MAXIMAL AMENABILITY OF THE RADIAL SUBALGEBRA IN FREE QUANTUM GROUP FACTORS

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Abstract. We show that the radial MASA in the orthogonal free quantum group algebra $\mathcal{L}(\mathbb{F}O_N)$ is maximal amenable if N is large enough, using the Asymptotic Orthogonality Property. This relies on a detailed study of the corresponding bimodule, for which we construct in particular a quantum analogue of Rădulescu's basis. As a byproduct we also obtain the value of the Pukánszky invariant for this MASA.

Contents

Introduction

The orthogonal free quantum groups $\mathbb{F}O_N$, for $N \in \mathbb{N}^*$, are discrete quantum groups which where introduced by Wang [Wan95] via their universal C^*-algebra defined by generators and relations:

$$C_u^*(\mathbb{F}O_N) = A_o(N) = C^*(u_{i,j}, 1 \le i,j \le N \mid u = \bar{u}, uu^* = u^*u = 1).$$

Here $u = (u_{i,j})_{i,j}$ is the matrix of generators, u^* is the usual adjoint in $M_N(C_u^*(\mathbb{F}O_N))$, and $\bar{u} = (u_{i,j}^*)_{i,j}$. There is a natural coproduct $\Delta : C_u^*(\mathbb{F}O_N) \to C_u^*(\mathbb{F}O_N) \otimes C_u^*(\mathbb{F}O_N)$ which encodes the quantum group structure, and which turns $C_u^*(\mathbb{F}O_N)$ into a Woronowicz C^*-algebra [Wor98]. In particular $C_u^*(\mathbb{F}O_N)$ is equipped with a canonical Δ-invariant tracial state h. In this article we are interested in the von Neumann algebra $\mathcal{L}(\mathbb{F}O_N) = \lambda(C_u^*(\mathbb{F}O_N))'' \subset B(H)$ generated by the image of $C_u^*(\mathbb{F}O_N)$ in the GNS representation λ associated with h. We still denote $u_{i,j} \in \mathcal{L}(\mathbb{F}O_N)$ the images of the generators.

The von Neumann algebras $\mathcal{L}(\mathbb{F}O_N)$, and their unitary variants $\mathcal{L}(\mathbb{F}U_N)$, can be seen as quantum, or matricial, analogues of the free group factors $\mathcal{L}(F_N)$. More precisely if we denote $FO_N = (\mathbb{Z}/2)^{*N}$, with canonical generators a_i, $1 \le i \le N$, we have a surjective $*$-homomorphism $\pi : C_u^*(\mathbb{F}O_N) \to C_u^*(FO_N)$, $u_{i,j} \mapsto \delta_{i,j} a_i$ compatible with coproducts. It turns out that this analogy is fruitful also at an analytical level: one can show that $\mathcal{L}(\mathbb{F}O_N)$ shares many properties with $\mathcal{L}(FO_N)$ and $\mathcal{L}(F_N)$, although the existence of π, which has a huge kernel, is useless to prove such properties. For instance, $\mathcal{L}(\mathbb{F}O_N)$ is non amenable for $N \ge 3$ [Ban97], and in fact it is a full and prime II_1 factor [VV07] without Cartan subalgebras [Iso15]. On the other hand it is not isomorphic to a free group factor [BV18].

The II_1 factor $M = \mathcal{L}(\mathbb{F}O_N)$ has a natural "radial" abelian subalgebra, $A = \chi_1'' \cap M$ where $\chi_1 = \chi_1^* = \sum_1^N u_{i,i}$ is the sum of the diagonal generators. It was shown, already in [Ban97],

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that $\chi_1/2$ is a semicircular variable with respect to h, in particular $\|\chi_1\| = 2$ in $\mathcal{L}(\mathbb{F}O_N)$. Since $\epsilon(\chi_1) = N$ in $C^*_u(\mathbb{F}O_N)$, this implies the non-amenability of $\mathbb{F}O_N$ for $N \geq 3$. The subalgebra $A \subset M$ is the quantum analogue of the radial subalgebra of $\mathcal{L}(F_N)$, generated by the sum $\chi_1 = \sum_1^N (a_i + a_i^*)$ of the generators $a_i \in F_N$ and their adjoints, which is known to be a maximal abelian subalgebra (MASA) since [Pyt81].

The position of A in M was already investigated in [FV16], where it was shown, for $N \geq 3$, to be a strongly mixing MASA. Note that $\mathbb{F}O_N$ admits deformations $\mathbb{F}O_Q$, where $Q \in M_N(\mathbb{C})$ is an invertible matrix such that $Q\bar{Q} = \pm I_N$. When Q is not unitary, the corresponding von Neumann algebra $M = \mathcal{L}(\mathbb{F}O_Q)$ is a type III factor, at least for small deformations [VV07]. One can still consider the abelian subalgebra $A = \chi_1'' \cap M$, but if Q is not unitary it is not maximal abelian anymore, as shown in [KW22]. More precisely, in this case the inclusion $A \subset M$ is quasi-split in the sense of [DL84].

The aim of the present article is to pursue the study of [FV16] in the non-deformed case. Our main result is the following theorem, proved in Section 6. Here, and in the rest of the article, we fix a free ultrafilter ω on $\mathbb{N}$, but the result also holds for the Fréchet filter $\omega = \infty$.

Theorem A. *There exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ the radial subalgebra $A \subset M = \mathcal{L}(\mathbb{F}O_N)$ satisfies the strong Asymptotic Orthogonality Property: if $B \subset A$ is a diffuse subalgebra, for every $y \in A^\perp \cap M$ and for every bounded sequence of elements $z_r \in A^\perp \cap M$ such that $\|[b, z_r]\|_2 \to_\omega 0$ for all $b \in B$, we have $(yz_r \mid z_r y) \to_\omega 0$.*

The Asymptotic Orthogonality Property (AOP) originates from Popa's seminal article [Pop83] where it was established for $A = a_1'' \subset \mathcal{L}(F_N)$, the generator MASA in free group factors, and proved to imply maximal amenability. Houdayer later showed that the strong AOP as above implies absorbing amenability, so that we can formulate the following corollary, which is a quantum analogue of the results of [Wen16] and [CFRW10] about the radial MASA in free group factors.

Corollary B. *There exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ the radial subalgebra $A \subset M = \mathcal{L}(\mathbb{F}O_N)$ has the absorbing amenability property: for any amenable subalgebra $P \subset M$ such that $P \cap A$ is diffuse, we have $P \subset A$. In particular A is maximal amenable.*

Proof. This results from [Hou14]. Indeed, denote $B = P \cap A$, which is amenable (in fact abelian). The inclusion $A \subset M$ is weakly mixing through B because B is diffuse and $A \subset M$ is strongly mixing [FV16, Theorem 5.7]. It has the AOP relative to B because according to Theorem A it has the strong AOP and B is diffuse. Then [Hou14, Theorem 8.1] shows that $P \subset A$. The result also follows from [Wen16, Proposition 1] since M is known to be strongly solid, cf [Iso15] and [FV15, Theorem 4.11]. Since A is diffuse, this applies in particular if $A \subset P \subset M$ with P amenable, so that A is maximal amenable. □

The proof of Theorem A follows a strategy which can also be traced back to Popa's work on the generator MASA of free group factors. One can identify the following ingredients:

(1) a good description of the A, A-bimodule $H = \ell^2(F_N)$;
(2) a decreasing sequence of subspaces $V_m \subset H$ such that, for $y \in A^\perp \cap M$ fixed and m big enough, $yV_m \perp V_m y$;
(3) a compactness property, with respect to the left or right actions of A, for two projections F_m, F'_m whose ranges span $V_m^\perp$.

Let us now explain how these ingredients are obtained in the classical and quantum cases, and the organization of the article, which starts with the preliminary Section 1.

The more precise goal for (1) is to exhibit an orthonormal basis W of the A, A-bimodule $A^\perp \cap H$ with good combinatorial properties, which will allow to carry out computations. In the case of the generator MASA in $\mathcal{L}(F_N)$, this basis is just given by the set of reduced words in F_N which do not start nor end with a_1 nor a_1^{-1}. In the case of the radial MASA in $\mathcal{L}(F_N)$, a convenient basis was constructed by Rădulescu [Răd91] to show that the radial MASA is singular with Pukánszky invariant $\{\infty\}$.

In Section 2 we construct an analogue $W = \bigsqcup_{k\geq 1} W_k$ of the Rădulescu basis for our free quantum groups. Surprisingly one has to take into account additional symmetries of H given by the rotation maps ρ_k which already played a (minor) role in [FV16]. Using this construction and a result from [FV16], we can already deduce (Corollary 2.13) that the Pukánszky invariant of the radial MASA in $\mathcal{L}(\mathbb{F}O_N)$ is $\{\infty\}$, a result that was missing in [FV16].

From $x \in W$ one can generate a natural $\mathbb{C}$-linear basis $(x_{i,j})_{i,j\in\mathbb{N}}$ of the cyclic submodule AxA. In Rădulescu's case, $(x_{i,j})$ is orthogonal as soon as $x \in W_k$ with $k \geq 2$, and for $k = 1$ it is nevertheless a Riesz basis. In our case, $(x_{i,j})$ is never orthogonal and we have to show that it is a Riesz basis, uniformly over $x \in W$. This is accomplished in Section 3, which is the most challenging technically, and we manage to reach this conclusion only if N is large enough.

The core of the strategy lies in ingredient (2). In the case of the generator MASA in $\mathcal{L}(\mathbb{F}O_N)$, V_m is simply the subspace of H generated by the reduced words of F_N that begin and end with a "large" power a_1^k of the generator, $|k| \geq m$, without being themselves a power of a_1. We have then clearly $V_m y \perp y V_m$ if $y \in A^\perp \cap M$ is supported on reduced words of length at most m.

In the case of the radial MASA in $\mathcal{L}(\mathbb{F}O_N)$, V_m is defined in terms of the Rădulescu basis as the subspace generated by the elements $x_{i,j}$, $x \in W$, i, $j \geq m$. We adopt the same definition in the quantum case, using our analogue of the Rădulescu basis, and we show in Section 4 that the orthogonality property $V_m y \perp y V_m$ holds in an approximate sense as $m \to \infty$. Note that we use one of the two main technical tools from [FV16], in an improved version (Lemma 1.6).

Given the way we defined the subspaces V_m, it is natural for step (3) to consider the orthogonal projection F_m (resp. F'_m) onto the subspace generated by the vectors $x_{i,j}$ with $i < m$ (resp. $j < m$). In the case of the generator MASA in $\mathcal{L}(F_N)$, we have a natural isomorphism of A, A-bimodules $A^\perp \cap H \simeq L^2(A) \otimes \ell^2(W) \otimes L^2(A)$, and in this identification the projection F_m can be written $f_m \otimes 1 \otimes 1$, where $f_m \in B(L^2(A))$ is a finite rank projection. In particular if $(u_j)_j$ is a sequence of unitaries in A which converges weakly to 0, we have $\|F_m u_j F_m\| \to_j 0$, where u_j acts from the left on $A^\perp \cap H$.

For the radial MASA in $\mathcal{L}(\mathbb{F}O_N)$, we still have an isomorphism $A^\perp \cap H \simeq L^2(A) \otimes \ell^2(W) \otimes L^2(A)$ but it is not explicit and we cannot determine how F_m acts through it. Instead we prove "by hand" in Section 5 that $\|F_m u_j F_m\| \to_j 0$. Among the technical tools that we use are classical estimates about the Jones-Wenzl projections, but also elementary arguments about Temperley-Lieb tangles which seem to be new in this context.

Finally in Section 6 we give the detailed proof of Theorem A, following the above strategy. Some care has to be taken due to the fact that the basis $(x_{i,j})$ is not orthogonal. For instance, if E_m denotes the orthogonal projection onto V_m, in fact we don't have $E_m + (F_m \vee F'_m) = 1$. We prove instead that $\|(1 - E_m)(1 - F_{2m} \vee F'_{2m})\|$ tends to 0 as $m \to \infty$, and this relies on an off-diagonal decay property for the inverse of the Gram matrix of $(x_{i,j})$, established in Section 3.

1. Preliminaries

In this article, a discrete quantum group $\mathbb{\Gamma}$ is given by a Woronowicz C^*-algebra $C^*(\mathbb{\Gamma})$ [Wor98], i.e. a unital C^*-algebra equipped with a unital $*$-homomorphism $\Delta : C^*(\mathbb{\Gamma}) \to C^*(\mathbb{\Gamma}) \otimes C^*(\mathbb{\Gamma})$ satisfying the following two axioms: i) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (co-associativity); ii) $\Delta(C^*(\mathbb{\Gamma}))(1 \otimes C^*(\mathbb{\Gamma}))$ and $\Delta(C^*(\mathbb{\Gamma}))(C^*(\mathbb{\Gamma}) \otimes 1)$ span dense subspaces of $C^*(\mathbb{\Gamma}) \otimes C^*(\mathbb{\Gamma})$ (bi-cancellation). This encompasses classical discrete groups, as well as duals of classical compacts groups G, given by $C^*(\mathbb{\Gamma}) = C(G)$.

In this setting Woronowicz proved existence and uniqueness of a bi-invariant state $h \in C^*(\mathbb{\Gamma})^*$, i.e. satisfying the relations $(h \otimes \mathrm{id})\Delta = 1h = (\mathrm{id} \otimes h)\Delta$. We can consider the GNS representation λ associated with h and we shall mainly work with the corresponding von Neumann algebra $M = \mathcal{L}(\mathbb{\Gamma}) = \lambda(C^*(\mathbb{\Gamma}))''$ represented on the Hilbert space $H = \ell^2(\mathbb{\Gamma})$. We still denote h the factorization of the invariant state to M. As the notation suggests, in the classical case $\mathcal{L}(\Gamma)$ is the usual group von Neumann algebra with its canonical trace, whereas for the dual of a compact group G we have $\mathcal{L}(\mathbb{\Gamma}) = L^\infty(G)$ with the Haar integral.

A corepresentation of $\mathbb{\Gamma}$ is an element $u \in M\bar{\otimes}B(H_u)$ such that $u_{13}u_{23} = (\Delta \otimes \mathrm{id})(u)$. We will work exclusively with unitary and finite-dimensional corepresentations. We denote $\mathrm{Hom}(u, v) \subset B(H_u, H_v)$ the space of intertwiners from u to v, i.e. maps T such that $(1 \otimes T)u = v(1 \otimes T)$. A corepresentation u is irreducible if $\mathrm{Hom}(u, u) = \mathbb{C}\mathrm{id}$; two corepresentations u, v are equivalent if $\mathrm{Hom}(u, v)$ contains a bijection. The tensor product of u and v is $u \otimes v := u_{12}v_{13}$, with $H_{u\otimes v} = H_u \otimes H_v$. We have defined in this way a tensor C^*-category denoted $\mathrm{Corep}(\mathbb{\Gamma})$ with a fiber functor to Hilbert spaces.

Let $u \in M \otimes B(H_u)$ be a corepresentation of $\mathbb{\Gamma}$. For ζ, $\xi \in H_u$ we can consider the corresponding coefficient $u_{\zeta,\xi} = (\mathrm{id} \otimes \zeta^*)u(\mathrm{id} \otimes \xi) = (\mathrm{id} \otimes \mathrm{Tr})(u(1 \otimes \xi\zeta^*)) \in M$. More generally for $X \in B(H_u)$ we denote $u(X) = (\mathrm{id} \otimes \mathrm{Tr})(u(1 \otimes X))$ — although it would perhaps be more natural to denote this element $u(\varphi)$ where $\varphi = \mathrm{Tr}(\,\cdot\, X) \in B(H_u)^*$.

In the present article we will work only with unimodular discrete quantum groups, equivalently, the canonical state h will be a trace. In this case the Peter-Weyl-Woronowicz orthogonality relations read, for u irreducible:

$$(u(X) \mid u(Y)) = (\dim u)^{-1}(X \mid Y), \tag{1.1}$$

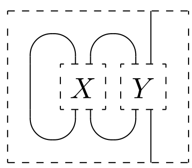
where we use on the left the scalar product associated with h, $(x \mid y) = h(x^*y)$, and on the right the Hilbert-Schmidt scalar product $(X \mid Y) = \mathrm{Tr}(X^*Y)$. On the other hand we have $(u(X) \mid v(Y)) = 0$ if u, v are irreducible and not equivalent.

The product in M can be computed according to the evident formula $u(X)v(Y) = (u \otimes v)(X \otimes Y)$. We have moreover $u(TX) = v(XT)$ for $X \in B(H_u, H_v)$ and $T \in \mathrm{Hom}(v, u)$. As a result, if we choose intertwiners $T_i \in \mathrm{Hom}(w_i, u \otimes v)$ such that $T_i^*T_i = \mathrm{id}$ and $\sum_i T_iT_i^* = \mathrm{id}$, we obtain the formula $u(X)v(Y) = \sum_i w_i(T_i^*(X \otimes Y)T_i)$, which we can use to compute the product of coefficients of irreducible corepresentations as a linear combination of coefficients of irreducible corepresentations.

In this article we consider the orthogonal free quantum groups $\mathbb{\Gamma} = \mathbb{F}O_N$ defined in the Introduction, and assume $N \geq 3$. Associated to N is the parameter $q \in]0, 1[$ such that $q+q^{-1} = N$, which plays an important role in the computations. We have $q \to 0$ as $N \to \infty$. Banica [Ban96] showed that the C^*-tensor category $\mathrm{Corep}(\mathbb{F}O_N)$ is equivalent, as an abstract tensor category, to the Temperley-Lieb category TL_δ at parameter $\delta = N$, and that $\mathbb{F}O_N$ is realized via Tannaka-Krein duality by the fiber functor $F : TL_N \to \mathrm{Hilb}$ which sends the generating object to $H_1 := \mathbb{C}^N$, with corepresentation $u = (u_{i,j})_{i,j}$ given by the canonical generators of $\mathcal{L}(\mathbb{F}O_N)$, and the generating morphism to $F(\cap) = t := \sum_i e_i \otimes e_i \in H_1 \otimes H_1$, where $(e_i)_i$ is the canonical basis of $\mathbb{C}^N$.

This means that we have a pictorial representation of elements $A \in \mathrm{Hom}(H_1^{\otimes k}, H_1^{\otimes l})$. More precisely, denote $NC_2(k, l)$ the set of non-crossing pair partitions of $k + l$ points. For each partition $\pi \in NC_2(k, l)$ there is a morphism $T_\pi \in \mathrm{Hom}(H_1^{\otimes k}, H_1^{\otimes l})$ whose matrix coefficients $(e_{i_1} \otimes \cdots \otimes e_{i_l} \mid T_\pi(e_{j_1} \otimes \cdots \otimes e_{j_k}))$ are equal to 1 if "the indices i_s, j_t agree in each block of π", and to 0 otherwise. Then, for $N \geq 3$ the maps T_π with $\pi \in NC_2(k, l)$ form a linear basis of $\mathrm{Hom}(H_1^{\otimes k}, H_1^{\otimes l})$. Elements $\pi \in NC_2(k, l)$, and the corresponding morphisms T_π, are usually depicted inside a rectangle with k numbered points on the upper edge and l numbered points on the bottom edge by drawing non-crossing strings joining the two elements in each block of π.

More generally, the collection of spaces $B(H_1^{\otimes k}, H_1^{\otimes l})$ is an (even) planar algebra, meaning that linear maps obtained by composing and tensoring given maps $X_i \in B(H_1^{\otimes k_i}, H_1^{\otimes l_i})$ with maps T_π can be represented by means of a rectangular Temperley-Lieb diagram as above with p internal boxes representing the maps X_i. For instance, if X, $Y \in B(H_1^{\otimes 2})$ we have, drawing dashed internal and external boxes, and solid Temperley-Lieb strings:

$(t^* \otimes t^* \otimes \mathrm{id})(\mathrm{id} \otimes X \otimes Y)(t \otimes t \otimes \mathrm{id}) =$  $\in B(H_1)$.

