $p_\alpha = id_{H_\alpha} \in S$ the central support of α.
We denote: $\alpha \otimes \beta = (\alpha \otimes \beta) \circ \delta$, $\bar{\alpha} \simeq {}^t \circ \alpha \circ \kappa$ (non unitarily).

A length is then an operator of the form $L = \Sigma\, \ell(\alpha)\, p_\alpha$ with $\ell : R \to \mathbb{R}_+$, $\ell(\varepsilon) = 0$, $\ell(\bar{r}) = \ell(r)$ and

$$r \subset s \otimes t \Rightarrow \ell(r) \leq \ell(s) + \ell(t)$$

Ex $(S,\delta) = (C_0(\Gamma), \delta_\Gamma)$, ℓ any length function on $\Gamma = R$.
Ex (S,δ) finitely generated ie R generated (wrt $\otimes$, $\subset$) by $R_0 \subset R$ finite. Assume $\bar{R}_0 = R_0$, $\varepsilon \notin R_0$. Word length:

$$\ell(\alpha) = \min\{k \mid \exists \beta_1, \dots, \beta_k \in R_0 \; r \subset \beta_1 \otimes \dots \otimes \beta_k\}$$

In other words $\ell(\alpha) = d(\alpha, \varepsilon)$ in the classical Cayley graph associated to (S, δ, R_0) (as opposed to the quantum one).
All the word lengths are equivalent in an appropriate sense.

We will denote by p_n the spectral projection of L associated to $]n-1, n]$: $p_n = \Sigma\{p_\alpha \mid n-1 < \ell(\alpha) \leq n\}$.

Question: can it be useful to consider non-central lengths?

For $a \in \mathcal{S}$, $\hat{a} \in \hat{\mathcal{S}}$ we put

$\|a\|_2 = \|\Lambda(a)\|$, $\|a\|_{2,s} = \|(1+L)^s a\|_2$

$\|\hat{a}\|_2 = \|\hat{\Lambda}(\hat{a})\|$, $\|\hat{a}\|_{2,s} = \|(1+L)^s \hat{\Lambda}(\hat{a})\|_2$

We denote by H_L^s the completion of $\mathcal{S}$ wrt $\|\cdot\|_{2,s}$.
It is a Hilbert space called the s-Sobolev space of (S,δ,L)

Def We say that (S,δ,L) has property RD if one of the following equivalent conditions is satisfied.

i) $\exists\, C, s > 0$ $\forall a \in \mathcal{S}$ $(\|a\|_2 \leq)$ $\|\mathcal{F}(a)\| \leq \|a\|_{2,s}$

ii) $\exists\, C, s > 0$ $\forall \hat{a} \in \hat{\mathcal{S}}$ $(\|\hat{a}\|_2 \leq)$ $\|\hat{a}\| \leq \|\hat{a}\|_{2,s}$

iii) $H_L^\infty := \bigcap_{s>0} H_L^s \subset \hat{S}_2$ (continuously incl. in H)

iv) $\exists P \in \mathbb{R}[X]$ $\forall n, a \in p_n \mathcal{S}$ $\|\mathcal{F}(a)\| \leq P(n)\|a\|_2$

v) $\exists P \in \mathbb{R}[X]$ $\forall n, a \in p_n \mathcal{S}$ $\forall k, l$ $\|p_l \mathcal{F}(a) p_k\| \leq P(n)\|a\|_2$

It is about controlling the norm of the reduced C^*-algebra of our q. discrete gp (something complicated!) by L^2-norms.
Classical case: goes back to Haagerup '79 (prop RD and a-T-menability for $\mathbb{F}_n$). Defined and studied in the general case by Jolissaint ('86, '90). Other examples: cocompact lattices in $SL_3(F)$ (F l.c., non discrete field), hyperbolic groups. Counter-example: $SL_3(\mathbb{Z})$; $\mathbb{Z}$ with the length $l(a) = \log|a|$.

One application:

Thm Assume that (S,δ,L) has prop RD and is unimodular (?). Then H_L^∞ is a dense subalgebra of $\hat{S}_2$ and the inclusion $H_L^\infty \hookrightarrow \hat{S}_2$ induces isomorphisms in K-theory.
NB: the hypothesis of unimodularity is perhaps unnecessary.

Necessary or sufficient conditions: growth.

Def We say that (S,δ,L) has polynomial growth if
$\exists P \in \mathbb{R}[X]$ $\forall n \in \mathbb{N}$ $h_R(p_n) \leq P(n)$.

Prop If (S,δ) is amenable and (S,δ,L) has RD, then (S,δ,L) has polynomial growth.

Prop If (S,δ,L) has polynomial growth and is unimodular, (S,δ,L) has property RD. (essential ??)

