KK$_S$

Hopf C^*-algebras

$\delta : S \to M(S\otimes S)$ non degenerate s.t. $\delta(S)(1\otimes S)$, $\delta(S)(S\otimes 1) \subset S\otimes S$, coassociative: $(\delta\otimes id)\delta = (id\otimes\delta)\delta$

Example: $S = C_0(G)$, $\delta(f)(r,s) = f(rs)$. Note that $S\otimes S = C_0(G\times G)$, $M(S\otimes S) = C_b(G\times G)$, and the condition $\delta(f)(S\otimes 1) \subset S\otimes S$ means that $\delta(f)(r,s)$ tends to 0 "when r is fixed and s goes to infinity".

Coactions

$\delta_A : A \to M(A\otimes S)$ non degenerate, $\delta(A)(1\otimes S) \subset A\otimes S$ and $(\delta_A\otimes id)\delta_A = (id\otimes\delta)\delta_A$.

Example: $S = C_0(G)$, G acting on A. Then $A\otimes S = C_0(G,A)$ and $\{x\in M(A\otimes S) / x(1\otimes S) \subset A\otimes S\} = C_b(G,A)$. $\delta_A(a)(g) = g\cdot a$.

A is a S-algebra if δ_A is injective and $\delta_A(A)(1\otimes S)$ is dense in $A\otimes S$ (automatic in the example).

On Hilbert B-modules: $\delta_E : E \to M(E\otimes S) = \mathcal{L}_{B\otimes S}(B\otimes S, E\otimes S)$ compatible with δ_B, δ, E... Then one gets a coaction $\delta_{K(E)}$ on $K_B(E)$.

Covariant representation: $\pi : A \to \mathcal{L}_B(E)$ s.t. $\delta_E(\pi(a)\xi) = (\pi\otimes id)\delta_A(a)\cdot\delta_E(\xi)$.

S-equivariant KK-theory:

A, B C^*-algebras endowed with coactions of a Hopf C^*-algebra (S,δ). $E_S(A,B)$: triples (E,π,F) s.t.

- E Hilbert B-module, countably generated, $\mathbb{Z}/2\mathbb{Z}$-graded, with an S-coaction of degree 0
- $\pi : A \to \mathcal{L}_B(E)^{(0)}$ covariant, $F \in \mathcal{L}_B(E)^{(1)}$
- $[\pi(A), F] + \pi(A)(F^2 - 1) + \pi(A)(F - F^*) \subset \mathcal{K}_B(E)$
- $(\pi(A) \otimes S)(F \otimes 1 - \delta_{\mathcal{K}(E)}(F)) \subset \mathcal{K}_{B \otimes S}(E \otimes S)$

Homotopy relation induced by $E_S(A, B([0,1]))$ $\rightsquigarrow$ $KK_S(A,B)$. The basic functorial properties of equivariant KK-theory hold, especially the Kasparov product.

|Quantum gps |Crossed Products $\longrightarrow$ slide

In the compact case the sub-C^*-algebra $A \rtimes_{red} \hat{S}$ can be characterized in the following way:

$$A \rtimes_{red} \hat{S} = \mathcal{K}_A(A \otimes H)^{S_{red}}$$

for a non trivial coaction of S_{red} on $A \otimes H$ (coming from the coaction on A and the right regular coaction on H).

With $F = 0$, we get $\beta \in KK_{S_{red}}((A \rtimes_{red} \hat{S})_1, A)$ where $(A \rtimes_{red} \hat{S})_1$ is the crossed product C^*-algebra endowed with the trivial coaction.

|Descent Morphisms |Green-Julg Theorem

|K-amenability $\longrightarrow$ slides

Julg-Valette Operator

Example on a particular tree: we choose an origin and use the orientation consisting in the edges going away from the origin, called the ascending orientation. Julg-Valette map: associates to each edge its extremity which is the furthest from the origin.
At the level of the ℓ^2-spaces one gets the Julg-Valette operator, which they used to prove the K-amenability of groups acting on trees with amenable stabilizers, generalizing a result of Cuntz on free groups.

Graphs

Oriented graphs

$\Theta \circlearrowright E \underset{s}{\overset{t}{\rightrightarrows}} V$, $\Theta^2 = 1$, no fixed point, $t\Theta = s$

Geometrical edges: $E_g = E/x \sim \Theta(x)$

Orientation: $E \supset E_+ \xrightarrow{\sim} E_g$.

Action of G: on E and V, commuting to Θ, s, t.

Typically we will write E_0 for a G-invariant orientation, and E_+ for an ascending orienta°

Hilbertian side

$\Theta \circlearrowright K \underset{S}{\overset{T}{\rightrightarrows}} H$, $K_g = \mathrm{Ker}(\Theta - \mathrm{id}) \subset K$

Representations of G on H and K

Orientation $\to K_+ \subset K$, $P_+ : K \xrightarrow{\perp} K_+$.

If K_+ is associated to the ascending orientation of a pointed tree, the Julg-Valette operator is $F = T \circ p_+ : K_g \to H$. It defines an element $\gamma \in KK_G(\mathbb{C}, \mathbb{C})$.

Remark. The C^*-algebra $C_0(V)$ acts on H by multiplication. Consider in particular the characteristic function p_n of the set of vertices at distance n to the origin: this gives a decomp. $H = \oplus p_n H$ which reflects the notion of "distance to the origin" at the Hilbertian level, and from which it is easy to recover the Julg-Valette operator.

Amalgamated free products

Classical Serre Tree

$G = G_1 *_H G_2 \to V = G/G_1 \sqcup G/G_2$, $E_g = E_0 = G/H$

$t : gH \to gG_1$, $s : gH \to gG_2$

G acting by left translation

Quantum Serre Tree, K-amenability → slide

Cayley Graph

Classical Cayley Graph

$1 \notin \Delta = \Delta^{-1} \subset \Gamma$ finite $E = \Gamma \times \Delta$, $V = \Gamma$

$t(g,h) = gh$ $s(g,h) = g$ $\theta(g,h) = (gh, h^{-1})$

$H = \ell^2(\Gamma)$, $K = H \otimes p_1 H$, p_1 a central projec° in $\hat{S}$

Quantum Cayley Graph, γ-element of $A_o(Q)$ → sl