The irreducible objects of the Temperley-Lieb category, and hence the irreducible corepresentations of $\mathbb{F}O_N$, can be labeled by integers $k \in \mathbb{N}$ up to equivalence, in such a way that $u_0 = 1 \otimes \mathrm{id}_{\mathbb{C}}$ is the trivial corepresentation, $u_1 = u$ is the generating object, and the following fusion rules are satisfied:

$$u_k \otimes u_l \simeq u_{|k-l|} \oplus u_{|k-l|+2} \oplus \cdots \oplus u_{k+l}.$$

We denote H_k the Hilbert space associated with u_k and denote $d_k = \dim H_k$. We write Tr_k, tr_k the standard and normalized traces on $B(H_k)$. Note that $d_0 = 1$ and $d_1 = N$. The remaining dimensions can be computed using the fusion rules and are given by q-numbers:

$$d_k = [k+1]_q := \frac{q^{k+1} - q^{-(k+1)}}{q - q^{-1}}. \tag{1.2}$$

The irreducible characters are $\chi_k = (\mathrm{id} \otimes \mathrm{Tr}_k)(u_k) \in M$. It follows from the fusion rules and the Peter-Weyl-Woronowicz formula that they form an orthonormal basis of the $*$-subalgebra $\mathcal{A}$ generated by $\chi_1 = \sum u_{i,i}$, which is weakly-$*$ dense in $A = \chi_1''$.

According to the fusion rules, u_k appears with multiplicity 1 as a subobject of $u_1^{\otimes k}$. We agree to take for H_k the corresponding subspace of $H_1^{\otimes k}$, and we denote $P_k \in B(H_1^{\otimes k})$ the orthogonal projection onto H_k: this is the kth Jones-Wenzl projection. We have $P_k(P_a \otimes P_b) = P_k$, i.e. H_k is a subspace of $H_a \otimes H_b$, as soon as $k = a + b$. We usually write id_k for the identity map of $B(H_1^{\otimes k})$.

As another consequence of the fusion rules, there is a unique line of fixed vectors in $H_k \otimes H_k$. We already know the generator $t = t_1$ of $\mathrm{Hom}(H_0, H_1 \otimes H_1)$. This map satisfies the *conjugate equations* $(\mathrm{id}_1 \otimes t^*)(t \otimes \mathrm{id}_1) = \mathrm{id}_1 = (t^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes t)$. We slightly abuse notation by defining recursively $t_1^1 = t_1$, $t_1^k = (\mathrm{id}_1^{\otimes k-1} \otimes t_1 \otimes \mathrm{id}_1^{\otimes k-1}) t_1^{k-1} \in \mathrm{Hom}(H_0, H_1^{\otimes 2k})$, so that $\mathrm{Hom}(H_0, H_k \otimes H_k)$ is generated by $t_k := (P_k \otimes P_k)t_1^k = (\mathrm{id}_k \otimes P_k)t_1^k = (P_k \otimes \mathrm{id}_k)t_1^k$. Note that we have then $t_k^*(X \otimes \mathrm{id}_k)t_k = \mathrm{Tr}_k(X)$ for $X \in B(H_k)$, in particular $\|t_k\| = \sqrt{d_k}$.

Using the intertwiner t one can also investigate more precisely the position of H_n in $H_{n-1} \otimes H_1$, and this gives rise for instance to the Wenzl recursion relation [Wen87, Prop. 1], see also [VV07, Notation 7.7]:

$$P_n = (P_{n-1} \otimes \mathrm{id}_1) + \sum_{l=1}^{n-1} (-1)^{n-l} \frac{d_{l-1}}{d_{n-1}} \left(\mathrm{id}_1^{\otimes(l-1)} \otimes t \otimes \mathrm{id}_1^{\otimes(n-l-1)} \otimes t^* \right) (P_{n-1} \otimes \mathrm{id}_1). \tag{1.3}$$

One can go further and define the *basic intertwiner* $V_m^{k,l} = (P_k \otimes P_l)(\mathrm{id}_{k-a} \otimes t^a \otimes \mathrm{id}_{l-a})P_m$ which spans $\mathrm{Hom}(H_m, H_k \otimes H_l)$, where $m = k + l - 2a$. It is not isometric but its norm can be computed explicitly, see [Ver07, Lemma 4.8]. Following [FV16], we denote $\kappa_m^{k,l} = \|V_m^{k,l}\|^{-1}$. This yields the following explicit formula to compute the product of coefficients of irreducible corepresentations:

$$u_k(X)u_l(Y) = \sum_{a=0}^{\min(k,l)} \left(\kappa_m^{k,l}\right)^2 u_m \left(V_m^{k,l*}(X \otimes Y)V_m^{k,l} \right), \tag{1.4}$$

where we still agree to write $m = k + l - 2a$. This motivates the following notation (which is indeed connected with the convolution product in $c_c(\mathbb{F}O_N)$ up to constants).

Notation 1.1. For $X \in B(H_k)$, $Y \in B(H_l)$, $m = k + l - 2a$ we consider the following element of $B(H_m)$:

$$X *_m Y = V_m^{k,l*}(X \otimes Y)V_m^{k,l} = P_m(\mathrm{id}_{k-a} \otimes t_a^* \otimes \mathrm{id}_{l-a})(X \otimes Y)(\mathrm{id}_{k-a} \otimes t_a \otimes \mathrm{id}_{l-a})P_m.$$

One can perform analysis in the tensor category $\mathrm{Corep}(\mathbb{F}O_N)$. Recall for instance Lemma 1.3 from [VV07] below, with some more precise information about constants.

Lemma 1.2. *For any $k \in \mathbb{N}$ we have $q^{-k} \leq d_k \leq q^{-k}/(1-q^2)$.*

Proof. Clear from (1.2). □

Lemma 1.3. *Fix $q_0 \in \,]0,1[$ and assume that $q \in \,]0,q_0]$. Then there exists a constant C depending only on q_0 such that $\|(P_{a+b} \otimes \mathrm{id}_c)(\mathrm{id}_a \otimes P_{b+c}) - P_{a+b+c}\| \le Cq^b$ for all a, b, $c \in \mathbb{N}$.*

Proof. This is [VV07, Lemma 1.4], we only have to check that the constant C remains bounded as $q \to 0$. The proof of [VV07] explicitly gives the following upper bound:

$$\|(P_{a+b} \otimes \mathrm{id}_c)(\mathrm{id}_a \otimes P_{b+c}) - P_{a+b+c}\| \le q^b \Big(\prod_0^\infty (1 + Dq^k)\Big)\Big(\sum_0^\infty Cq^k\Big),$$

where C and D a priori depend on q. Let us show that one can choose C and D uniformly over $]0,q_0]$. Using Lemma 1.2 we have

$$q^{-b-c}\frac{[2]_q[a]_q}{[a+b+c+1]_q} \le q^{-b-c}\frac{q^{-1}q^{-a+1}}{q^{-a-b-c}(1-q^2)^2} \le \frac{1}{(1-q_0^2)^2}.$$

Similarly:

$$\begin{aligned} q^{-b-c}\Big|\frac{[2]_q[a+b]_q}{[a+b+c+1]_q} - \frac{[2]_q[b]_q}{[b+c+1]_q}\Big| &= q^{-b-c}\frac{[2]_q[a]_q[c]_q}{[a+b+c+1]_q[b+c+1]_q} \\ &\le q^{-b-c}\frac{q^{-1}q^{-a+1}q^{-c+1}}{q^{-a-b-c}q^{-b-c}(1-q^2)^3} \le \frac{q_0^{b+1}}{(1-q_0^2)^3} \le \frac{1}{(1-q_0^2)^3}. \end{aligned}$$

In [VV07], the only constraint on C is to be an upper bound for these two quantities, hence it can indeed be chosen to depend only on q_0. On the other hand, D should be an upper bound for

$$q^{-c}\frac{[2]_q[b]_q}{[b+c+1]_q} \le q^{-c}\frac{q^{-1}q^{-b+1}}{q^{-b-c}(1-q^2)^2} \le \frac{1}{(1-q_0^2)^2},$$

hence it can also be chosen to depend only on q_0. ☐

We also have estimates on the constants κ, already proved in [Ver07]. The formulae for $\kappa_m^{k,l}$ show that, again, the constant C is uniform for q varying in an interval $]0,q_0]$ with $q_0 < 1$, but we will not need this fact.

Lemma 1.4. *There exists a constant C, depending only on q, such that we have $1 \le \sqrt{d_a}\kappa_m^{k,l} \le C$ for all k, l and $m = k+l-2a$.*

Proof. See the proof of [Ver07, Lemma 4.8], [BVY21, p. 1583], [BC18, Equation (6) and Proposition 3.1]. ☐

The following estimate appeared also in connection with Property RD [Ver07]. Recall that $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm on matrix spaces.

Lemma 1.5. *Consider integers such that $m = k+l-2a$. Then for any $X \in B(H_1^{\otimes k})$, $Y \in B(H_1^{\otimes l})$ we have $\|(\mathrm{id}_{k-a} \otimes t_a^* \otimes \mathrm{id}_{l-a})(X \otimes Y)(\mathrm{id}_{k-a} \otimes t_a \otimes \mathrm{id}_{l-a})\|_2 \le \|X\|_2\|Y\|_2$ and $\|(\mathrm{id} \otimes \mathrm{Tr}_a)(X)\|_2 \le \sqrt{d_a}\|X\|_2$.*

Proof. The proof of [Ver07, Theorem 4.9] applies, although it was there used only for $X \in B(H_k)$, $Y \in B(H_l)$. Let us repeat it. Consider an orthonormal basis $(e_i)_i$ of H_a, then the basis $(\bar e_i)_i$ defined by putting $t_a = \sum_i e_i \otimes \bar e_i$ is orthonormal as well. Put $E_I = e_i e_j^*$ and $\bar E_I = \bar e_i \bar e_j^* \in B(H_a)$ for $I = (i,j)$, these are orthonormal bases of $B(H_a)$ for the Hilbert-Schmidt structure and we have $t_a^*(E_I \otimes \bar E_J)t_a = \delta_{I,J}$. Decompose $(\mathrm{id}_{k-a} \otimes P_a)X(\mathrm{id}_{k-a} \otimes P_a) = \sum X_I \otimes E_I$ with $X_I \in B(H_1^{\otimes k-a})$ and similarly $(P_a \otimes \mathrm{id}_{l-a})Y(P_a \otimes \mathrm{id}_{l-a}) = \sum \bar E_J \otimes Y_J$. We have then $\sum \|X_I\|_2^2 = \|(\mathrm{id}_{k-a} \otimes P_a)X(\mathrm{id}_{k-a} \otimes P_a)\|_2^2 \le \|X\|_2^2$ and similarly $\sum \|Y_J\|_2^2 \le \|Y\|_2^2$. Finally we have by the triangle inequality and Cauchy-Schwarz :

$$\begin{aligned} \|(\mathrm{id} \otimes t_a^* \otimes \mathrm{id})(X \otimes Y)(\mathrm{id} \otimes t_a \otimes \mathrm{id})\|_2^2 &= \|\textstyle\sum_{I,J} t_a^*(E_I \otimes \bar E_J)t_a \times (X_I \otimes Y_J)\|_2^2 \\ &\le \big(\textstyle\sum_I \|X_I\|_2\|Y_I\|_2\big)^2 \\ &\le (\textstyle\sum_I \|X_I\|_2^2)(\textstyle\sum_I \|Y_I\|_2^2) \le \|X\|_2^2\|Y\|_2^2. \end{aligned}$$

The second inequality of this lemma follows by taking $l = a$ and $Y = \mathrm{id}_a$, but can also be proved more directly by noticing that in the canonical isometric isomorphism $B(K \otimes L) \simeq K \otimes L \otimes \bar{L} \otimes \bar{K}$, the partial trace $\mathrm{id} \otimes \mathrm{Tr}_L$ corresponds to the map $\mathrm{id} \otimes t_L^* \otimes \mathrm{id}$, where $t_L^* : \mathbb{C} \to L \otimes \bar{L}$ is the canonical duality vector whose norm is $\sqrt{\dim L}$. □

We will use again one of the two main estimates from [FV16] about $\mathrm{Corep}(\mathbb{F}O_N)$. For a, b, $c \in \mathbb{N}$ consider $\Pi_{a,b,c} = (\mathrm{id}_a \otimes \mathrm{tr}_b \otimes \mathrm{id}_c)(P_{a+b+c}) \in B(H_a \otimes H_c)$ — this time the analysis deals with $\mathrm{Corep}(\mathbb{F}O_N)$ together with its canonical fiber functor. Proposition 3.2 of [FV16] shows that $\Pi_{a,b,c}$ is almost scalar as $b \to \infty$. We give below an improvement of the corresponding constants.

Lemma 1.6. *For every $q_0 \in]0,1[$ there exists constants $C > 0$, $\alpha \in]0,1[$ such that, for all a, b, $c \in \mathbb{N}$ and $q \in]0, q_0]$ we have $\|\Pi_{a,b,c} - \lambda(\mathrm{id}_a \otimes \mathrm{id}_c)\| \leq C q^{\lfloor \alpha b \rfloor}$ for some scalar $\lambda \in \mathbb{C}$.*

Proof. Proposition 3.2 of [FV16] uses the scalar $\lambda = \lambda_{a,c}$ explicitly given by $\lambda_{a,c} = q^{-a-c}/d_a d_c$. Consider $\Pi'_{a,b,c} = d_b \Pi_{a,b,c} - d_b \lambda_{a,c} \mathrm{id}_{a \otimes c}$. A direct computation shows that

$$\mathrm{Tr}(\Pi'_{a,b,c}) = d_{a+b+c} - q^{-a-c} d_b = q^{b+2} \frac{q^{-a-c} - q^{a+c}}{1-q^2}$$
$$\leq \frac{1}{1-q^2} \sqrt{d_{a+c} d_a d_c} = \frac{\sqrt{d_{a+c}}}{1-q^2} (\mathrm{Tr}\, \mathrm{id}_{a \otimes c})^{1/2}.$$

On the other hand [FV16] shows the existence of constants $D_{a,c}$ such that $|\,\mathrm{Tr}(\Pi'_{a,b,c} f)| \leq D_{a,c} (\mathrm{Tr}\, f^* f)^{1/2}$ for $f \in B(H_a) \otimes B(H_c)$ with $\mathrm{Tr}(f) = 0$. This implies

$$|\,\mathrm{Tr}(\Pi'_{a,b,c} f) \leq (d_{a+c}/(1-q^2)^2 + D_{a,c}^2)^{1/2} (\mathrm{Tr}\, f^* f)^{1/2}$$

for any $f \in B(H_a) \otimes B(H_c)$, hence $\|\Pi_{a,b,c} - \lambda_{a,c} \mathrm{id}\| \leq (d_{a+c}/(1-q^2)^2 + D_{a,c}^2)^{1/2} d_b^{-1}$.

Moreover, it is explicitly stated in the proof of [FV16, Prop. 3.2] that one can take the constants $D_{a,c}$ defined by induction over c as follows: $D_{a,0} = 0$ and

$$D_{a,c} = K_c \max(d_1^{1/2} D_{a,c-1} + d_1^{3/2} d_{a-1}, d_{a+c}^{1/2}),$$

where $1 \leq K_c = 1/(1-q^c) \leq K := 1/(1-q)$. In particular $d_{a+c} \leq D_{a,c}^2$ if $c \geq 1$. Putting $C_1 = \sqrt{2}/(1-q_0^2)$ we have thus $\|\Pi_{a,b,c} - \lambda_{a,c} \mathrm{id}\| \leq C_1 D_{a,c} q^b$.

One can then show by induction that the constants $D_{a,c}$ satisfy the estimate $D_{a,c} \leq (2NK)^{a+c}$, where $N = d_1 = q + q^{-1}$. Indeed $K_c d_{a+c}^{1/2} \leq K N^{(a+c)/2} \leq (2KN)^{a+c}$, and for $c \in \mathbb{N}^*$ we have by induction

$$K_c (d_1^{1/2} D_{a,c-1} + d_1^{3/2} d_{a-1}) \leq K N^{1/2} (2NK)^{a+c-1} + K N^{3/2} N^{a-1} \leq (2NK)^{a+c}.$$

Of course this estimate is quite bad, but one can improve it using [VV07, Lemma A.4].

More precisely, let $\alpha > 0$ be such that $(2KN)^{2\alpha} q = q^\alpha$. Take a, $c \geq \alpha b$. Denote C_0 the constant given by Lemma 1.3. Then we have

$$P_{a+b+c} \simeq (P_a \otimes \mathrm{id}_b \otimes P_c)(\mathrm{id}_{a-\lfloor \alpha b \rfloor} \otimes P_{b+2\lfloor \alpha b \rfloor} \otimes \mathrm{id}_{c-\lfloor \alpha b \rfloor})$$

up to $2C_0 q^{\lfloor \alpha b \rfloor}$ in operator norm. Applying $\mathrm{id} \otimes \mathrm{tr}_b \otimes \mathrm{id}$, which is contracting, to this estimate we obtain

$$\Pi_{a,b,c} \simeq (P_a \otimes P_c)(\mathrm{id}_{a-\lfloor \alpha b \rfloor} \otimes \Pi_{\lfloor \alpha b \rfloor, b, \lfloor \alpha b \rfloor} \otimes \mathrm{id}_{c-\lfloor \alpha b \rfloor}) \simeq \lambda(\mathrm{id}_a \otimes \mathrm{id}_c)$$

up to $2C_0 q^{\lfloor \alpha b \rfloor} + C_1 D_{\lfloor \alpha b \rfloor, \lfloor \alpha b \rfloor} q^b \leq 2C_0 q^{\lfloor \alpha b \rfloor} + C_1 (2NK)^{2\lfloor \alpha b \rfloor} q^b$ in operator norm. Since $q \leq 1 \leq 2NK$ we have moreover $(2NK)^{2\lfloor \alpha b \rfloor} q^b \leq (2NK)^{2\alpha b} q^b = q^{\alpha b} \leq q^{\lfloor \alpha b \rfloor}$ by definition of α. This yields $\|\Pi_{a,b,c} - \lambda(\mathrm{id}_a \otimes \mathrm{id}_c)\| \leq (2C_0 + C_1) q^{\lfloor \alpha b \rfloor}$. This estimate is also valid if a, $c < \alpha b$ because in this case $D_{a,c} q^b \leq (2KN)^{2\alpha b} q^b = q^{\alpha b}$. It holds also in the remaining cases by using Lemma 1.3 only one side.

Finally we have shown the existence of $D_0 > 0$, depending only on q_0, and $\alpha > 0$ such that for all a, b, c there exists a constant λ such that $\|\Pi_{a,b,c} - \lambda(\mathrm{id}_a \otimes \mathrm{id}_c)\| \leq D_0 q^{\lfloor \alpha b \rfloor}$. One should be careful that α depends on q. In fact it can be computed explicitly from the defining relation $(2KN)^{2\alpha}q = q^\alpha$, with $K = 1/(1-q)$ and $N = q + q^{-1}$: one gets

$$\alpha = \frac{1}{3}\left[1 - \frac{2\ln 2}{3\ln q} - \frac{2}{3\ln q}\ln\left(\frac{1+q^2}{1-q}\right)\right]^{-1}.$$

From this it follows that α is decreasing from $1/3$ to 0 as q varies from 0 to 1, and the result follows. ☐

Remark 1.7. For instance one can take $\alpha = 1/4$ for $q_0 = 0.16$ (or $N_0 = 7$). We also have $q^\alpha \sim Lq^{1/3}$ as $q \to 0$, where $L = \exp(2\ln(2)/9)$.

We will need in the next section one last tool about the representation category of $\mathbb{F}O_N$. The Wenzl recursion relation (1.3), applied twice, yields the following bilateral version.

Lemma 1.8. *For $n \geq 4$ we have the bilateral Wenzl recursion relation:*

$$\begin{aligned} P_n = (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) + \\ & - \frac{d_{n-2}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-2)})(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & - \frac{d_{n-2}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes n-2} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & + \frac{(-1)^{n-1}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_1^{\otimes n-2} \otimes t^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & + \frac{(-1)^{n-1}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t^* \otimes \mathrm{id}_1^{\otimes n-2} \otimes t)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ & + \frac{d_1 + d_{n-3}d_{n-2}}{d_{n-1}d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes n-4} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1). \end{aligned}$$

For $n = 3$ the formula still holds, without the last term.