NB: One has $h_R(p_n) = \sum\{m_\alpha^2 \mid n-1 < \ell(\alpha) \leq n\}$. Denote by $s_n = \#\{\alpha \in R \mid n-1 < \ell(\alpha) \leq n\}$ and
$M_n = \max\{m_\alpha \mid n-1 < \ell(\alpha) \leq n\}$.
Then (S,δ,L) has polynomial growth $\Leftrightarrow$ (s_n) and (M_n) have polynomial growth. Recall that $M_n = 1$ in the discrete case (whereas $m_\alpha = \dim \alpha$ in the compact case).

Examples 1: duals of compact groups and their deformations. We denote by $\hat{G} = (S,\delta)$ the DQG associated to a compact group G.
$\hat{G}$ is amenable ($\hat{\varepsilon}: f \mapsto f(e)$ co-unit on $\hat{S}_{r}$) and unimodular ($\kappa^2 = 1 \Leftrightarrow h_R = h_L \Leftrightarrow h$ tracial), hence $\hat{G}$ has RD iff it has polynomial growth iff (s_n) and (M_n) have polynomial growth.
On the other hand, by definition $\hat{G}$ has RD iff $H_L^\infty(\hat{G}) \subset C(G)$. Morally, $H_L^\infty(\hat{G})$ is a space of smooth functions (function with rapidly decreasing Fourier coefficients, cf the case $G = \mathbb{T}^n$) so this inclusion seems very natural...

Case of $G = SU(2)$.

Let us denote by α_k the k^{th} irreducible representation, one has $\bar{\alpha}_k \simeq \alpha_k$, $\alpha_0 = 1$, α_1 is the fundamental representation, and $\alpha_k \otimes \alpha_1 \simeq \alpha_{k-1} \oplus \alpha_{k+1}$. As a result, if we choose α_1 as generator of R, $l(\alpha_k) = k$. In particular $\forall n\ s_n = 1$.

On the other hand $2M_k = \dim \alpha_k \otimes \alpha_1 = \dim \alpha_{k-1} \oplus \dim \alpha_{k+1} = M_{k-1} + M_{k+1}$ "hence" $\forall k\ M_k = k+1$. So $\widehat{SU(2)}$ has polynomial growth and prop RD.

Now look at Woronowicz' deformations $G_q = SU_q(2)$: $\widehat{G_q}$ is still amenable, but not unimodular ($q \in]0,1[$). Moreover the semi-ring of reps of G_q is the same as the one of G, so that $\forall n\ s_n = 1$ and $M_1 M_n = M_{n-1} + M_{n+1}$. But this time $M_1 = q + q^{-1} > 2$ hence (M_n) grows geometrically so that $\widehat{SU_q(2)}$ doesn't have property RD.

This can be generalized to the case of $SU(N)$ and its quantum deformations (and probably to the other simple compact Lie groups):

<u>Prop</u> For any $N \geq 2$, $\widehat{SU(N)}$ has property RD and $\widehat{SU_q(N)}$ doesn't have property RD ($q \in]0,1[$).

This relies on the knowledge of the theory of representations of $SU(N)$ and easy combinatorial considerations about Young diagrams.

Examples 2: free quantum groups.

We have seen examples of "real" DQG (neither discrete nor compact) that don't have RD. Now:

<u>Thm</u> For any $N \geq 2$, the orthogonal free quantum groups $\widehat{A_o(I_N)}$ have property RD.

NB: It seems that the $\widehat{A_o(Q)}$ with $Q^*Q \notin \mathbb{C}I_N$ (and $Q\bar{Q} \in \mathbb{C}I_N$ as usual) don't have RD. It seems that these results are also valid for A_u.

For $N=2$, one recovers the case of $\widehat{SU(2)}$ and $\widehat{SU_q(2)}$. In fact, $A_o(Q) = (\hat{S}, \hat{\delta})$ is classically defined by generators and relations, like the C^*-algebra of the free group, but there is also a "universal property": any compact quantum group that has the same semi-ring of representations as $SU(2)$ is isomorphic to one $A_o(Q)$. In particular $A_o(Q)$ has also only one irreducible repr α_k with $l(\alpha_k) = k$ (so that $\forall u$ $\Delta_u = 1$), and one still has $M_u M_1 = M_{u+1} + M_{u-1}$, with M_1 determined by Q. This implies that $\widehat{A_o(Q)}$ has polynomial growth for $N \geq 3$. However for $N \geq 3$ $A_o(Q)$ is not amenable.

In fact the proof of the Thm relies on the technical study of version v) of the Definition, ie of $\|p_\ell F(a) p_k\|$ for $a \in p_u S$. As usual this amounts to questions concerning the "fine" repr theory of $A_o(Q)$ (relative positions of decomposable tensors in $\alpha_u \otimes \alpha_k$ and of the subspace $\alpha_\ell \subset \alpha_u \otimes \alpha_k$ ...).