Proof. We first multiply the relation (1.3) on the left by $(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)$. All terms except $l = 1$ and $l = n-1$ vanish because they involve $P_{n-2}(\mathrm{id}_i \otimes t \otimes \mathrm{id}_j)$, and we are left with :

$$\begin{aligned} P_n = (P_{n-1} \otimes \mathrm{id}_1) + \frac{(-1)^{n-1}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_1^{\otimes n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1) \\ - \frac{d_{n-2}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes n-2} \otimes tt^*)(P_{n-1} \otimes \mathrm{id}_1). \end{aligned}$$

Let us denote A, B, C the three terms on the right-hand side above, without the numeric coefficients. We apply the left version Wenzl's recursion to the projections P_{n-1}:

$$P_{n-1} = (\mathrm{id}_1 \otimes P_{n-2}) + \sum_{k=1}^{n-2}(-1)^{n-1-k}\frac{d_{k-1}}{d_{n-2}}\left(t^* \otimes \mathrm{id}_1^{\otimes(n-k-2)} \otimes t \otimes \mathrm{id}_1^{\otimes(k-1)}\right)(\mathrm{id}_1 \otimes P_{n-2}).$$

Multiplying on the left by $(\mathrm{id}_1 \otimes P_{n-2})$ this yields

$$A = (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) - \frac{d_{n-3}}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes(n-2)})(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1).$$

We proceed similarly with B: only the terms $k = 1$, $k = 2$ have a non-vanishing contribution and we obtain, applying the conjugate equation:

$$\begin{aligned} B = (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_1^{\otimes n-2} \otimes t^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ + \frac{(-1)^n}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes n-2})(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) \\ + \frac{(-1)^{n-1}d_1}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes n-4} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1). \end{aligned}$$

Finally for C only the terms $k=1$, $k=n-2$ survive, yielding:

$$C = (\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes n-2} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) + \frac{(-1)^n}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(t^* \otimes \mathrm{id}_1^{\otimes n-2} \otimes t)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1) - \frac{d_{n-3}}{d_{n-2}}(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1)(tt^* \otimes \mathrm{id}_1^{\otimes n-4} \otimes tt^*)(\mathrm{id}_1 \otimes P_{n-2} \otimes \mathrm{id}_1).$$

The result follows by gathering A, B and C with their coefficients and using the relation $d_{n-3}d_{n-1}+1 = d_{n-2}^2$. □

2. Decomposition of the Bimodule

In this section we consider the GNS space $H = \ell^2(\mathbb{F})$ of $M = \mathcal{L}(\mathbb{F})$ with respect to the Haar trace h. We identify M with a dense subspace of H. We shall study H as an A,A-bimodule for $A = \chi_1'' \cap M$. We will more specifically consider the orthogonal $H^\circ \subset H$ of the trivial bimodule $A \subset H$, and we shall decompose it into simpler, pairwise orthogonal submodules generated by natural elements, see Proposition 2.12. Moreover we will exhibit for each of these cyclic submodules $\mathcal{A}x\mathcal{A}$ a linear basis $(x_{i,j})_{i,j}$, see Proposition 2.9 and Corollary 2.14. Recall that $\mathcal{A}$ is the canonical dense sub-$*$-algebra of A generated by χ_1.

We denote $p_k \in B(H)$ the orthogonal projection onto the subspace $p_kH = u_k(B(H_k))$ spanned by coefficients of u_k. Note that p_k belongs in fact to the dual algebra $\ell^\infty(\mathbb{F})$, and that the projection $P_k \in B(H_1^{\otimes k})$ introduced in the preceding section is the image of p_k under the natural representation of $\ell^\infty(\mathbb{F})$ on the corepresentation space $H_1^{\otimes k}$.

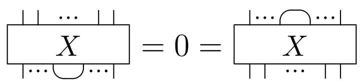
The space H° is spanned by its subspaces p_kH° and we have $p_kH^\circ = H^\circ \cap p_kH = u_k(B(H_k)^\circ)$ where $B(H_k)^\circ = \{X \in B(H_k) \mid \mathrm{Tr}(X) = 0\}$. In the case of the classical generator MASA $a_1'' \subset \mathcal{L}(F_N)$, the subspace analogous to p_kH° is spanned by reduced words of length k, different from $a_1^{\pm k}$. We introduce below a subspace $H^{\circ\circ} \subset H^\circ$ which is the quantum replacement for the set of words $g \in F_N$ that do not start nor end with a_1.

Notation 2.1. For $n \geq 1$ we denote

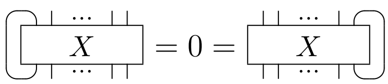
$$B(H_n)^{\circ\circ} = \{X \in B(H_n) \mid (\mathrm{Tr}_1 \otimes \mathrm{id})(X) = 0 = (\mathrm{id} \otimes \mathrm{Tr}_1)(X)\}.$$

We denote $H^{\circ\circ}$ the closed linear span of the subspaces $u_n(B(H_n)^{\circ\circ})$ in H°.

Remark 2.2. It is well-known that $H_n \subset H_1^{\otimes n}$ is the subspace of vectors $\zeta \in H_1^{\otimes n}$ such that $(\mathrm{id}_i \otimes t^* \otimes \mathrm{id}_{n-i-2})(\zeta) = 0$ for all $i = 0, \dots, n-2$. As a consequence, an element $X \in B(H_1^{\otimes n})$ arises from an element of $B(H_n)$ **iff** we have $(\mathrm{id}_i \otimes t^* \otimes \mathrm{id}_{n-i-2})X = 0$ and $X(\mathrm{id}_i \otimes t \otimes \mathrm{id}_{n-i-2}) = 0$ for all i. Graphically this means we have $X \in B(H_n)$ **iff** we obtain 0 by applying to X any planar tangle which connects two consecutive points on the lower or upper edge of the internal box corresponding to X:



Since $(\mathrm{Tr}_1 \otimes \mathrm{id})(X) \in B(H_1^{\otimes n-1})$ (resp $(\mathrm{id} \otimes \mathrm{Tr}_1)(X)$) is obtained from X by applying the planar tangle connecting the upper left and lower left (resp. upper right and lower right) points of the internal box, we conclude that $X \in B(H_1^{\otimes n})$ belongs to $B(H_n)^{\circ\circ}$ **iff** we obtain 0 by applying to X any planar tangle which connects any two consecutive points of the internal box corresponding to X. Diagrammatically this is represented by the additional constraints:



Now we compute the dimension of $B(H_n)^{\circ\circ}$, see Proposition 2.5. This will be useful to prove that the families $(x_{i,j})_{i,j}$ are linearly independent at Corollary 2.14. The latter also follows from the stronger results of Section 3, but there we will have to assume that N is large enough and the proofs are much more involved. Note however that the proof below is not optimal either, in the sense that the underlying technical result established at Lemma 2.4 does not hold if $q+q^{-1} \in]2, 2.41[$, which can occur for the non unimodular groups $\mathbb{F}O_Q$. We believe that Lemma 2.3 and Proposition 2.5 hold true for any group $\mathbb{F}O_Q$ with $q+q^{-1}>2$, i.e. excluding the duals of $SU(2)$ and $SU_{-1}(2)$.

In the statement below we use the leg numbering notation: $t^*_{1,n} = \sum_i e^*_i \otimes \mathrm{id}_1 \otimes \cdots \otimes \mathrm{id}_1 \otimes e^*_i$.

Lemma 2.3. *Assume $N \geq 3$. For $n \geq 3$ the map $t^*_{1,n} : H_n \to H_{n-2}$ is surjective.*

Proof. We apply $t^*_{1,n} \cdot t_{1,n}$ to the bilateral Wenzl recursion formula from Lemma 1.8. Using the conjugate equations we have in $B(H_1^{\otimes n-2})$:

$$\begin{aligned}
&t^*_{1,n}(tt^* \otimes \mathrm{id}_1^{\otimes n-2})t_{1,n} = \mathrm{id}_1^{\otimes n-2} = t^*_{1,n}(\mathrm{id}_1^{\otimes n-2} \otimes tt^*)t_{1,n},\\
&t^*_{1,n}(t \otimes \mathrm{id}_1^{\otimes n-2} \otimes t^*)t_{1,n} = C^{-2}_{n-2},\\
&t^*_{1,n}(t^* \otimes \mathrm{id}_1^{\otimes n-2} \otimes t)t_{1,n} = C^{2}_{n-2},\\
&t^*_{1,n}(tt^* \otimes \mathrm{id}_1^{\otimes n-4} \otimes tt^*)t_{1,n} = t_{1,n-2}t^*_{1,n-2},
\end{aligned}$$

where $C_{n-2} : \xi \otimes \zeta \mapsto \zeta \otimes \xi$ for $\xi \in H_1$, $\zeta \in H_1^{n-3}$. Thus we obtain

$$\begin{aligned}
t^*_{1,n}P_n t_{1,n} = &\left(d_1 - \tfrac{2d_{n-2}}{d_{n-1}}\right) P_{n-2} + \\
&+ \frac{(-1)^{n-2}}{d_{n-1}} P_{n-2}(C^2_{n-2} + C^{-2}_{n-2})P_{n-2} + \frac{d_1 + d_{n-3}d_{n-2}}{d_{n-1}d_{n-2}} P_{n-2}t_{1,n-2}t^*_{1,n-2}P_{n-2}.
\end{aligned}$$

Observe moreover that $\|P_{n-2}C^{\pm 2}_{n-2}P_{n-2}\| \leq 1$ and $\|P_{n-2}t_{1,n-2}t^*_{1,n-2}P_{n-2}\| \leq d_1$ by composition. Now, the inequality established in the next Lemma shows that $t^*_{1,n}P_n t_{1,n} \geq \epsilon P_{n-2}$ for some $\epsilon > 0$. As a result $t^*_{1,n}P_n t_{1,n} \in B(H_{n-2})$ is invertible and the result follows. □

Lemma 2.4. *Still assuming $N \geq 3$, we have for any $n \geq 3$:*

$$d_1 - \frac{2d_{n-2}}{d_{n-1}} > \frac{2}{d_{n-1}} + d_1 \frac{d_1 + d_{n-3}d_{n-2}}{d_{n-1}d_{n-2}}.$$

Proof. Denote $e_n = d_{n-1} - d_{n-2}$, $f_n = d_{n-5} + 1 + d_1$, with the convention $d_k = 0$ if $k < 0$. For $n = 3$ we have $e_3 = N^2 - N - 1$, $f_3 = 1 + N$ and since $N \geq 3 > 1 + \sqrt{3}$ we have $e_3 > f_3$.

On the other hand we have, using the identity $Nd_{n-1} = d_n + d_{n-2}$ valid for $n \in \mathbb{Z}^*$:

$$e_{n+1} - e_n = d_n - 2d_{n-1} + d_{n-2} = (N-2)d_{n-1} \geq d_{n-1} \geq d_{n-4} - d_{n-5} = f_{n+1} - f_n.$$

An easy induction then shows that we have $e_n > f_n$ for every $n \geq 3$.

Multiplying this inequality by d_1 we find

$$\begin{aligned}
& d_1d_{n-1} - d_1d_{n-2} > d_1d_{n-5} + d_1 + d_1^2\\
\iff\ & d_1d_{n-1} - (d_1 - 1)d_{n-2} > d_1d_{n-5} + d_{n-2} + d_1 + d_1^2\\
\implies\ & d_1d_{n-1} - 2d_{n-2} > d_1d_{n-3} + 2 + d_1^2/d_{n-2},
\end{aligned}$$

using the facts $d_1 \geq 3$, $d_1d_{n-5} + d_{n-2} \geq d_{n-4} + d_{n-2} \geq d_1d_{n-3} - 1$, and $d_{n-2} \geq 1$. Note that the inequality $d_1d_{n-5} \geq d_{n-4}$, resulting from the fusion rules, does not hold for $n = 4$, but one can check directly that in this case $d_1d_{n-5} + d_{n-2} = d_1d_{n-3} - 1$. □

Proposition 2.5. *Still assume $N \geq 3$. For $n \geq 2$ we have $\dim B(H_n)^{\circ\circ} = \dim p_nH^{\circ\circ} = d_{2n} - d_{2n-2}$. For $n = 1$ we have $\dim B(H_1)^{\circ\circ} = \dim p_1H^{\circ\circ} = d_2$.*
On the other hand if $N = 2$ we have $\dim B(H_n)^{\circ\circ} = \dim p_nH^{\circ\circ} = d_{2n}$ for all $n \geq 1$.

Proof. Recall the identification $B(H_n) \simeq H_n \otimes H_n$ via $X \mapsto x = (X \otimes \mathrm{id})t_n$. In this identification the condition $(\mathrm{id} \otimes \mathrm{Tr}_1)(X) = 0$ reads $(\mathrm{id}_{n-1} \otimes t^* \otimes \mathrm{id}_{n-1})(x) = 0$ and the corresponding kernel is $H_{2n} \subset H_n \otimes H_n$. This holds as well if $N = 2$. Then the condition $(\mathrm{Tr}_1 \otimes \mathrm{id})(X) = 0$ reads $t^*_{1,2n}(x) = 0$, so that the result follows from the rank theorem and Lemma 2.3. For $n = 1$ both conditions coincide and we have $B(H_1)^{\circ\circ} = B(H_1)^\circ \simeq H_2$. In the case $N = 2$ one can check that $t^*_{1,2n}$ vanishes on H_{2n} for all n (but we will not use this case in the present article). □

Recall then the "rotation operators" $\rho : B(H_k) \to B(H_k)$ already considered in [FV16] and defined as follows: $\rho(X) = (P_k \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes P_k)$. It follows from [FV16, Lemma 3.1] that ρ stabilizes the subspace $B(H_n)^\circ$ and contracts the Hilbert-Schmidt norm. On $B(H_n)^{\circ\circ}$ it behaves even better: as the next lemma shows, it is a finite order unitary — in particular, it is diagonalizable.

Lemma 2.6. *The map ρ is a bijection from $B(H_n)^{\circ\circ}$ to itself. Moreover we have $\rho^{2n} = \mathrm{id}$ and $\rho^* = \rho^{-1}$ on $B(H_n)^{\circ\circ}$.*

Proof. We first note that for $X \in B(H_n)^{\circ\circ}$ the element $Y = (\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1} \otimes \mathrm{id}_1)$ of $B(H_{n-1} \otimes H_1, H_1 \otimes H_{n-1})$ is directly equal to $\rho(X)$. This is clear if $n = 1$ since then $\mathrm{id}_1 \otimes \mathrm{id}_{n-1} = \mathrm{id}_1 = P_1$. Assume $n \geq 2$. Since $t^*(\mathrm{id} \otimes A)t = \mathrm{Tr}_1(A)$ for any $A \in B(H_1)$ we have $(t^* \otimes \mathrm{id}_{n-2})Y = (\mathrm{id} \otimes t^*)[(\mathrm{Tr}_1 \otimes \mathrm{id})(X) \otimes \mathrm{id}_1)] = 0$, and similarly $Y(\mathrm{id}_{n-2} \otimes t) = 0$, so that $Y = P_n Y P_n = \rho(X)$. Then we compute $(\mathrm{Tr}_1 \otimes \mathrm{id})(Y)$ using again the morphism t. Thank to the conjugate equation we have

$$\begin{aligned}(\mathrm{Tr}_1 \otimes \mathrm{id})(Y) &= (t^* \otimes \mathrm{id}_{n-1})(\mathrm{id}_1 \otimes \mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes \mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)\\ &\qquad (\mathrm{id}_1 \otimes t \otimes \mathrm{id}_{n-1} \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-2} \otimes \mathrm{id}_1)\\ &= (\mathrm{id}_{n-1} \otimes t^*)(X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-2} \otimes \mathrm{id}_1) = 0,\end{aligned}$$

since $X \in B(H_n)$. Similarly $(\mathrm{id} \otimes \mathrm{Tr}_1)(Y) = 0$ and this proves $\rho(B(H_n)^{\circ\circ}) \subset B(H_n)^{\circ\circ}$. The conjugate equation also implies that $(t^* \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes Y \otimes \mathrm{id}_1)(\mathrm{id}_n \otimes t) = X$ so that ρ is a bijection with $\rho^{-1}(X) = (t^* \otimes P_n)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(P_n \otimes t)$. This holds as well for $n = 1$.

Let us check that ρ^{-1} is the adjoint of ρ with respect to the Hilbert-Schmidt scalar product. Using twice the conjugate equation we have, for X, $Y \in B(H_n)^{\circ\circ}$:

$$\begin{aligned}\mathrm{Tr}_n(\rho^{-1}(X)^*Y) &= (\mathrm{Tr}_1 \otimes \mathrm{Tr}_{n-1})[(\mathrm{id}_n \otimes t^*)(\mathrm{id}_1 \otimes X^* \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_n)Y]\\ &= \mathrm{Tr}_{n-1}[(t^* \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes \mathrm{id}_1 \otimes X^* \otimes \mathrm{id}_1)\\ &\qquad (\mathrm{id}_1 \otimes t \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes Y)(t \otimes \mathrm{id}_{n-1})]\\ &= \mathrm{Tr}_{n-1}[(\mathrm{id}_{n-1} \otimes t^*)(X^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes Y)(t \otimes \mathrm{id}_{n-1})]\\ &= \mathrm{Tr}_{n-1}[(\mathrm{id}_{n-1} \otimes t^*)(X^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^* \otimes \mathrm{id}_1)\\ &\qquad (\mathrm{id}_1 \otimes Y \otimes \mathrm{id}_1 \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1} \otimes t)]\\ &= (\mathrm{Tr}_{n-1} \otimes \mathrm{Tr}_1)[X^*(\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes Y \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1})]\\ &= \mathrm{Tr}_n(X^*\rho(Y)).\end{aligned}$$

Recall the notation $t_1^n \in \mathrm{Hom}(\mathbb{C}, H_1^{\otimes n} \otimes H_1^{\otimes n})$ from the Preliminaries and consider the associated antilinear map $j^n : H_1^{\otimes n} \to H_1^{\otimes n}$ given by $j^n(\zeta) = (\zeta^* \otimes \mathrm{id})t^n$. If $(e_i)_i$ is the canonical basis of $H_1 = \mathbb{C}^N$ we have $j^n(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_n} \otimes \cdots \otimes e_{i_1}$ so that $j^n \circ j^n = \mathrm{id}$ and $j^n(\zeta) = (\mathrm{id} \otimes \zeta^*)t^n$. Using the fact that $\rho(X) = (\mathrm{id}_1 \otimes \mathrm{id}_{n-1} \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_{n-1} \otimes \mathrm{id}_1)$ for $X \in B(H_n)^{\circ\circ}$ we have easily $\rho^n(X) = (\mathrm{id}_n \otimes t^{n*})(\mathrm{id}_n \otimes X_n \otimes \mathrm{id})(t^n \otimes \mathrm{id}_n)$, which yields $(\zeta \mid \rho^n(X)\xi) = (j^n\xi \mid Xj^n\zeta)$ for all ζ, $\xi \in H_n$. Applying this identity a second time we get $\rho^{2n}(X) = X$. □

We shall now analyze the submodule AxA when x belongs to $H^{\circ\circ}$. In the analogy with the generator MASA $a_1'' \subset \mathcal{L}(F_N)$ in a free group factor, the vectors $x_{i,j}$ below play the role of the words $a_1^i g a_1^j \in F_N$, where $g \in F_N$ does not start nor end with a.

Notation 2.7. For $x \in H$ and i, $j \in \mathbb{N}$ we denote $x_{i,j} = \sum_n p_{i+n+j}(\chi_i p_n(x) \chi_j)$. For $X \in B(H_n)$ we denote $X_{i,j} = P_{i+n+j}(\mathrm{id}_i \otimes X \otimes \mathrm{id}_j)P_{i+n+j} \in B(H_{i+n+j})$.