Question: is non-unimodularity an obstruction to property RD? ($A_o(Q)$ is unimod. iff $Q^*Q \in \mathbb{C}I_N$)

Some proofs...

Def. $n \Rightarrow o$

We write $k \overset{n}{\sim} l$ if $\delta(p_n)(p_k \otimes p_l) \neq 0$. One knows from the theory of SQG that $k \overset{n}{\sim} l \Leftrightarrow l \overset{n}{\sim} k \Leftrightarrow k \overset{l}{\sim} n$ and $p_l \mathcal{F}(a) p_k \neq 0$ with $a \in p_n S \Rightarrow k \overset{n}{\sim} l$.

Moreover we have $(i-1)p_i \leq L p_i \leq i p_i$ hence

$$\delta(L)\delta(p_n)(p_k \otimes p_l) \geq (n-1)\,\delta(p_n)(p_k \otimes p_l) \text{ and}$$
$$\delta(L)\delta(p_n)(p_k \otimes p_l) \leq (L\otimes 1 + 1 \otimes L)\,\delta(p_n)(p_k \otimes p_l)$$
$$\leq (k+l)\,\delta(p_n)(p_k \otimes p_l)$$

Hence if $k \overset{n}{\sim} l$ we have $n-1 \leq k+l$. Applying this to $k \overset{l}{\sim} n$ and $l \overset{k}{\sim} n$ we get

$$|k-l|-1 \leq n \leq k+l+1$$

In particular $\#\{n \mid k \overset{n}{\sim} l\} \leq k+l+1-(|k-l|-1)+1$
$$\leq 2\min(k,l)+3$$

Now, take $a \in p_n S$ and $\xi \in H$. We have

$$\|\mathcal{F}(a)\xi\|^2 = \Sigma_l \|p_l \mathcal{F}(a)\xi\|^2 \leq \Sigma_l \left(\Sigma_k \|p_l \mathcal{F}(a) p_k \xi\|\right)^2$$
$$\leq P(n)^2 \|a\|_2^2\, \Sigma_l \left(\Sigma_{k \overset{n}{\sim} l} \|p_k \xi\|\right)^2$$

Finally, by Cauchy-Schwarz

$$\Sigma_l \left(\Sigma_{k \overset{n}{\sim} l} \|p_k \xi\|\right)^2 \leq \Sigma_l \left[(2n+3)\,\Sigma_{k \overset{n}{\sim} l} \|p_k \xi\|^2\right]$$
$$\leq (2n+3)\,\Sigma_k \Sigma_{l \overset{n}{\sim} k} \|p_k \xi\|^2$$
$$\leq (2n+3)^2\, \Sigma_k \|p_k \xi\|^2 = (2n+3)^2 \|\xi\|^2$$

Hence $\|\mathcal{F}(a)\| \leq (2n+3)\,P(n)\,\|a\|_2$. CQFD

Thm on K-theory

Put $D_L(\hat{a}) = [L, \hat{a}]$ unbounded, densely defined on H, for $\hat{a} \in \hat{S}_r$. Like in the classical case, the idea is to prove that $H_L^\infty = \bigcap_k \mathrm{Dom}\, D_L^k$ (H_L^∞ and $\hat{S}_r$ are embedded in H). Then results on closed derivations automatically prove that H_L^∞ (which is clearly dense) is a subalgebra of $\hat{S}_r$ which is stable under holomorphical functional calculus in $\hat{S}_r$. This implies that the inclusion induces isomorphisms in K-theory.

One only uses prop RD to prove $H_L^\infty = \bigcap_k \mathrm{Dom}\, \Delta_L^k$, but this is much more involved than in the classical case...

<u>amen. + RD $\Rightarrow$ poly. growth</u>

Amenability means that the co-unit $\hat{\varepsilon} : \hat{S} \to \mathbb{C}$ extends to $\hat{S}_r$ or, equivalently, that there exists a character $\hat{\varepsilon}$ on $\hat{S}_r$ such that $(\hat{\varepsilon} \otimes id)(V) = id_H$. In particular $\hat{\varepsilon} \circ \mathcal{F} = h_R$.
If prop RD is verified one gets $\forall a \in S$

$$|h_R(a)| = |\hat{\varepsilon}\mathcal{F}(a)| \le \|\mathcal{F}(a)\| \le C\|a\|_{2,s}$$

$$\Rightarrow h_R(p_n) \le C\|(1+L)^s p_n\| \le C(1+n)^s \left(h_R(p_n^* p_n)\right)^{1/2}$$

$$\Rightarrow h_R(p_n) \le C^2(1+n)^{2s}.$$