Remark 2.8. The sum in the definition of $x_{i,j}$ indeed converges in H, since its terms are pairwise orthogonal an satisfy the inequality $\|\chi_i p_n(x) \chi_j\| \leq \|\chi_i\| \|\chi_j\| \|p_n(x)\|$. This yields a map $(x \mapsto x_{i,j})$ which is linear and bounded from H to H. We will mostly use the notation $x_{i,j}$ in the case when x belongs to one of the subspaces $p_n H$.

Note also that we have by construction $u_n(X)_{i,j} = u_{i+n+j}(X_{i,j})$ for $X \in B(H_n)$. Indeed, denote $x = u_n(X)$ and recall that $\chi_i = u_i(\mathrm{id}_i)$, $\chi_j = u_j(\mathrm{id}_j)$. To compute the component $p_{i+n+j}(\chi_i x \chi_j)$ one has to use an orthonormal basis of isometric intertwiners $T : H_{i+n+j} \to$

$H_i \otimes H_n \otimes H_j$. But according to the fusion rules there is only one such intertwiner up to a phase, and by construction of the spaces H_k we can take for it the canonical inclusion of H_{i+n+j} into $H_i \otimes H_n \otimes H_j \subset H_1^{\otimes i+n+j}$, whose adjoint is given by P_{i+n+j}.

Finally, we record the fact that $X_{i,j}$ is the orthogonal projection of $\mathrm{id}_i \otimes X \otimes \mathrm{id}_j \in B(H_1^{\otimes i+n+j})$ onto $B(H_{i+n+j})$, with respect to the Hilbert-Schmidt scalar product — indeed for any Y, $Z \in B(H_1^{\otimes i+n+j})$ we have $\mathrm{Tr}(Y^* P_{i+n+j} Z P_{i+n+j}) = \mathrm{Tr}((P_{i+n+j} Y P_{i+n+j})^* Z)$.

Proposition 2.9. *Fix $k \in \mathbb{N}^*$, $X \in B(H_k)^{\circ\circ}$ an eigenvector of ρ and $x = u_k(X) \in H^{\circ\circ}$. Then we have $\mathcal{A}x\mathcal{A} = \mathrm{Span}\{x_{i,j} \mid i, j \in \mathbb{N}\}$.*

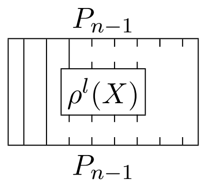
Proof. Let us prove by induction over $i + j = n - k$ that $x_{i,j} \in \mathcal{A}x\mathcal{A}$. Assume that $x_{p,q} \in \mathcal{A}x\mathcal{A}$ if $p + q \leq i + j$ and compute $\chi_1 x_{i,j}$. We have $p_{n-1}(\chi_1 x_{i,j}) = (\kappa_{n-1}^{1,n})^2 u_{n-1}(\mathrm{id}_1 *_{n-1} X_{i,j})$ and

$$\begin{aligned} \mathrm{id}_1 *_{n-1} X_{i,j} &= (t^* \otimes P_{n-1})(\mathrm{id}_1 \otimes X_{i,j})(t \otimes P_{n-1}) \\ &= P_{n-1}(t^* \otimes \mathrm{id}_{n-1})(\mathrm{id}_1 \otimes P_n)(\mathrm{id}_{i+1} \otimes X \otimes \mathrm{id}_j)(\mathrm{id}_1 \otimes P_n)(t \otimes \mathrm{id}_{n-1})P_{n-1}. \end{aligned} \tag{2.1}$$

Since the Jones-Wenzl projections P_n are intertwiners, we can expand them into linear combinations of Temperley-Lieb diagrams, so that $\mathrm{id}_1 *_{n-1} X_{i,j}$ is a linear combination of maps of the form $P_{n-1} T_\pi(X) P_{n-1}$, where π is a Temperley-Lieb diagram with $n-1$ upper and lower points and an internal box with $2k$ points, and $T_\pi : B(H_1^{\otimes k}) \to B(H_1^{\otimes n-1})$ is the associated map. Since we multiply on the left and on the right by P_{n-1} and $X = P_k X P_k$ belongs to $B(H_k)^{\circ\circ}$, the term associated with π vanishes as soon as a string of π connects two upper points, or two lower points, or two internal points.

Now consider the string originating from the first top left external point in a diagram π such that $P_{n-1} T_\pi(X) P_{n-1} \neq 0$. If it is not connected to the internal box, it has to connect the top left point to the first bottom left external point, otherwise some other string would have to connect two upper or two lower external point, because of the non-crossing constraint. We can re-apply this reasoning to the following top left external points, until we find an external point connected to X, say with index $p + 1$ on the top external edge. Moreover up to replacing X by its image $\rho^l(X)$ under some iterated rotation we can assume that this external point is connected to the first top left point of the internal box by a vertical edge. Thus our diagram has the following form:

$$P_{n-1} T_\pi(X) P_{n-1} =$$



.

Continuing with the same reasoning we see that in fact the only possibility for a non-vanishing diagram is $P_{n-1}(\mathrm{id}_p \otimes \rho^l(X) \otimes \mathrm{id}_q) P_{n-1}$, with $l \in \mathbb{Z}$ and $p+q = n-k-1$. Since X is an eigenvector of ρ, this shows that $p_{n-1}(\chi_1 x_{i,j})$ is a linear combination of vectors $x_{p,q}$ with $p+q < i+j$, which belong to $\mathcal{A}x\mathcal{A}$ by the induction hypothesis. Note that if $i = j = 0$ we have $p_{n-1}(\chi_1 x_{i,j}) = 0$; this can also be checked directly because (2.1) then equals $(\mathrm{Tr}_1 \otimes \mathrm{id})(X)$.

We have $p_{n+1}(\chi_1 x_{i,j}) = p_{n+1}(\chi_1 \chi_i x \chi_j) = x_{i+1,j}$ because $p_{n+1}(\chi_1 y) = 0$ if $y \in p_k H$ with $k < n$. We have thus $x_{i+1,j} = \chi_1 x_{i,j} - p_{n-1}(\chi_1 x_{i,j})$ and it follows that $x_{i+1,j}$ belongs to $\mathcal{A}x\mathcal{A}$. One can proceed in the same way on the right to show that $x_{i,j+1}$ belongs to $\mathcal{A}x\mathcal{A}$. By induction we have proved $x_{i,j} \in \mathcal{A}x\mathcal{A}$ for all i, j. Moreover from the identities $\chi_1 x_{i,j} = x_{i+1,j} + p_{n-1}(\chi_1 x_{i,j})$, $x_{i,j}\chi_1 = x_{i,j+1} + p_{n-1}(x_{i,j}\chi_1)$ and the fact that $p_{n-1}(\chi_1 x_{i,j})$, $p_{n-1}(x_{i,j}\chi_1)$ are linear combinations of vectors $x_{p,q}$ it also follows that $\mathrm{Span}\{x_{i,j}\}$ is stable under the left and right actions of $\mathcal{A}$. ☐

Remark 2.10. Using the Jones-Wenzl recursion relations, one can prove more precisely that $p_{n-1}(x_{i,j}\chi_1)$ is a linear combination of $x_{i-1,j}$, $\rho^{\pm 1}(x)_{i,j-1}$ and $x_{i+1,j-2}$, where we abusively write $\rho(u_k(X)) := u_k(\rho(X))$.

Notation 2.11. Choose for all $k \geq 1$ a basis of eigenvectors of ρ in $B(H_k)^{\circ\circ}$ and denote W_k its image in $p_k H^{\circ\circ}$. Put as well $W = \bigcup_k W_k$, which is a linearly independent family in $H^{\circ\circ}$. For $x \in W$ we denote $H(x) = \overline{AxA}$, and for $k \in \mathbb{N}^*$, $H(k) = \overline{AW_kA}$. The previous lemma shows that the vectors $x_{i,j}$ span a dense subspace of $H(x)$.

Proposition 2.12. *The family W spans H° as a closed A,A-bimodule. Moreover, for $x \neq y \in W$ we have $H(x) \perp H(y)$.*

Proof. Denote $L_n = \mathrm{Span}\{X_{i,j} \mid X \in W_k, k \leq n, i+j+k = n\} \subset B(H_n)^\circ$, and let us show by induction over $n \geq 1$ that $L_n = B(H_n)^\circ$. For $n = 1$ we have by definition $L_1 = \mathrm{Span}\, W_1 = B(H_1)^{\circ\circ} = B(H_1)^\circ$. Assume that $L_n = B(H_n)^\circ$ and take $Y \in L_{n+1}^\perp \cap B(H_{n+1})^\circ$. We want to show that $Y = 0$. We consider first $(\mathrm{Tr}_1 \otimes \mathrm{id})(Y)$. For any generator $X_{i,j}$ of L_n we have

$$\mathrm{Tr}_n(X_{i,j}^*(\mathrm{Tr}_1 \otimes \mathrm{id})(Y)) = (\mathrm{Tr}_1 \otimes \mathrm{Tr}_n)(P_{n+1}(\mathrm{id}_1 \otimes X_{i,j}^*)P_{n+1}Y) = \mathrm{Tr}_{n+1}(X_{i+1,j}^* Y) = 0,$$

by assumption on Y. Since $L_n = B(H_n)^\circ$, this implies $(\mathrm{Tr}_1 \otimes \mathrm{id})(Y) = 0$. Similarly, $(\mathrm{id} \otimes \mathrm{Tr}_1)(Y) = 0$. As a result, $Y \in B(H_{n+1})^{\circ\circ}$. But $B(H_{n+1})^{\circ\circ} \subset L_{n+1}$, and $Y \perp L_{n+1}$, so that we have indeed proved $Y = 0$. Taking into account Proposition 2.9, this proves that $p_n H^\circ \subset \mathrm{Span}\, \mathcal{A} W \mathcal{A}$ for every n and the first result follows.

For the second part of the statement, take $x \in W_k$, $y \in W_l$ distinct, with $k \leq l$. The subspaces $H(x)$, resp. $H(y)$ are spanned by vectors $\chi_1^p x \chi_1^q$, resp. $\chi_1^r y \chi_1^s$. We have

$$(\chi_1^p x \chi_1^q \mid \chi_1^r y \chi_1^s) = (x \mid \chi_1^{p+r} y \chi_1^{q+s}) = (x \mid p_k(\chi_1^{p+r} y \chi_1^{q+s})).$$

But $\chi_1^{p+r} y \chi_1^{q+s} \in \mathrm{Span}\{y_{i,j}\}$ and $y_{i,j} \in p_{i+l+j} H$. Since $k \leq l$ this implies that $p_k(\chi_1^{p+r} y \chi_1^{q+s}) \in \mathbb{C} y$ and the result follows since $x \perp y$. ☐

Corollary 2.13. *The bimodule H° is isomorphic to $L^2(A) \otimes \ell^2(W) \otimes L^2(A)$. In particular the Pukánszky invariant of $A \subset M$ is $\{\infty\}$.*

Proof. Indeed the proof of in [FV16, Theorem 5.10] shows that the measure induced on $[-2,2] \times [-2,2]$ induced by a given $\zeta \in H^\circ \cap \mathcal{A}$ and the action of $A \otimes A$ on H° is equivalent to the Lebesgue measure — in fact it has a non-zero analytic density. As a result, the corresponding cyclic bimodule $H(\zeta)$ is isomorphic to the coarse bimodule $L^2(A) \otimes L^2(A)$. This applies to $\zeta = x \in W$. Now, Proposition 2.12 shows that we have an isomorphism of A, A-bimodules

$$H^\circ \simeq \bigoplus_{x \in W} L^2(A) \otimes L^2(A) \simeq L^2(A) \otimes \ell^2(W) \otimes L^2(A).$$

As a result $(A \otimes A)' \cap B(H^\circ) \simeq A \bar{\otimes} B(\ell^2(W)) \bar{\otimes} A$ and the value of the Pukánszky invariant follows since W is infinite. ☐

Corollary 2.14. *For $x \in W$ the vectors $x_{i,j}$ are linearly independent.*

Proof. Since the subspaces $p_n H^\circ$ are pairwise orthogonal, it suffices to consider a subfamily $(x_{i,j})$ with $i+k+j = n$ fixed. Note that $\#\{x_{i,j} \mid i+k+j = n\} = n-k+1$. According to Proposition 2.12 we have

$$p_n H^\circ = \bigoplus_{k \leq n} \bigoplus_{x \in W_k} \mathrm{Span}\{x_{i,j} \mid i+k+j = n\},$$

so that $\dim p_n H^\circ \leq \sum_{k=1}^n (n-k+1) \# W_k$. We will prove that this estimate is an equality, so that $\dim \mathrm{Span}\{x_{i,j} \mid i+k+j = n\} = n-k+1$ for all $x \in W_k$, $k \leq n$, which implies the linear independence.

Recall from Proposition 2.5 that $\# W_k = \dim B(H_k)^{\circ\circ} = d_{2k} - d_{2k-2}$ for $k \geq 2$, and $\# W_1 = d_2$. We have then

$$\begin{aligned}\textstyle\sum_{k=1}^n (n-k+1)\# W_k &= \textstyle\sum_{k=1}^n (n-k+1) d_{2k} - \sum_{k=2}^n (n-k+1) d_{2k-2} \\ &= \textstyle\sum_{k=1}^n (n-k+1) d_{2k} - \sum_{k=1}^{n-1} (n-k) d_{2k} \\ &= \textstyle\sum_{k=1}^n d_{2k} = d_n^2 - 1 = \dim B(H_n)^\circ = \dim p_n H^\circ.\end{aligned}$$

The computation of the sum in the last line follows from the decomposition of $u_n \otimes u_n$ given by the fusion rules. ☐

Remark 2.15. As a result, the map $\Phi : c_c(\mathbb{N}) \otimes H^{\circ\circ} \otimes c_c(\mathbb{N}) \to H^\circ$, $\delta_i \otimes x \otimes \delta_j \mapsto x_{i,j}$ is injective with dense image. It is however not an isometry. We will see in the next section that, at least for "large N", it extends to an isomorphism from $\ell^2(\mathbb{N}) \otimes H^{\circ\circ} \otimes \ell^2(\mathbb{N})$ to H°.

We end this section with two further properties of elements of $H^{\circ\circ}$ which are established using the action of planar tangles and will be used in sections 4 and 5.

Proposition 2.16. *For any $\zeta \in H(k)$, $\zeta' \in H(k')$ and $y \in p_n H$ with $n < |k - k'|$ we have $y\zeta \perp \zeta'$. If $k > n$ we have $y\zeta \in H^\circ$.*

Proof. By bilinearity one can assume $\zeta = x_{i,j} = u_{i+k+j}(X_{i,j})$, $\zeta'^* = x'_{i',j'} = u_{i'+k'+j'}(X'_{i',j'})$, with $X \in B(H_k)^{\circ\circ}$, $X' \in B(H_{k'})^{\circ\circ}$. Denote also $y = u_n(Y)$ with $Y \in B(H_n)$. Then the product $y\zeta$ is a linear combination of elements $u_m(Y *_m X_{i,j})$ with $m = n + i + k + j - 2a$. Using the Peter-Weyl relations (1.1) it thus suffices to prove that $\mathrm{Tr}(X'_{i',j'}(Y *_m X_{i,j})) = 0$, with $m = i' + k' + j' = i + k + j - 2a$.

By definition, the element in the trace is computed by the following formula:

$$P_m(\mathrm{id}_{i'} \otimes X' \otimes \mathrm{id}_{j'})P_m(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+j+k-a})(\mathrm{id}_n \otimes P_{i+k+j})$$
$$(Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j)(\mathrm{id}_n \otimes P_{i+k+j})(\mathrm{id}_{n-a} \otimes t_a \otimes \mathrm{id}_{i+j+k-a})P_m.$$

Since P_m, P_{i+k+j}, t_a are morphisms, this element is a linear combination of planar tangles on m lower and upper points, with 3 inside boxes, applied to X, X', Y. Since $\mathrm{Tr}(Z) = t_m^*(Z \otimes \mathrm{id}_m)t_m$, the scalar $\mathrm{Tr}(X'_{i',j'}(Y *_m X_{i,j}))$ is itself a linear combination of such planar tangles T, without external points, applied to X, X', Y.

Fix one of this tangles and consider the strings starting at one of the $2k$ points on the internal box corresponding to X. These strings can have their second ends on X, X' or Y. If $2k > 2k' + 2n$, the first possibility must happen at least once, i.e. there is a string connecting two points of X. Since the strings are non crossing, this implies that there is even a string connecting two consecutive points of the internal box corresponding to X. But then the value of the tangle applied to X, X', Y is 0 since $X \in B(H_k)^{\circ\circ}$: see Remark 2.2.

If $k < k' - n$ we proceed in the same way by considering strings starting on the internal box corresponding to X'. The last assertion of the statement amounts to considering the trace $\mathrm{Tr}(Y *_m X_{i,j})$ which is again a linear combination of planar tangles without external points applied to Y and X, and if $k > n$ the same argument as above applies. □

Lemma 2.17. *For $X \in B(H_k)^{\circ\circ}$ and any $a \in \mathbb{N}$ we have $(\mathrm{Tr}_{a+1} \otimes \mathrm{id}_{k-1})(X_{a,0}) = 0$.*

Proof. As in the previous proof, $(\mathrm{Tr}_{a+1} \otimes \mathrm{id}_{k-1})(X_{a,0}) = (\mathrm{Tr}_{a+1} \otimes \mathrm{id}_{k-1})(P_{a+k}(\mathrm{id}_a \otimes X)P_{a+k})$ is a linear combination of planar tangles with $k - 1$ upper and lower points, and one internal box filled with X which has k upper and lower points. For any such tangle, we cannot connect all points of the internal box to external points. As a result, two points of the internal box have to be connected, and again by Remark 2.2 this implies that the tangle vanishes when applied to $X \in B(H_k)^{\circ\circ}$. □

3. Invertibility of the Gram Matrix

In this section we fix $k \in \mathbb{N}^*$, $x = u_k(X) \in p_k H^{\circ\circ}$ with $X \in B(H_k)^{\circ\circ}$ an eigenvector of ρ with associated eigenvalue μ, $|\mu| = 1$. Recall the notation $x_{i,j} = p_{i+k+j}(\chi_i x \chi_j)$, $X_{i,j} = P_{i+k+j}(\mathrm{id}_i \otimes X \otimes \mathrm{id}_j)P_{i+k+j}$. We know from the previous section that $(x_{i,j})$ spans a dense subspace of the bimodule AxA. Our aim is now to show that it is a Riesz basis, i.e. it implements an isomorphism between $H(x) = \overline{AxA}$ in H and $\ell^2(\mathbb{N} \times \mathbb{N})$. We will only achieve this for small q, i.e. large N. We thus consider the associated Gram matrix, which is block diagonal since $p_m H \perp p_n H$ for $m \neq n$. Let us formalize this as follows:

Notation 3.1. We fix $k \in \mathbb{N}^*$ and $x = u_k(X) \in W_k$. We denote $G = G(x)$ the Gram matrix of the family $(x_{i,j})_{i,j} \subset H$, and $G_n = G_n(x)$ its diagonal block corresponding to indices (i, j) such that $x_{i,j} \in p_n H$, i.e. $i + k + j = n$. Since k is fixed we drop the second index j and denote $x_{n;i} = x_{i,j}$, $X_{n;i} = X_{i,j}$. For $i, p \in \{0, \ldots, n\}$ we denote accordingly

$$G_{n;i,p} = (x_{n;i} \mid x_{n;p}) = d_n^{-1}(X_{n;i} \mid X_{n;p}).$$

The second equality follows from the Peter-Weyl-Woronowicz orthogonality relations, using the Hilbert-Schmidt scalar product in $B(H_n)$. Let us record the following symmetry properties of G:

Lemma 3.2. *For any $n = i + k + j = p + k + q$ we have*

$$G_{n;i,p}(x) = \overline{G_{n;p,i}(x)} = G_{n;q,j}(x^*) = G_{n;j,q}(S(x)).$$

Proof. As a Gram matrix, G_n is self-adjoint, which corresponds to the first equality. Define maps J, $U : H \to H$ by $J(x) = x^*$, $U(x) = S(x)$ where S is the antipode. The maps are surjective isometries because we are in the Kac case, and since u_n is orthogonal they stabilize $p_n H$ and send χ_n to itself. We have then

$$\begin{aligned} G_{n;i,p}(x) &= (p_n(\chi_i x \chi_j) \mid p_n(\chi_p x \chi_q)) = (J p_n(\chi_p x \chi_q) \mid J p_n(\chi_i x \chi_j)) \\ &= (p_n(\chi_q x^* \chi_p) \mid p_n(\chi_j x^* \chi_i)) = G_{n;q,j}(x^*) \\ &= (U p_n(\chi_i x \chi_j) \mid U p_n(\chi_p x \chi_q)) = (p_n(\chi_j S(x) \chi_i) \mid p_n(\chi_q S(x) \chi_p)) = G_{n;j,q}(S(x)). \quad \square \end{aligned}$$

Our main aim is then to show the existence of a constant C such that $\|G_n\|$, $\|G_n^{-1}\| \le C$ for all n. In fact we even want the constant C to be uniform over k and $x \in W_k$, so that the map Φ from Remark 2.15 will indeed be an isomorphism.

We shall first show that the Gram matrix $G = G(x)$ is bounded as an operator on $\ell^2(\mathbb{N} \times \mathbb{N})$. We start with an easy estimate, which is not sufficient for this purpose but will be useful later. We then prove an off-diagonal decay estimate for the coefficients of the Gram matrix, see Lemma 3.4, using the improvement of the main estimate of [FV16] established at Lemma 1.6. These two results easily imply the boundedness of $G(x)$ on $\ell^2(\mathbb{N} \times \mathbb{N})$, which we record at Proposition 3.5. Note however that the constant C obtained in this way depends on k, so that one cannot deduce the boundedness of the whole Gram matrix. This will be improved later.

Lemma 3.3. *We have $\|x_{n;i}\|_2 \le (1-q^2)^{-3/2}\|x\|_2$ for all n, i, hence $|G_{n;i,p}| \le (1-q^2)^{-3}\|x\|_2^2$ for all n, i, p.*

Proof. We have

$$\begin{aligned} \|X_{n;i}\|_2^2 &= \mathrm{Tr}(P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)P_n(\mathrm{id}_i \otimes X \otimes \mathrm{id}_j)P_n) \\ &\le \mathrm{Tr}(P_n(\mathrm{id}_i \otimes X^*X \otimes \mathrm{id}_j)P_n) \le \mathrm{Tr}(P_i \otimes X^*X \otimes P_j) = d_i d_j \|X\|_2^2, \end{aligned}$$

hence $\|x_{n;i}\|_2^2 \le (d_i d_j d_k / d_n)\|x\|_2^2$. The result then follows from Lemma 1.2. $\square$

Lemma 3.4. *For every $q_0 \in \,]0,1[$ there exists $\alpha \in \,]0,1[$ and $C > 0$ depending only on q_0 such that $|G_{n;i,p}| \le C q^{\alpha(|p-i|-k)-2-k}\|x\|_2^2$ for all n, i, p such that $|p-i| \ge k$, as soon as $q \in \,]0,q_0]$.*

Proof. The reader will find after the proof a graphical "explanation" of the computations. Write $n = i+k+j = p+k+q$. We have $(x_{n;i} \mid x_{n;p}) = \mathrm{Tr}(X_{i,j}^* X_{p,q})/d_n$. We first assume $p - i \ge k$ and put $a = \lfloor (p-i-k)/2 \rfloor$. By Lemma 1.3 we have $\|P_n - (\mathrm{id}_{i+k+a} \otimes P_{j-a})(P_p \otimes \mathrm{id}_{k+q})\| \le Dq^{p-(i+k+a)} \le Dq^a$. This yields

$$\begin{aligned} (x_{n;i} \mid x_{n;p}) &= d_n^{-1} \mathrm{Tr}_n[P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)P_n(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)P_n] \\ &\simeq d_n^{-1} \mathrm{Tr}_n[P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_a \otimes P_{j-a})(P_p \otimes X \otimes \mathrm{id}_q)P_n] \\ &= d_n^{-1}(\mathrm{Tr}_{i+k} \otimes \mathrm{Tr}_a \otimes \mathrm{Tr}_{j-a})[(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_a \otimes P_{j-a})(P_p \otimes X \otimes \mathrm{id}_q)P_n]. \end{aligned} \tag{3.1}$$

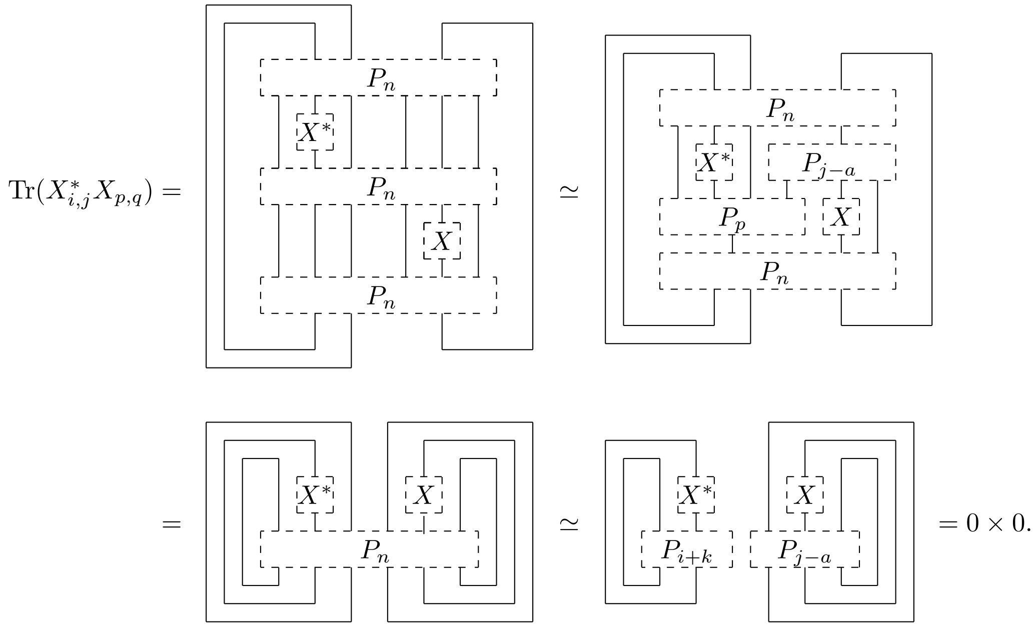
Since $d_n^{-1} \mathrm{Tr}_n(P_n \cdot P_n)$ is a state, the error is bounded by $Dq^a\|X\|^2$. In the last expression, the projection $P_p \otimes \mathrm{id}_k \otimes \mathrm{id}_q$ is absorbed in P_n, and since $j - a \ge k + q$ the partial trace $(\mathrm{id}_{i+k} \otimes \mathrm{Tr}_a \otimes \mathrm{id}_{j-a})(P_n)$ appears. We know from Lemma 1.6 that this partial trace is equal to a multiple λ of the identity up to $Ed_a q^{\lfloor \beta a \rfloor}$ if $q \in \,]0,q_0]$, for some $\beta \in \,]0,1[$ and $E > 0$ depending only on q_0. Applying the remaining traces and dividing by d_n the total error is controlled by

$$\begin{aligned} Dq^a\|X\|^2 + Eq^{\lfloor \beta a \rfloor}\frac{d_{i+k}d_a d_{j-a}}{d_n}\|X\|^2 &\le q^{\lfloor \beta a \rfloor}(D + E/(1-q_0^2)^3)\|X\|_2^2 \\ &\le Cq^{\alpha(p-i-k)-2-k}\|x\|_2^2, \end{aligned}$$

for $C = [D + E/(1-q_0^2)^3]/(1-q_0)$ and $\alpha = \beta/2$ — recall that $\|X\|_2^2 = d_k\|x\|_2^2 \le q^{-k}\|x\|_2^2/(1-q)$. But if we replace $(\mathrm{id}_{i+k} \otimes \mathrm{Tr}_a \otimes \mathrm{id}_{j-a})(P_n)$ by $\lambda(P_{i+k} \otimes P_{j-a})$ in (3.1) we can see the trace $\mathrm{Tr}((\mathrm{id}_i \otimes X^*)P_{i+k})$ which vanishes (as well as $\mathrm{Tr}((\mathrm{id}_{a'} \otimes X \otimes \mathrm{id}_q)P_{j-a})$, where $a' = \lceil (p-i-k)/2 \rceil$).

This proves the result if $p-i \geq k$. If $i-p \geq k$ we can proceed in the same way "on the other side" and the result follows because then $q-j=|p-i| \geq k$. □

We give below a graphical version of the above proof, for the convenience of the reader, in the case $p-i \geq k$. Of course it is still necessary to carry out the quantitative bookkeeping of approximations, as we deed above. It is possible to draw similar graphical computations for many lemmata in this section and the following ones.



Proposition 3.5. *Fix $q \in \,]0,1[$ and assume that $q \in \,]0,q_0]$. There exists a constant $C>0$, depending on k and q_0, such that $\|G_n\| \leq C\|x\|_2^2$ for all n. In particular $G(x)$ is bounded.*

Proof. Take the constants α, C provided by Lemma 3.4. Put $l = k + \lceil (2+k)/\alpha \rceil$, so that $\alpha k + 2 + k \leq \alpha l$, and decompose $G_n = \hat{G}_n + \check{G}_n$, where $\hat{G}_{n;i,p} = \delta_{|i-p|\leq l} G_{n;i,p}$. From Lemma 3.4 we have $|\check{G}_{n;i,p}| \leq C q^{\alpha(|p-i|-l)} \|x\|_2^2$ and it is then a standard fact that $\check{G}$ is bounded. More precisely for any $\lambda \in \ell^2(\mathbb{N})$ we have by Cauchy-Schwarz

$$\begin{aligned}\Big|\textstyle\sum_{i,p} \bar{\lambda}_i \lambda_p \check{G}_{n;i,p}\Big| &\leq (\textstyle\sum_{i,p} |\lambda_i|^2 |\check{G}_{n;i,p}|)^{1/2} (\textstyle\sum_{i,p} |\lambda_p|^2 |\check{G}_{n;i,p}|)^{1/2} \\ &\leq C\|x\|_2^2 \textstyle\sum_i |\lambda_i|^2 \textstyle\sum_{|p-i|>l} q^{\alpha(|p-i|-l)} \leq \dfrac{2q^\alpha C\|x\|_2^2}{1-q^\alpha} \|\lambda\|^2.\end{aligned}$$

This shows that $\|\check{G}_n\| \leq 2q_0^\alpha C \|x\|_2^2/(1-q_0^\alpha)$ for all n and $q \in \,]0,q_0]$.

On the other hand by Lemma 3.3 we have $|G_{n;i,p}| \leq (1-q_0^2)^{-3}\|x\|_2^2$ for all n, i, p and it follows easily $\|\hat{G}_n\| \leq (2l+1)(1-q_0^2)^{-3}\|x\|_2^2$. □

Now we want to prove that G has a bounded inverse and obtain uniform estimates with respect to k. This requires a finer analysis of the band matrices $\hat{G}_n$ of the previous proof. We first show that for $m<n$ the diagonal blocks of size $m-k+1$ of G_n "resemble" G_m, with a better approximation order for blocks that are far away from the "borders" of G_n. This will allow to reduce the analysis of $\hat{G}_n$ to that of a "fixed size" matrix G_m (in fact m will depend on k, but not on n).

Lemma 3.6. *Fix $q_0 \in \,]0,1[$ and assume that $q \in \,]0,q_0]$. Then there exist a constant C depending only on q_0 such that, writing $n = m+a+b$ and $m = i+k+j = p+k+q$ we have:*

$$|G_{m;i,p} - G_{n;i+a,p+a}| \leq \begin{cases} C\|x\|_2^2 \, q^{\max(j,q)-k} & \text{if } a=0, \\ C\|x\|_2^2 \, q^{\max(i,p)-k} & \text{if } b=0. \end{cases}$$

We also have $|G_{m;i,p} - G_{n;i+a,p+a}| \leq C\|x\|_2^2 \, q^{\min(i,j,p,q)-k}$ for a, b arbitrary.

Proof. By definition we have $n = (i+a)+k+(j+b) = (p+a)+k+(q+b)$. The case $b=0$ follows from the case $a=0$ by symmetry. The "general case" follows from the first two cases by going first from m to $n' = m+b$ and then from n' to $n = n'+a$, and observing that $Cq^{\max(j,q)} + Cq^{\max(i,p)} \leq 2Cq^{\min(i,j,p,q)}$. So we assume $a=0$.

According to Lemma 1.3 we have $\|P_n - (\mathrm{id}_{i+k}\otimes P_{j+b})(P_m\otimes \mathrm{id}_b)\| \leq Cq^j$, which yields

$$\begin{aligned}(x_{n;i} \mid x_{n;p}) &= d_n^{-1}\operatorname{Tr}_n[P_n(\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j+b})P_n(\mathrm{id}_p\otimes X\otimes \mathrm{id}_{q+b})P_n] \\ &\simeq d_n^{-1}\operatorname{Tr}_n[P_n(\mathrm{id}_i\otimes X^*\otimes P_{j+b})(P_m\otimes \mathrm{id}_b)(\mathrm{id}_p\otimes X\otimes \mathrm{id}_{q+b})P_n] \\ &= d_n^{-1}(\operatorname{Tr}_m\otimes \operatorname{Tr}_b)[P_n(\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j+b})(P_m\otimes \mathrm{id}_b)(\mathrm{id}_p\otimes X\otimes \mathrm{id}_{q+b})]\end{aligned}$$

up to $Cq^j\|X\|^2 \leq Cq^j d_k\|x\|_2^2$, since P_{j+b} is absorbed in P_n. Since we have $(\mathrm{id}\otimes\operatorname{Tr}_b)(P_n) = (d_n/d_m)P_m$ this reads

$$(x_{n;i} \mid x_{n;p}) \simeq d_m^{-1}\operatorname{Tr}_m[P_m(\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)P_m(\mathrm{id}_p\otimes X\otimes \mathrm{id}_q)] = (x_{m;i} \mid x_{m;j})$$

up to $Cq^j d_k\|x\|_2^2 \leq Cq^{j-k}\|x\|_2^2/(1-q_0)$. If $j \leq q$ we proceed in the same way starting with the estimate $P_n \simeq (P_m\otimes \mathrm{id}_b)(\mathrm{id}_{p+k}\otimes P_{q+b})$ up to Cq^q. ◻

In the next Theorem we show that the blocks G_n of the Gram matrix G are related by a recursion formula, which allows at Lemma 3.8 to obtain estimates on G_m with a good behavior as $k\to\infty$, improving the "naive" Lemma 3.3.

Theorem 3.7. *Fix $n > k > 0$ and $x = u_k(X) \in W_k$ with $\rho(X) = \mu X$. For $0 \leq i < n-k$ and $0 \leq p \leq n-k$ we have:*

$$G_{n;i,p} = \delta_{p<n-k}(1-A_p^n)G_{n-1;i,p} + \delta_{p>0}B_p^n G_{n-1;i,p-1} + \delta_{p>1}C_p^n G_{n-1;i,p-2} \text{ where}$$
$$A_p^n = \frac{d_{p+k}d_{p+k-1}}{d_n d_{n-1}},\ B_p^n = 2(-1)^k\operatorname{Re}(\mu)\frac{d_{p+k-1}d_{p-1}}{d_n d_{n-1}},\ C_p^n = -\frac{d_{p-1}d_{p-2}}{d_n d_{n-1}}.$$

Note that $A_p^n = 1$ if $p = n-k$, $B_p^n = 0$ if $p=0$ and $C_p^n = 0$ if $p=0$ or 1, if one puts $d_{-l} = 0$ for $l > 0$. Hence the corresponding terms vanish "naturally" from the recursion equation.

Proof. Record the fact that by assumption $j \geq 1$, but we allow $q = n-p-j = 0$.
Step 1. We have $G_{n;i,p} = d_n^{-1}\operatorname{Tr}_1^{\otimes n}(T)$ with $T = (\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)P_n(\mathrm{id}_p\otimes X\otimes \mathrm{id}_q)P_n$. We will apply Wenzl's recursion relation (1.3) to both occurrences of P_n in this expression. Let us start with the first one, by the adjoint of (1.3) we have $T = \sum_{l=1}^n (-1)^{n-l}(d_{l-1}/d_{n-1})T_l$ where

$$\begin{aligned}T_n &= (\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)(P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_p\otimes X\otimes \mathrm{id}_q)P_n \qquad \text{and} \\ T_l &= (\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)(P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_{n-2}\otimes t) \\ &\qquad (\mathrm{id}_{l-1}\otimes t^*\otimes \mathrm{id}_{n-l-1})(\mathrm{id}_p\otimes X\otimes \mathrm{id}_q)P_n \qquad \text{for } l < n.\end{aligned}$$

Step 2. Denote $M = T_n$. Recall that $(\mathrm{id}\otimes\operatorname{Tr}_1)(P_n) = (d_n/d_{n-1})P_{n-1}$, so that if $q \geq 1$ we have $d_n^{-1}\operatorname{Tr}_1^{\otimes}(M) = d_{n-1}^{-1}\operatorname{Tr}_1^{\otimes n-1}(X_{i,j-1}^* X_{p,q-1}) = G_{n-1;i,p}$. If $q=0$ we have to apply (1.3) to the second occurrence of P_n. This yields $M = \sum_{l=1}^n (-1)^{n-l}\,(d_{l-1}/d_{n-1})M_l$ where

$$\begin{aligned}M_n &= (\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)(P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_p\otimes X)(P_{n-1}\otimes \mathrm{id}_1) \qquad \text{and} \\ M_l &= (\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_j)(P_{n-1}\otimes \mathrm{id}_1)(\mathrm{id}_p\otimes X) \\ &\qquad (\mathrm{id}_{l-1}\otimes t\otimes \mathrm{id}_{n-l-1})(\mathrm{id}_{n-2}\otimes t^*)(P_{n-1}\otimes \mathrm{id}_1) \qquad \text{for } l < n.\end{aligned}$$

In $(\mathrm{id}\otimes\operatorname{Tr}_1)(M_n)$ we can factor $(\mathrm{id}\otimes\operatorname{Tr}_1)(X) = 0$ so this term disappears. Moreover all terms M_l vanish because t hits $X = XP_k$ or P_{n-1}, except M_p. By the conjugate equation we have

$$\begin{aligned}(\mathrm{id}\otimes\operatorname{Tr}_1)(M_p) &= (\mathrm{id}_{n-1}\otimes t^*)(M_p\otimes \mathrm{id}_1)(\mathrm{id}_{n-1}\otimes t) \\ &= (\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_{n-1}\otimes t^*)(\mathrm{id}_p\otimes X\otimes \mathrm{id}_1)(\mathrm{id}_{p-1}\otimes t\otimes \mathrm{id}_{n-p})P_{n-1}\end{aligned}$$

and we recognize $(\mathrm{id}\otimes\operatorname{Tr}_1)(M_p) = (\mathrm{id}_i\otimes X^*\otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_{p-1}\otimes \rho(X))P_{n-1}$. Altogether we can thus write $d_n^{-1}\operatorname{Tr}(M) = \delta_{p<n-k}G_{n-1;i,p} + (-1)^k\delta_{p=n-k}\mu(d_{p-1}/d_n)G_{n-1;i,p-1}$.

Step 3. Now we come back to the terms T_l with $l < n$. Most of them vanish because t^* hits either $X = P_k X$ or P_n. The only remaining terms are $M' := T_{p+k}$, which appears only if $p < n-k$ (i.e. $q \geq 1$), and $M'' := T_p$, which appears if $p \geq 1$. For these terms we apply as well (1.3) to the second occurrence of P_n. This yields $M' = \sum_{l=1}^{n}(-1)^{n-l}(d_{l-1}/d_{n-1})M'_l$ where

$$\begin{aligned} M'_n = {} & (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t) \\ & (\mathrm{id}_{p+k-1} \otimes t^* \otimes \mathrm{id}_{q-1})(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)(P_{n-1} \otimes \mathrm{id}_1) \quad \text{and} \\ M'_l = {} & (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)(\mathrm{id}_{p+k-1} \otimes t^* \otimes \mathrm{id}_{q-1}) \\ & (\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)(\mathrm{id}_{l-1} \otimes t \otimes \mathrm{id}_{n-l-1})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1) \quad \text{for } l < n. \end{aligned}$$

One can simplify $(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_n) = (\mathrm{id}_{n-1} \otimes t^*)(M'_n \otimes \mathrm{id}_1)(\mathrm{id}_{n-1} \otimes t)$ using the conjugate equation:

$$(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_n) = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_{p+k-1} \otimes t^* \otimes \mathrm{id}_q)(\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q+1})(P_{n-1} \otimes t).$$

This vanishes if $q \geq 2$ because in this case t hits P_{n-1}. If $q = 1$ applying once again the conjugate equation we recognize $(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_n) = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q-1})P_{n-1}$ so that $\mathrm{Tr}_1^{\otimes n}(M'_n) = d_{n-1}G_{n-1;i,p}$. Finally we have $d_n^{-1}\,\mathrm{Tr}_1^{\otimes n}(M'_n) = \delta_{p=n-k-1}(d_{p+k}/d_n)G_{n-1;i,p}$.

Again most of the terms M'_l with $l < n$ vanish because the last t hits either $X = XP_k$ or P_{n-1}. The first non-vanishing term, if $p \geq 1$, is M'_p and we recognize $(\mathrm{id}_k \otimes t^*)(\mathrm{id}_1 \otimes X \otimes \mathrm{id}_1)(t \otimes \mathrm{id}_k) = \rho(X) = \mu X$. By the conjugate equation we have $(\mathrm{id} \otimes \mathrm{Tr}_1)(L \otimes tt^*) = L \otimes \mathrm{id}_1$ so that

$$\begin{aligned} (\mathrm{id} \otimes \mathrm{Tr}_1)(M'_p) &= \mu(\mathrm{id} \otimes \mathrm{Tr}_1)[(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t) \\ &\qquad (\mathrm{id}_{p-1} \otimes X \otimes \mathrm{id}_{q-1})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1)] \\ &= \mu(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_{p-1} \otimes X \otimes \mathrm{id}_q)P_{n-1}. \end{aligned}$$

As a result we have $d_n^{-1}\,\mathrm{Tr}_1^{\otimes n}(M'_p) = \mu(d_{n-1}/d_n)G_{n-1;i,p-1}$.

The second non-vanishing term is M'_{p+k}, but it contains $(\mathrm{id}_{k-1} \otimes t^*)(X \otimes \mathrm{id}_1)$ $(\mathrm{id}_{k-1} \otimes t) = (\mathrm{id} \otimes \mathrm{Tr}_1)(X)$ hence it vanishes as well. Finally we have M'_{p+k+1} which appears if $q \geq 2$ and by the conjugate equation can also be written

$$\begin{aligned} M'_{p+k+1} = {} & (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t) \\ & (\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q-2})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1). \end{aligned}$$

As for M'_p we have the further simplification

$$(\mathrm{id} \otimes \mathrm{Tr}_1)(M'_{p+k+1}) = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q-1})P_{n-1}$$

so that $d_n^{-1}\,\mathrm{Tr}_1^{\otimes n}(M'_{p+k+1}) = (d_{n-1}/d_n)G_{n-1;i,p}$.

Step 4. We proceed similarly for M'', writing $M'' = \sum_{l=1}^{n}(-1)^{n-l}(d_{l-1}/d_{n-1})M''_l$ with

$$\begin{aligned} M''_n = {} & (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t) \\ & (\mathrm{id}_{p-1} \otimes t^* \otimes \mathrm{id}_{k+q-1})(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)(P_{n-1} \otimes \mathrm{id}_1) \quad \text{and} \\ M''_l = {} & (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)(\mathrm{id}_{p-1} \otimes t^* \otimes \mathrm{id}_{k+q-1}) \\ & (\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)(\mathrm{id}_{l-1} \otimes t \otimes \mathrm{id}_{n-l-1})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1) \quad \text{for } l < n. \end{aligned}$$

As in the case of M'_n we find

$$(\mathrm{id} \otimes \mathrm{Tr}_1)(M''_n) = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_{p-1} \otimes t^* \otimes \mathrm{id}_{q+k})(\mathrm{id}_p \otimes X \otimes \mathrm{id}_{q+1})(P_{n-1} \otimes t),$$

which vanishes as soon as $q \geq 1$ because t then hits P_{n-1}. If $q = 0$ we recognize $(\mathrm{id} \otimes \mathrm{Tr}_1)$ $(M''_n) = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_{p-1} \otimes \rho^*(X))P_{n-1}$. Altogether we have $d_n^{-1}\,\mathrm{Tr}_1^{\otimes n}(M''_n) = \delta_{p=n-k}\bar{\mu}(d_{n-1}/d_n)G_{n-1;i,p-1}$.

The first non-vanishing term M''_l is M''_{p-1}, if $p \geq 2$, which by the conjugate equation reads $M''_{p-1} = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)(\mathrm{id}_{p-2} \otimes X \otimes \mathrm{id}_q)(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1)$. As for M' it follows $(\mathrm{id} \otimes \mathrm{Tr}_1)(M''_{p-1}) = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_{j-1})P_{n-1}(\mathrm{id}_{p-2} \otimes X \otimes \mathrm{id}_{q+1})P_{n-1}$ and $d_n^{-1}\,\mathrm{Tr}_1^{\otimes n}(M''_{p-1}) = (d_{n-1}/d_n)G_{n-1;i,p-2}$. The second non-vanishing term would be M''_p but it contains $(\mathrm{Tr}_1 \otimes \mathrm{id})(X)$ hence in fact it vanishes. The last term to consider is M''_{p+k} which appears if $q = n-p-k > 0$ and we recognize

$$M''_{p-k} = (\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j)(P_{n-1} \otimes \mathrm{id}_1)(\mathrm{id}_{n-2} \otimes t)$$
$$(\mathrm{id}_{p-1} \otimes \rho^*(X) \otimes \mathrm{id}_{q-1})(\mathrm{id}_{n-2} \otimes t^*)(P_{n-1} \otimes \mathrm{id}_1)$$

which yields as before $d_n^{-1} \mathrm{Tr}_1^{\otimes n}(M''_{p-k}) = \bar\mu(d_{n-1}/d_n) G_{n-1;i,p-1}$.

Step 5. Finally we can collect all terms as follows:

$$G_{n;i,p} = \frac{1}{d_n} \mathrm{Tr}_1^{\otimes n} \Bigg[M + (-1)^{n-p-k}\delta_{p<n-k} \frac{d_{p+k-1}}{d_{n-1}} M'_n + (-1)^k \delta_{n-k>p\geq 1} \frac{d_{p+k-1}d_{p-1}}{d_{n-1}^2} M'_p$$
$$- \delta_{p<n-k-1} \frac{d_{p+k-1}d_{p+k}}{d_{n-1}^2} M'_{p+k+1} + (-1)^{n-p}\delta_{p\geq 1} \frac{d_{p-1}}{d_{n-1}} M''_n$$
$$-\delta_{p\geq 2} \frac{d_{p-1}d_{p-2}}{d_{n-1}^2} M''_{p-1} + (-1)^k \delta_{n-k>p\geq 1} \frac{d_{p-1}d_{p+k-1}}{d_{n-1}^2} M''_{p+k} \Bigg].$$

According to the computations carried above we obtain:

$$G_{n;i,p} = \delta_{p<n-k} G_{n-1;i,p} + (-1)^k \delta_{p=n-k}\mu \frac{d_{p+k-1}d_{p-1}}{d_{n-1}d_n} G_{n-1;i,p-1}$$
$$- \delta_{p=n-k-1} \frac{d_{p+k-1}d_{p+k}}{d_{n-1}d_n} G_{n-1;i,p} + (-1)^k \delta_{n-k>p\geq 1}\mu \frac{d_{p+k-1}d_{p-1}}{d_{n-1}d_n} G_{n-1;i,p-1}$$
$$- \delta_{p<n-k-1} \frac{d_{p+k-1}d_{p+k}}{d_{n-1}d_n} G_{n-1;i,p} + (-1)^k \delta_{p=n-k}\bar\mu \frac{d_{p-1}d_{p+k-1}}{d_n d_{n-1}} G_{n-1;i,p-1}$$
$$- \delta_{p\geq 2} \frac{d_{p-1}d_{p-2}}{d_{n-1}d_n} G_{n-1;i,p-2} + (-1)^k \delta_{n-k>p\geq 1}\bar\mu \frac{d_{p-1}d_{p+k-1}}{d_{n-1}d_n} G_{n-1;i,p-1}.$$

Merging cases together as appropriate this yields the expression in the statement. ☐

Lemma 3.8. *Fix $q_0 \in \,]0,1[$ and assume that $q \in \,]0, q_0]$. Then there exists a constant $C > 0$, depending only on q_0, such that*

$$|G_{m;i,p}| \leq C(m-k+1)q^{k+1}\|x\|_2^2 \qquad \text{and}$$
$$G_{m;p,p} \geq (C^{-1} - C(m-k)q^{k+1})\|x\|_2^2$$

if $x \in W_k$ and $0 \leq i \neq p \leq m-k$.

Proof. Since G_m is symmetric we can assume $i < p \leq m-k$. We have $|\mathrm{Re}(\mu)| \leq 1$ and for $p \leq n-k$ Lemma 1.2 shows that we have $|B_p^n| \leq 2q^{k+1}/(1-q^2)^2$, $|C_p^n| \leq q^{2(k+1)}/(1-q^2)^2$. Since moreover $A_p^n \in [0,1]$, the recursion formula of Theorem 3.7 and Lemma 3.3 imply

$$|G_{m;i,p}| \leq \delta_{p<m-k}|G_{m-1;i,p}| + 3q^{k+1}(1-q^2)^{-5}\|x\|_2^2.$$

We iterate this inequality $m-p-k+1$ times, until $m = p+k$, in which case the first term disappears. This yields the first estimate with $C = 3/(1-q_0^2)^5$.

For the second one, let us start with $G_{m;0,0}$. In the recursion relation of Theorem 3.7 only the first term is non zero when $i = p = 0$. By an easy induction we have thus

$$G_{m;0,0} = G_{k;0,0} \prod_{l=k+1}^{m} (1 - A_0^l) = \|x\|_2^2 \prod_{l=k+1}^{m} \left(1 - \frac{d_k d_{k-1}}{d_l d_{l-1}}\right).$$

Using the explicit expression of the dimensions d_i and the fact that $1-q^{2k} \leq 1-q^{2l}$ if $l \geq k$ we obtain the following lower bound, which depends only on q_0:

$$G_{m;0,0} = \|x\|_2^2 \prod_{l=k+1}^{m} \left(1 - q^{2(l-k)} \frac{(1-q^{2k+2})(1-q^{2k})}{(1-q^{2l+2})(1-q^{2l})}\right)$$
$$\geq \|x\|_2^2 \prod_{l=k+1}^{\infty} \left(1 - q^{2(l-k)}\right) \geq \|x\|_2^2 \prod_{i=1}^{\infty} (1-q_0^{2i}) \geq C^{-1}\|x\|_2^2,$$

increasing C if necessary. Since $\|x\|_2 = \|x^*\|_2$, the same estimate is true for $G_{m;m-k,m-k}$ by Lemma 3.2.

For the other diagonal terms we use again the recursion equation, which yields for $p < m-k$:

$$G_{m;p,p} \geq (1 - A_p^m) G_{m-1;p,p} - 3q^{k+1}(1-q^2)^{-5}\|x\|_2^2.$$

Again we iterate until $m = p+k+1$, obtaining

$$G_{m;p,p} \geq G_{p+k;p,p} \textstyle\prod_{l=p+k+1}^m (1 - A_p^l) - 3(m-p-k)q^{k+1}(1-q^2)^{-5}\|x\|_2^2.$$

The coefficients $1 - A_p^l$ do not appear in the second term since they are dominated by 1. We have already proved above that $\prod_{l=p+k+1}^m (1 - A_p^l) \geq C^{-1}$ (replace k by $k+p$) and so we obtain $G_{m;p,p} \geq C^{-2}\|x\|_2^2 - C(m-k)q^{k+1}\|x\|_2^2$. □

The estimates we have obtained about the "fixed size" matrix G_m will be sufficient for our purposes only in the $q \to 0$ limit. This corresponds to letting $q + q^{-1} = N \to \infty$, and apparently we are thus varying the spaces H_1, H_k. However, let us note that the numbers

$$G_{n;i,p} = d_n^{-1} \operatorname{Tr}_1^{\otimes n}[P_n(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j) P_n(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)]$$

do not really depend on the precise form of the matrix $X \in B(H_k)^{\circ\circ}$, but only on k, $\|x\|_2$ and on the eigenvalue μ of the rotation operator ρ corresponding to X. Indeed, we can expand the projections P_n into linear combination of Temperley-Lieb diagrams π, whose coefficients depend on n, π and the parameter q. Moreover, after this expansion the evaluation of

$$\operatorname{Tr}_1^{\otimes n}[T_\pi(\mathrm{id}_i \otimes X^* \otimes \mathrm{id}_j) T_{\pi'}(\mathrm{id}_p \otimes X \otimes \mathrm{id}_q)]$$

is given by the evaluation of a Temperley-Lieb tangle at X^* and X. Non-vanishing terms necessarily correspond to tangles where strings cannot start and end on the same internal box, and so they are of the form $\operatorname{Tr}_k[\rho^r(X)^*\rho^s(X)] = \mu^{s-k}\|X\|_2^2 = d_k\mu^{s-k}\|x\|_2^2$. As a result $G_{n;i,p}$ can as well be considered as a function of k, μ, $\|x\|_2$ and q. Then it makes sense to take a limit $q \to 0$, and this will allow to prove results "for large N".

The following elementary Lemma, which will be used for the last part of Theorem 3.10, is probably known. Below, and in the proof of Theorem 3.10, we denote $M \preceq B$ the *entrywise* inequality between matrices, and $|M|$ the *entrywise* absolute value.

Lemma 3.9. *For $r \geq 0$, consider the matrix $B_r \in M_n(\mathbb{C})$ given by $B_{r;i,j} = -r^{|i-j|}$ for $i \neq j$ and $B_{r;i,i} = 1$. Then, if $r < \frac{1}{3}$, there exist $s \in]0,1[$ and $C > 0$, independent of n, such that $|B_r^{-1}| \preceq C|B_s|$.*

Proof. We put $A_r = 2 - B_r = (r^{[i-j|})_{i,j} \in M_n(\mathbb{C})$ and we compare with the bi-infinite Toeplitz matrices $\mathcal{A}_r = 2 - \mathcal{B}_r = (r^{[i-j|})_{i,j} \in M_{\mathbb{Z}}(\mathbb{C})$. We will see that $\|\mathcal{A}_r\| < 2$ if $|r| < \frac{1}{3}$. Then we have as well $\|A_r\| < 2$ and we can write

$$B_r^{-1} = \tfrac{1}{2}\textstyle\sum_{k=0}^\infty (\tfrac{1}{2}A_r)^k, \qquad \mathcal{B}_r^{-1} = \tfrac{1}{2}\textstyle\sum_{k=0}^\infty (\tfrac{1}{2}\mathcal{A}_r)^k.$$

Since the coefficients of A_r, $\mathcal{A}_r$ are positive we have $(A_r^k)_{i,j} \leq (\mathcal{A}_r^k)_{i,j}$ and it follows that $(B_r^{-1})_{i,j} \leq (\mathcal{B}_r^{-1})_{i,j}$. Thus it suffices to prove the result for the infinite matrix $\mathcal{B}_r$.

Now of course in the canonical isomorphism $\ell^2(\mathbb{Z}) \simeq L^2(\mathbb{T})$ the matrix $\mathcal{A}_r$ corresponds to multiplication by the Poisson kernel

$$P_r(z) = \textstyle\sum_{k=-\infty}^{+\infty} r^{|k|} z^k = \dfrac{1 - r^2}{1 - rw + r^2},$$

where $z \in \mathbb{T}$ and $w = z + \bar{z}$. As asserted above, we have indeed $\|P_r\|_\infty = (1+r)/(1-r) < 2$ if $r < \frac{1}{3}$. Moreover $\mathcal{B}_r^{-1}$ corresponds to multiplication by

$$(2 - P_r(z))^{-1} = \frac{1 - rw + r^2}{1 - 2rw + 3r^2} = \frac{(1+s^2)(1 - rw + r^2)}{(1-s^2)(1+3r^2)} P_s(w),$$

as an easy calculation shows. Here s is the unique element in $]0,1[$ such that $s + s^{-1} = \frac{3}{2}r + \frac{1}{2}r^{-1} \in]2,+\infty[$. Multiplication by the first factor above corresponds on $\ell^2(\mathbb{Z})$ to a tridiagonal matrix with bounded entries. This preserves the off-diagonal decay of the Toeplitz matrix $\mathcal{A}_s$ associated with P_s and the result is proved. □

Theorem 3.10.

(1) For all $q_0 \in]0,1[$ there exists $C > 0$ such that, assuming $q \le q_0$, we have $\|G_n\| \le C\|x\|_2^2$ for all $x \in W$ and all n.

(2) There exists $q_1 \in]0,1[$ and $D > 0$ such that, assuming $q \le q_1$, we have $\|G_n^{-1}\| \le D\|x\|_2^{-2}$ for all $x \in W$ and all n.

(3) One can choose q_1 and find $\beta > 0$ such that, assuming $q \le q_1$, we have $|(G_n^{-1})_{i,p}| \le Dq^{\beta|p-i|}$ for all n, i, p, $x \in W$.

This shows in particular that $\{x_{i,j} \mid x \in W, i,j \in \mathbb{N}\}$ is a Riesz basis of $H^{\circ\circ}$ if $q \le q_1$, and that the map $\Phi : \ell^2(\mathbb{N}) \otimes H^{\circ\circ} \otimes \ell^2(\mathbb{N}) \to H^\circ$, $\delta_i \otimes x \otimes \delta_j \mapsto x_{i,j}$ from Remark 2.15 is an isomorphism.

Proof. Fix $q_0 \in]0,1[$ and assume $q < q_0$. In this proof C denotes a "generic constant" depending on q_0, that we will only modify a finite number of times. We take the constants $C > 0$ and $\alpha > 0$ of Lemma 3.4 and we fix the "cut-off width" $l = k + \lceil (3+k)/\alpha \rceil$. We will distinguish three regimes for the coefficients of our Gram matrix: the diagonal entries, for which we have the trivial estimate of Lemma 3.3 and the lower bound of Lemma 3.8 ; the entries $G_{n;i,p}$ with $0 < |i-p| < 2l$ for which we have the uniform estimate of Lemma 3.8 with a good behavior as $k \to \infty$; and the entries such that $|i-p| \ge 2l$ for which we have the off-diagonal decay estimate of Lemma 3.4 with a bad behavior as $k \to \infty$.

Recall that Lemma 3.4 shows that $|G_{n;i,p}| \le Cq^{\alpha(|p-i|-k)-2-k}\|x\|_2^2$ if $|p-i| \ge k$, which by definition of l yields $|G_{n;i,p}| \le Cq^{1+\alpha(|p-i|-l)}\|x\|_2^2$. In particular for $|p-i| \ge 2l$ we obtain $|G_{n;i,p}| \le Cq^{1+\alpha|p-i|/2}\|x\|_2^2$.

We then deal with the entries such that $0 < |p-i| < 2l$. First assuming $n > 2l+5k+1$, we approximate each such entry $G_{n;i,p}$ by a corresponding entry $G_{m;i-a,p-a}$ of the smaller matrix G_m with $m-k = 2l+4k+1$, using Lemma 3.6. Write $n = i+k+j = p+k+q = m+a+b$. If i, j, p, $q > 2k$ we can choose a, b such that $i-a$, $p-a$, $j-b$, $q-b \ge 2k+1$ — we can e.g. take $a = \min(i,p) - 2k - 1$, and since $|i-p| < 2l$ we have $i-a < 2l+2k+1$ hence $j-b = 2l+4k-(i-a)+1 > 2k$. We have then

$$|G_{n;i,p} - G_{m;i-a,p-a}| \le Cq^{1+k}\|x\|_2.$$

If $i \le 2k$ or $p \le 2k$ we use the case $a = 0$, we have then $j-b \ge 2k+1$ (resp. $q-b \ge 2k+1$) and the estimate still holds. Similarly if $j \le 2k$ or $q \le 2k$ we use the case $b = 0$.

Now if $i \ne p$ Lemma 3.8 shows that $|G_{m;i-a,p-a}| \le C(m-k+1)q^{k+1}\|x\|_2^2$. Altogether we have obtained the estimate $|G_{n;i,p}| \le C(2l+4k+3)q^{1+k}\|x\|_2^2$ if $0 < |p-i| < 2l$. It holds also if $n \le 2l+5k+1$ by applying directly Lemma 3.8 with $m = n$. Observe moreover that $l \le (1+\alpha^{-1})k+1+3\alpha^{-1} \le 6\alpha^{-1}k$. In particular the sequence $v_k = (2l+4k+2)q_0^{k/2}$ is bounded, hence we can modify C so that $|G_{n;i,p}| \le Cq^{1+k/2}\|x\|_2^2$ for $0 < |p-i| < 2l$. In that case we have $|p-i| < 12\alpha^{-1}k$, hence we have as well $|G_{n;i,p}| \le Cq^{1+\alpha|p-i|/24}\|x\|_2^2$. Merging this with the case $|p-i| \ge 2l$ we have finally $|G_{n;i,p}| \le Cq^{1+\alpha|p-i|/24}\|x\|_2^2$ for all $i \ne p$, where C and α depend only on q_0.

Decompose $G_n = \hat{G}_n + \check{G}_n$, where $\hat{G}_n$ is diagonal with the same diagonal entries as G_n. The previous estimate shows that $\check{G}_n$ is bounded, more precisely for any $\lambda \in \ell^2(\mathbb{N})$ we have by Cauchy-Schwarz

$$\begin{aligned}\left|\textstyle\sum_{i,p}\bar{\lambda}_i\lambda_p\check{G}_{n;i,p}\right| &\le (\textstyle\sum_{i,p}|\lambda_i|^2|\check{G}_{n;i,p}|)^{1/2}(\textstyle\sum_{i,p}|\lambda_p|^2|\check{G}_{n;i,p}|)^{1/2}\\ &\le Cq\|x\|_2^2 \textstyle\sum_i|\lambda_i|^2\textstyle\sum_{|p-i|>1}q^{\alpha|p-i|/24} \le \dfrac{2q^{1+\alpha/24}C\|x\|_2^2}{1-q^{\alpha/24}}\|\lambda\|^2.\end{aligned}$$

This shows that $\|\check{G}_n\| \le Cq\|x\|_2^2$ for all n and x, after dividing C by $2/(1-q_0^{\alpha/24})$. On the other hand we also have $\|\hat{G}_n\| \le \|x\|_2^2/(1-q_0^2)^3$ by Lemma 3.3 and the first assertion is proved.

For the inverse of G, we need a lower bound on the diagonal entries. We proceed as above, approximating each coefficient $G_{n;p,p}$ by a diagonal coefficient $G_{m;p-a,p-a}$ of a smaller matrix

G_m, with $m-k=1+4k$, and either $a=0$, $b=0$, or $p-a=2k+1=q-b$. This yields $|G_{n;p,p}-G_{m;p-a,p-a}|\leq Cq^{k+1}\|x\|_2$. Then we use the lower bound of Lemma 3.8, obtaining

$$G_{n;p,p}\geq C^{-1}\|x\|_2^2-C(m-k+1)q^{k+1}\|x\|_2\geq C^{-1}\|x\|_2^2-Cq(4k+2)q_0^k\|x\|_2.$$

Since the sequence $v_k=(4k+2)q_0^k$ is bounded, we can modify C so as to obtain $G_{n;p,p}\geq(C^{-1}-Cq)\|x\|_2$. For q small enough, $C^{-1}-Cq>0$ and this shows $\hat{G}_n^{-1}\leq(C^{-1}-Cq)^{-1}\|x\|_2^{-2}\,\mathrm{I}$.

Now we write $G_n=(\mathrm{I}+\check{G}_n\hat{G}_n^{-1})\hat{G}_n$. The estimates obtained above show that we have $\|\check{G}_n\hat{G}_n^{-1}\|\leq D(q):=Cq/(C^{-1}-Cq)$. For q_1 small enough and $q\leq q_1$ we have $D(q)\leq D(q_1)<1$ so that G_n is invertible. Moreover we have

$$G_n^{-1}=\hat{G}_n^{-1}\textstyle\sum_{i=0}^{\infty}(-1)^i[\check{G}_n\hat{G}_n^{-1}]^i \tag{3.2}$$

so that $\|G_n^{-1}\|\leq(C^{-1}-Cq_1)^{-1}(1-D(q_1))^{-1}\|x\|_2^{-2}$.

Finally we establish the off-diagonal decay for G^{-1}. Starting back from the entrywise estimate $|G_{n;i,p}|\leq Cq^{1+\alpha|p-i|/24}\|x\|_2^2$, for $i\neq p$, we obtain $|\check{G}_n\hat{G}_n^{-1}|\preceq D(q)|\check{B}_r|$, using the notation of Lemma 3.9 with $r=q^{\alpha/24}$. For $q\leq q_1$ we have $D(q)<1$, the sum in Equation (3.2) converges in operator norm, hence also in the entrywise sense, and it yields

$$\begin{aligned}|G_n^{-1}|\preceq(C^{-1}-Cq)^{-1}\|x\|_2^{-2}\textstyle\sum_{i=0}^{\infty}|\check{B}_r|^i&=(C^{-1}-Cq)^{-1}\|x\|_2^{-2}(I-|\check{B}_r|)^{-1}\\&=(C^{-1}-Cq)^{-1}\|x\|_2^{-2}B_r^{-1}.\end{aligned}$$

If q_1 is small enough we have $r<\frac{1}{3}$ and Lemma 3.9 yields the last statement of the theorem, with $q^\beta=s$. ☐

Remark 3.11. Using the recursion relation of Theorem 3.7 and the symmetry properties of G_n, one can compute G_n by induction on n. Numerical experiments then show the existence, for all $q\in\,]0,1[$, of a constant $C>0$ such that $\|G_n\|\leq C\|x\|_2^2$, $\|G_n^{-1}\|\leq C\|x\|_2^{-2}$ for all n, k, $x\in W_k$. Thus our proof is far from optimal and we strongly believe that the results of Theorem 3.10 hold for all $q\in\,]0,1[$ (with constants depending on q).

4. An Orthogonality Property

Recall from Sections 2 and 3 that we have an isomorphism of normed spaces $\Phi:\ell^2(\mathbb{N})\otimes H^{\circ\circ}\otimes\ell^2(\mathbb{N})\to H^\circ$. In this section we shall establish a crucial asymptotic orthogonality property of the following subspaces:

Notation 4.1. For every $m\in\mathbb{N}$ we consider the following subspace of H°:

$$V_m=\Phi\left(\ell^2(\mathbb{N}_{\geq m})\otimes H^{\circ\circ}\otimes\ell^2(\mathbb{N}_{\geq m})\right)=\overline{\mathrm{Span}}\{x_{i,j}\mid x\in H^{\circ\circ},i,j\geq m\}.$$

In the rest of this section we will prove that for $y\in p_n(H^\circ)\subset M$ the scalar product $(\zeta y\mid y\zeta)$ becomes small, uniformly on unit vectors $\zeta\in V_m$, as $m\to\infty$, cf Theorem 4.9. We start by computations in $\mathrm{Corep}(\mathbb{F}O_N)$ which culminate in the "local estimate" of Theorem 4.5. In these computations $x=u_k(X)$ is a fixed element of $p_kH^{\circ\circ}$, which is not assumed to be an eigenvector of the rotation map ρ. We then assemble the pieces to come back to $H(k)$ and finally H°.

Recall from Section 1 that by Tannaka-Krein duality products $x_{i,j}y$, $yx_{i,j}$ can be computed from the elements $X_{i,j}*_mY$, $Y*_mX_{i,j}\in B(H_m)$ if $x=u_k(X)$ and $y=u_n(Y)$. Recall also that we use the Hilbert-Schmidt norm $\|X\|_2:=\mathrm{Tr}(X^*X)^{1/2}$ on $B(H_k)$. We have $\|AXB\|_2\leq\|A\|\|X\|_2\|B\|$, where $\|A\|$, $\|B\|$ are the operator norms of A, $B\in B(H_k)$. This yields for instance the inequality $\|X_{i,j}*_mY\|_2\leq d_a\|X_{i,j}\|_2\|Y\|_2$, where $m=i+k+j+n-2a$, and we recall moreover that $\|X_{i,j}\|_2\leq\sqrt{d_id_j}\|X\|_2$, see e.g. the proof of Lemma 3.3. We still make repeated use of Lemma 1.2 which allows to replace d_l with q^{-l} up to multiplicative constants.

Lemma 4.2. *Fix k, $n\in\mathbb{N}$ and $X\in B(H_k)^{\circ\circ}$, $Y\in B(H_n)^\circ$. Then for all $i\geq n$, $r\in\mathbb{N}$, $j\geq 2r$, $m=n+i+k+j-2a$ with $0\leq a\leq n$, there exists $Z\in B(H_{m-2r})^\circ$ such that*

$$\|Y*_mX_{i,j}-P_m(Z\otimes\mathrm{id}_{2r})P_m\|_2\leq Cd_aq^{i+k+j-a-2r}\sqrt{d_id_j}\|X\|_2\|Y\|_2,$$

where C is a constant depending only on q, and $\|Z\|_2\leq d_a\sqrt{d_id_{j-2r}}\|X\|_2\|Y\|_2$.

Proof. We have by definition

$$Y *_m X_{i,j} = P_m(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a})(\mathrm{id}_n \otimes P_{i+k+j})(Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j)$$
$$(\mathrm{id}_n \otimes P_{i+k+j})(\mathrm{id}_{n-a} \otimes t_a \otimes \mathrm{id}_{i+k+j-a})P_m.$$

We use the estimate from Lemma 1.3 as follows: $P_{i+k+j} \simeq (\mathrm{id}_a \otimes P_{i+k+j-a})\ (P_{i+k+j-2r} \otimes \mathrm{id}_{2r})$ up to $Cq^{i+k+j-a-2r}$ in operator norm. Since $(\mathrm{id}_{n-a} \otimes P_{i+k+j-a})$ is absorbed by P_m we can write

$$Y *_m X_{i,j} \simeq P_m(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a})(\mathrm{id}_n \otimes P_{i+k+j-2r} \otimes \mathrm{id}_{2r})(Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j)$$
$$(\mathrm{id}_n \otimes P_{i+k+j-2r} \otimes \mathrm{id}_{2r})(\mathrm{id}_{n-a} \otimes t_a \otimes \mathrm{id}_{i+k+j-a})P_m$$

up to $2C\|t_a\|^2 q^{i+k+j-a-2r}\|Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_j\|_2 = 2Cd_a q^{i+k+j-a-2r}\sqrt{d_i d_j}\|X\|_2\|Y\|_2$ in HS norm.

This yields the result with

$$Z = P_{m-2r}(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a-2r})(\mathrm{id}_n \otimes P_{i+k+j-2r})(Y \otimes \mathrm{id}_i \otimes X \otimes \mathrm{id}_{j-2r})$$
$$(\mathrm{id}_n \otimes P_{i+k+j-2r})(\mathrm{id}_{n-a} \otimes t_a^* \otimes \mathrm{id}_{i+k+j-a-2r})P_{m-2r}$$

which satisfies the right norm estimate. Note that we have $Z = Y *_{m-2r} X_{i,j-2r}$. □

Lemma 4.3. *Fix k, $n \in \mathbb{N}$ and $X \in B(H_k)^{\circ\circ}$, $Y \in B(H_n)^{\circ}$. Then for all $i \geq n$, $p \in \mathbb{N}$, $j \geq 2n+3p$, $m = n+i+k+j-2a$ with $0 \leq a \leq n$, we have*

$$\|(\mathrm{id} \otimes \mathrm{Tr}_{n+2p})(X_{i,j} *_m Y)\|_2 \leq Cq^{\alpha p}q^{-p}q^{-a}q^{-(i+j+n)/2}\|X\|_2\|Y\|_2,$$

where $C > 0$, $\alpha \in]0,1[$ are constants depending only on q.

Proof. In this proof C denotes a generic constant, depending only on q and that we will modify only a finite number of times.

We write $\mathrm{Tr}_{n+2p} = (\mathrm{Tr}_p \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a})\ (P_{n+2p} \cdot P_{n+2p})$. Applying this to $X_{i,j} *_m Y$, the projections $\mathrm{id} \otimes P_{n+2p}$ are absorbed in P_m:

$$(\mathrm{id} \otimes \mathrm{Tr}_{n+2p})(X_{i,j} *_m Y) = (\mathrm{id}_{m-2p-n} \otimes \mathrm{Tr}_p \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a})[$$
$$P_m(\mathrm{id}_{m-n+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})(X_{i,j} \otimes Y)(\mathrm{id}_{m-n+a} \otimes t_a \otimes \mathrm{id}_{n-a})P_m]$$

We shall proceed to three successive approximations to show that this quantity is almost zero.

We first use the estimate $P_m \simeq (\mathrm{id}_{m-p-n} \otimes P_{p+n})(P_{m-n+a} \otimes \mathrm{id}_{n-a})$ up to Cq^{p+a} in operator norm, from Lemma 1.3. The projections $P_{m-n+a} \otimes \mathrm{id}$ are absorbed by $X_{i,j}$ so that

$$P_m(\mathrm{id}_{m-n+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})(X_{i,j} \otimes Y)(\mathrm{id}_{m-n+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})P_m \simeq$$
$$\simeq (\mathrm{id}_{m-p-n} \otimes P_{p+n})(\mathrm{id}_{m-n+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})$$
$$(X_{i,j} \otimes Y)(\mathrm{id}_{m-n+a} \otimes t_a \otimes \mathrm{id}_{n-a})(\mathrm{id}_{m-p-n} \otimes P_{p+n}),$$

with an error controlled by $2Cq^{p+a}\|t_a\|^2\|X_{i,j} \otimes Y\|_2 \leq 2Cq^{p+a}d_a\sqrt{d_i d_j}\|X\|_2\|Y\|_2$ in Hilbert-Schmidt norm. Observing that Tr_p hits only $X_{i,j}$, we obtain

$$(\mathrm{id} \otimes \mathrm{Tr}_{2p+n})(X_{i,j} *_m Y) \simeq (\mathrm{id}_{m-2p-n} \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a})[(\mathrm{id}_{m-2p-n} \otimes P_{p+n})$$
$$(\mathrm{id}_{m-n-p+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})(Z \otimes Y)(\mathrm{id}_{m-n-p+a} \otimes t_a \otimes \mathrm{id}_{n-a})(\mathrm{id}_{m-2p-n} \otimes P_{p+n})],$$

where $Z = (\mathrm{id}_{m-2p-n} \otimes \mathrm{Tr}_p \otimes \mathrm{id}_{p+2a})(X_{i,j})$. We denote the right-hand side $\Phi(Z)$, with $\Phi : B(H_{m-2p-n} \otimes H_{p+2a}) \to B(H_{m-2p-n})$. After applying the partial trace $\mathrm{Tr}_p \otimes \mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a}$, see e.g. Lemma 1.5, the error is controlled as follows:

$$\|(\mathrm{id} \otimes \mathrm{Tr}_{2p+n})(X_{i,j} *_m Y) - \Phi(Z)\|_2 \leq 2Cq^{p+a}d_a\sqrt{d_p d_{p+a} d_{n-a} d_i d_j}\|X\|_2\|Y\|_2$$
$$\leq 2Cq^{-(i+j+n)/2}\|X\|_2\|Y\|_2,$$

up to dividing C by the appropriate power of $1/(1-q^2)$, cf Lemma 1.2. This error is less that the upper bounded in the statement.

Then we use the estimate $P_{i+j+k} \simeq (P_{i+k+p} \otimes \mathrm{id}_{j-p})(\mathrm{id}_{i+k} \otimes P_j)$, up to Cq^p in operator norm, in the expression of Z. We have by assumption $j \geq 3p+2a$, and in particular we can write

$$Z \simeq (\mathrm{id}_{i+k+j-2p-2a} \otimes \mathrm{Tr}_p \otimes \mathrm{id}_{p+2a})[(P_{i+k+p} \otimes \mathrm{id}_{j-p})(\mathrm{id}_i \otimes X \otimes P_j)(P_{i+k+p} \otimes \mathrm{id}_{j-p})]$$
$$= (P_{i+k+p} \otimes \mathrm{id}_{j-2p})(\mathrm{id}_i \otimes X \otimes P'_j)(P_{i+k+p} \otimes \mathrm{id}_{j-2p}) =: Z'$$

where $P'_j = (\mathrm{id}_{j-2p-2a} \otimes \mathrm{Tr}_p \otimes \mathrm{id}_{p+2a})(P_j) \in B(H_{j-2p-2a} \otimes H_{p+2a})$. The error in Z is controlled in HS norm by $2Cq^p\sqrt{d_p}\|P_i \otimes X \otimes P_j\|_2 = 2Cq^p\sqrt{d_p d_i d_j}\|X\|_2$, so that

$$\begin{aligned}\|\Phi(Z) - \Phi(Z')\|_2 &\leq 2Cq^p d_a \sqrt{d_p d_{p+a} d_{n-a} d_i d_j}\|X\|_2\|Y\|_2 \\ &\leq 2Cq^{-a} q^{-(i+j+n)/2}\|X\|_2\|Y\|_2.\end{aligned}$$

Again this is better than the estimate we are trying to prove.

Now Lemma 1.6 shows that $P'_j \simeq \lambda(\mathrm{id}_{j-2p-2a} \otimes \mathrm{id}_{p+2a})$ up to $Cq^{\alpha p}d_p$ in operator norm, for some constant λ depending on all parameters (and $\alpha > 0$ depending only on q). In HS norm we can control this error by $Cq^{\alpha p}d_p\sqrt{d_{j-2p-2a}d_{p+2a}}$. This yields

$$Z' \simeq Z'' := \lambda[(P_{i+k+p} \otimes \mathrm{id}_{j-3p-2a})(\mathrm{id}_i \otimes X \otimes \mathrm{id}_{j-2p-2a})(P_{i+k+p} \otimes \mathrm{id}_{j-3p-2a})] \otimes \mathrm{id}_{p+2a},$$

and we have the control

$$\begin{aligned}\|\Phi(Z') - \Phi(Z'')\|_2 &\leq Cq^{\alpha p} d_a d_p \sqrt{d_i d_{j-2p-2a} d_{p+2a} d_{p+a} d_{n-a}}\|X\|_2\|Y\|_2 \\ &\leq Cq^{\alpha p} q^{-a} q^{-p} q^{-(i+j+n)/2}\|X\|_2\|Y\|_2,\end{aligned}$$

which corresponds to the estimate in the statement.

We finally arrived at

$$\begin{aligned}\Phi(Z'') = \lambda&(P_{i+k+p} \otimes \mathrm{id}_{j-3p-2a})(\mathrm{id}_i \otimes X \otimes \mathrm{id}_{j-2p-2a})(P_{i+k+p} \otimes \mathrm{id}_{j-3p-2a}) \times \\ &\times (\mathrm{Tr}_{p+a} \otimes \mathrm{Tr}_{n-a})[(\mathrm{id}_{p+a} \otimes t_a^* \otimes \mathrm{id}_{n-a})(\mathrm{id}_{p+2a} \otimes Y)(\mathrm{id}_{p+a} \otimes t_a \otimes \mathrm{id}_{n-a})P_{p+n}].\end{aligned}$$

We claim that the second line above vanishes. Indeed $(\mathrm{Tr}_{p+a} \otimes \mathrm{id}_{n-a})(P_{p+n})$ is a multiple of id_{n-a}, since it is an intertwiner of H_{n-a}. We are then left with

$$\mathrm{Tr}_{n-a}[(t_a^* \otimes \mathrm{id}_{n-a})(\mathrm{id}_a \otimes Y)(t_a \otimes \mathrm{id}_{n-a})] = (\mathrm{Tr}_a \otimes \mathrm{Tr}_{n-a})(Y),$$

which vanishes because $y \in p_n H^\circ$. Hence $\Phi(Z'') = 0$ and the result is proved. □

Lemma 4.4. *For $Z \in B(H_{m-2r})$, $S = P_m(Z \otimes \mathrm{id}_{2r})P_m$ and $T \in B(H_m)$ we have*

$$|(S \mid T)| \leq \sqrt{d_r}\|(\mathrm{id} \otimes \mathrm{Tr}_r)(T)\|_2\|Z\|_2 + C\|Z\|_2\|T\|_2,$$

for some constant C depending only on q.

Proof. Recall once again from Lemma 1.3 that $P_m \simeq (P_{m-r} \otimes \mathrm{id}_r)(\mathrm{id}_{m-2r} \otimes P_{2r})$ up to Cq^r, where C is a constant depending only on q. Since $T(\mathrm{id}_{m-2r} \otimes P_{2r}) = T = P_m T$ we have

$$\begin{aligned}(S \mid T) &= \mathrm{Tr}_m(P_m(Z^* \otimes \mathrm{id}_{2r})P_m T) \\ &\simeq (\mathrm{Tr}_{m-2r} \otimes \mathrm{Tr}_{2r})((\mathrm{id}_{m-2r} \otimes P_{2r})(P_{m-r} \otimes \mathrm{id}_r)(Z^* \otimes \mathrm{id}_{2r})T) \\ &= (\mathrm{Tr}_{m-2r} \otimes \mathrm{Tr}_r \otimes \mathrm{Tr}_r)((P_{m-r} \otimes \mathrm{id}_r)(Z^* \otimes \mathrm{id}_r \otimes \mathrm{id}_r)T) \\ &= \mathrm{Tr}_{m-r}[P_{m-r}(Z^* \otimes \mathrm{id}_r)P_{m-r}(\mathrm{id} \otimes \mathrm{Tr}_r)(T)].\end{aligned}$$

By Cauchy-Schwarz the last quantity is dominated by $\sqrt{d_r}\|Z\|_2\|(\mathrm{id} \otimes \mathrm{Tr}_r)(T)\|_2$. Moreover the error term in the second line is similarly bounded by $Cq^r\|Z^* \otimes \mathrm{id}_r \otimes \mathrm{id}_r\|_2\|T\|_2 = Cq^r\sqrt{d_r d_r}\|Z\|_2 \|T\|_2 \leq C'\|Z\|_2\|T\|_2$. □

Theorem 4.5. *Fix k, k', $n \in \mathbb{N}$ and $X \in B(H_k)^{\circ\circ}$, $X' \in B(H_{k'})^{\circ\circ}$, $Y \in B(H_n)^\circ$. Then for all i, j, i', $j' \geq 10n$ and $m = n + i + k + j - 2a = n + i' + k' + j' - 2a'$ with $0 \leq a, a' \leq n$, we have*

$$|(X_{i,j} *_m Y \mid Y *_m X'_{i',j'})| \leq Cd_m(q^{\alpha(i'+j')} + q^{\alpha\min(j,j')})q^{(k+k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2,$$

where $\alpha > 0$ is a constant depending only on q, and C is a constant depending on q and n.

Proof. We put $p = \lfloor \min(j,j')/10 \rfloor - n$ and $r = n + 2p$. Thank to the assumption on j, j' we have $m \geq 2r$. We first apply Lemma 4.2 to find $Z \in B(H_{m-2r})^\circ$ such that $\|Y *_m X'_{i',j'} - S\|_2 \leq Cq^{i'+k'+j'-a'-2r}d_{a'}\sqrt{d_{i'}d_{j'}}\|X'\|_2\|Y\|_2$ with $S = P_m(Z \otimes \mathrm{id}_{2r})P_m$. The condition $j' \geq 2r$ is satisfied since $p \leq \frac{1}{10}j' - n$. We have then

$$|(X_{i,j} *_m Y \mid Y *_m X'_{i',j'})| \leq |(X_{i,j} *_m Y \mid S)| + \|X_{i,j} *_m Y\|_2\|Y *_m X'_{i',j'} - S\|_2.$$

Note that $\|X_{i,j} *_m Y\|_2 \le d_a\sqrt{d_i d_j}\|X\|_2\|Y\|_2$, and since $2r \le j'/2$ we have

$$\begin{aligned}\|X_{i,j} *_m Y\|_2\|Y *_m X'_{i',j'} - S\|_2 &\le Cq^{i'+k'+j'-a'-2r} d_a d_{a'}\sqrt{d_i d_j d_{i'} d_{j'}}\|X\|_2\|X'\|_2\|Y\|_2^2\\ &\le Cq^{i'/2}q^{-(i+j)/2}q^{k'}q^{-a-2a'}\|X\|_2\|X'\|_2\|Y\|_2^2\\ &\le C_n d_m q^{i'+j'/2}q^{(k+3k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2,\end{aligned}$$

were C_n is a constant depending on n and q. We apply then Lemma 4.4 to $T = X_{i,j} *_m Y$ and our S. This yields

$$|(X_{i,j} *_m Y \mid S)| \le \sqrt{d_r}\|(\mathrm{id}\otimes\mathrm{Tr}_r)(X_{i,j} *_m Y)\|_2\|Z\|_2 + C\|Z\|_2\|X_{i,j} *_m Y\|_2.$$

Lemma 4.2 also provides a bound on $\|Z\|_2$, in particular the second term on the right-hand side above is bounded by

$$\begin{aligned}Cd_{a'}d_a\sqrt{d_i d_j d_{i'} d_{j'-2r}}\|X\|_2\|X'\|_2\|Y\|_2^2 &\le Cq^{-a-a'}q^{-(i+j+i'+j')/2}q^r\|X\|_2\|X'\|_2\|Y\|_2^2\\ &\le C'_n d_m q^{\frac{1}{5}\min(j,j')}q^{(k+k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2,\end{aligned}$$

since we have $r \ge \frac{1}{5}\min(j,j') - n - 2$.

We finally apply Lemma 4.3. Again the condition $j \ge 2n+3p$ is satisfied because $p \le \frac{1}{10}j - n$. This yields

$$\begin{aligned}\sqrt{d_r}\|(\mathrm{id}\otimes\mathrm{Tr}_r)(T)\|_2\|Z\|_2 &\le C\sqrt{d_r}q^{\alpha_0 p}q^{-p}q^{-a}q^{-(i+j+n)/2}d_{a'}\sqrt{d_{i'}d_{j'-2r}}\|X\|_2\|X'\|_2\|Y\|_2^2\\ &\le Cq^{\alpha_0 p}q^{-p+r/2}q^{-a-a'}q^{-(i+j+n)/2}q^{-(i'+j')/2}\|X\|_2\|X'\|_2\|Y\|_2^2\\ &\le C''_n d_m q^{\alpha_0 p}q^{(k+k')/2}\|X\|_2\|X'\|_2\|Y\|_2^2.\end{aligned}$$

Since $p \ge \frac{1}{10}\min(j,j') - n - 1$, this yields the result, with $\alpha = \min(\alpha_0/10, 1/5)$. ☐

Corollary 4.6. *Fix k, k', $n \in \mathbb{N}$ and $x \in p_k H^{\circ\circ}$, $x' \in p_{k'}H^{\circ\circ}$, $y \in p_n H^\circ$. Then for i, i', j, $j' \ge 10n$ we have*

$$|(x_{i,j}y \mid yx'_{i',j'})| \le C(q^{\alpha(i+j)} + q^{\alpha(i'+j')} + q^{\alpha\max(\min(i,i'),\min(j,j'))})\|x\|_2\|x'\|_2,$$

where $\alpha > 0$ is a constant depending only on q, and C is a constant depending on q, n and y.

Proof. We have $x = u_k(X)$, $x' = u_{k'}(X')$, $y = u_n(Y)$ with $X \in B(H_k)^{\circ\circ}$, $X' \in B(H_{k'})^{\circ\circ}$, $y \in B(H_n)^\circ$. Recall from Remark 2.8 that we have then $x_{i,j} = u_{i+k+j}(X_{i,j})$. Following the reminder in Section 1 — specifically Equation (1.4) and Notation 1.1 — we obtain $x_{i,j}y = \sum_{a=0}^n(\kappa_m^{i+k+j,n})^2 u_m(X_{i,j} *_m Y)$, where $m = i+k+j+n-2a$ as usual. The same holds for $yx'_{i',j'}$, and the Peter–Weyl–Woronowicz Equation (1.1) yields

$$(x_{i,j}y \mid yx'_{i',j'}) = \sum_{a=0}^n \frac{1}{d_m}\left(\kappa_m^{i+k+j,n}\kappa_m^{n,i'+k'+j'}\right)^2(X_{i,j} *_m Y \mid Y *_m X'_{i',j'}).$$

According to Lemma 1.4, the constants κ are uniformly bounded by a constant depending only on q. Applying Theorem 4.5 and noticing that $q^{k/2}\|X\|_2 = q^{k/2}\sqrt{d_k}\|x\|_2 \le C\|x\|_2$ we obtain

$$|(x_{i,j}y \mid yx'_{i',j'})| \le C(q^{\alpha(i'+j')} + q^{\alpha\min(j,j')})\|x\|_2\|x'\|_2.$$

The estimate in the statement follows from this one by symmetry. Indeed, by switching left and right in Lemmata 4.2, 4.3 and 4.4, we obtain $|(x_{i,j}y \mid yx'_{i',j'})| \le C(q^{\alpha(i+j)} + q^{\alpha\min(i,i')})$. More precisely, recall that the antipode S is isometric on $\ell^2(\mathbb{\Gamma})$ in the Kac case, and observe that $S(\chi_i) = \chi_i$, so that $S(x_{i,j}) = S(x)_{j,i}$. Applying the first part of this proof we thus get

$$\begin{aligned}|(x_{i,j}y \mid yx'_{i',j'})| = |(yx'_{i',j'} \mid x_{i,j}y)| &= |(S(x')_{j',i'}S(y) \mid S(y)S(x)_{j,i})|\\ &\le C(q^{\alpha(i+j)} + q^{\alpha\min(i',i)})\|x\|_2\|x'\|_2.\end{aligned}$$

Taking the best of this estimate and the previous one yields the result. ☐

To pass from the "local" result of Corollary 4.6 to the "global" results of Proposition 4.8 and Theorem 4.9 we will need to analyze the kernel appearing on the right-hand side in Corollary 4.6. We state separately the following elementary lemma which will be useful for this purpose.

Lemma 4.7. *Let A, $B \in \ell^2(\mathbb{N}\times\mathbb{N})$ and put $q_{p;i,k} = q^{\max(\min(i,k),\min(p-i,p-k))}$. Then there exists a constant $C > 0$ depending only on q such that*

$$\sum_{i\geq 0}\sum_{k\geq 0}\sum_{p\geq i,k} q_{p;i,k}|A_{i,p-i}B_{k,p-k}| \leq C\|A\|_2\|B\|_2.$$

Proof. Denote $T = \{(i,k,p)\in\mathbb{N}^3 \mid p\geq i, p\geq k\}$. We start by applying Cauchy-Schwarz:

$$\left(\sum_T q_{p;i,k}|A_{i,p-i}B_{k,p-k}|\right)^2 \leq \sum_T q_{p;i,k}|A_{i,p-i}|^2 \;\times\; \sum_T q_{p;i,k}|B_{k,p-k}|^2.$$

By the symmetry in i and k it suffices to prove that $\sum_T q_{p;i,k}|A_{i,p-i}|^2 \leq C\|A\|_2^2$, which we can also write $\sum_{i=0}^\infty\sum_{p=i}^\infty s_{p;i}|A_{i,p-i}|^2 \leq C\|A\|_2^2$ with $s_{p;i} := \sum_{k=0}^p q_{p;i,k}$. This holds for all A if and only if $s_{p;i}$ is bounded independently of i and p. Since $q_{p;i,k} = q_{p;p-i,p-k}$ we have $s_{p;i} = s_{p;p-i}$, thus we can assume $0\leq i\leq p-i$. We write then

$$\begin{aligned} s_{p;i} = \sum_{k=0}^p q_{p;i,k} &= \left(\sum_{k=0}^{i-1} + \sum_{k=i}^{p-i} + \sum_{k=p-i+1}^p\right) q_{p;i,k} \\ &= \sum_{k=0}^{i-1} q^{\max(k,p-i)} + \left(\sum_{k=i}^{p-i} + \sum_{k=p-i+1}^p\right) q^{\max(i,p-k)} \\ &= iq^{p-i} + \left(\sum_{k=i}^{p-i} q^{p-k}\right) + iq^i \leq 2\sup_i(iq^i) + \tfrac{1}{1-q}. \end{aligned}$$

□

Recall from Notation 2.11 that for $w\in W$, $k\in\mathbb{N}^*$ we denote $H(w)$ resp. $H(k)$ the closure of AwA resp. AW_kA in H°, where W is our privileged basis of $H^{\circ\circ}$. Recall from Notation 3.1 that we denote $G(w)$ the Gram matrix of the family of vectors $w_{i,j}$, for $w\in W$.

Proposition 4.8. *Fix k, k', $n\in\mathbb{N}^*$ and $y\in p_nH^\circ$. Assume that we have a common upper bound $\|G(w)^{-1}\| \leq D\|w\|_2^{-2}$, $\|G(w)\|\leq D\|w\|_2$ for all $w\in W_k\cup W_{k'}$. Then for any $m\geq 10n$ and $\zeta\in V_m\cap H(k)$, $\zeta'\in V_m\cap H(k')$ we have $|(\zeta y\mid y\zeta')| \leq CDq^{\alpha(m-|k-k'|)}\|\zeta\|\|\zeta'\|$, where $\alpha>0$ is a constant depending only on q, and C is a constant depending on q, n and y.*

Proof. By assumption the map $(w,i,j)\mapsto w_{ij}$ induces a bicontinuous isomorphism between $p_kH^{\circ\circ}\otimes\ell^2(\mathbb{N}\times\mathbb{N})$ and $H(k)$. More precisely, since $AwA\perp Aw'A$ for $w\neq w'$ in W_k, the Gram matrix $G(k)$ of $(w_{i,j})_{i,j,w}$, with $w\in W_k$, i, $j\in\mathbb{N}$, is block diagonal with $G(w)$, $w\in W_k$, as diagonal blocks, and thus it is bounded with bounded inverse by hypothesis. We can in particular decompose $\zeta = \sum_{i,j} x(i,j)_{i,j}$ with $x(i,j)\in p_kH^{\circ\circ}$ and, denoting $x = (x(i,j))_{i,j}$, we have $\|x\|_2^2 = \sum_{i,j}\|x(i,j)\|^2 \leq D\|\zeta\|^2$. Similarly we write $\zeta' = \sum_{i,j} x'(i,j)_{i,j}$ with $x'(i,j)\in p_{k'}H^{\circ\circ}$ and $\|x'\|_2^2\leq D\|\zeta'\|^2$. We have then by Corollary 4.6:

$$\begin{aligned} |(\zeta y\mid y\zeta')| &\leq \sum_{i,j}\sum_{i',j'}|(x(i,j)_{i,j}y\mid yx'(i',j')_{i',j'})| \\ &\leq C\sum_{i,j}\sum_{i',j'}(q^{\alpha(i+j)} + q^{\alpha(i'+j')} + q^{\alpha\max(\min(i,i'),\min(j,j'))})\|x(i,j)\|\|x'(i',j')\| \end{aligned} \tag{4.1}$$

where C depends on q, n and y. Since ζ, $\zeta'\in V_m$ we have $x(i,j) = x'(i',j') = 0$ unless i, j, i', $j'\geq m$. Moreover the scalar product $(x(i,j)_{i,j}y\mid yx'(i',j')_{i',j'})$ vanishes unless $u_{i+k+j}\otimes u_n$ and $u_n\otimes u_{i'+k'+j'}$ have a common subobject, which entails $|i+k+j-i'-k'-j'|\leq 2n$. We remove from (4.1) the terms that do not satisfy these conditions. Moreover we regroup the three powers of q that appear in (4.1) into three distinct sums S_1, S_2, S_3 over i, i', j, j'.

We start with S_3. Denote $p = i+j-2m$, $p' = i'+j'-2m$, $l = p-p'$. Put $\underline{i} = i-m$, $\underline{i}' = i'-m+l$ and note that $j-m = p-\underline{i}$ and $j'-m = p-\underline{i}'$, so that

$$\begin{aligned} \max(\min(i,i'),\min(j,j')) &= \max(\min(\underline{i},\underline{i}'-l),\min(p-\underline{i},p-\underline{i}')) + m \\ &\geq \max(\min(\underline{i},\underline{i}'),\min(p-\underline{i},p-\underline{i}')) + m - |l|. \end{aligned}$$

As a result, $q^{\alpha\max(\min(i,i'),\min(j,j'))} \leq q^{\alpha(m-|l|)}q^\alpha_{p;\underline{i},\underline{i}'}$ using the notation of Lemma 4.7. Reorganizing S_3 we obtain

$$S_3 \leq \sum_l Cq^{\alpha(m-|l|)}\sum_{\underline{i}=0}^\infty\sum_{\underline{i}'=l}^\infty\sum_{p\geq\underline{i},\underline{i}'}\|x(\underline{i}+m,p-\underline{i}+m)\|\|x'(\underline{i}'+m-l,p-\underline{i}'+m)\|\; q^\alpha_{p;\underline{i},\underline{i}'}.$$

Consider one fixed value of l, with $l\geq 0$. By adding vanishing terms to the sum we can assume that the sum over $\underline{i}'$ starts at $\underline{i}' = 0$ and we apply Lemma 4.7 with $A_{r,s} = \|x(r+m,s+m)\|$,

$B_{r,s} = \|x'(r+m-l,s+m)\|$ which satisfy $\|A\|_2 = \|x\|_2$, $\|B\|_2 = \|x'\|_2$. If $l \leq 0$ we would rather put $\underline{i} = i - m - l$, $\underline{i}' = i' - m$ so that $j - m = p' - \underline{i}$ and $j' - m = p' - \underline{i}'$, and sum over $\underline{i} \geq -l \geq 0$, $\underline{i}' \geq 0$, $p' \geq \underline{i}, \underline{i}'$. Observe finally that $|l - (k' - k)| = |i + k + j - i' - k' - j'| \leq 2n$, so that l takes at most $4n + 1$ values and $|l| \leq |k - k'| + 2n$. Lemma 4.7 thus yields the following upper bound:

$$S_3 \leq CC'(4n+1)q^{\alpha(m-2n-|k-k'|)}\|x\|_2\|x'\|_2 \leq C''Dq^{\alpha(m-|k-k'|)}\|\zeta\|\|\zeta'\|$$

with C'' depending on q, n and y.

The case of S_1 (and of S_2) is similar but the counterpart of Lemma 4.7 is simpler. We put $\underline{i} = i - m$, $\underline{j} = j - m$, $\underline{i}' = i' - m$, $\underline{j}' = j' - m$. Observe that for non-vanishing terms in the sum we have $i + j - 2m = \frac{1}{2}(p + p' + l) = \frac{1}{2}(\underline{i} + \underline{j}) + \frac{1}{2}(\underline{i}' + \underline{j}') + \frac{1}{2}l$, and still $|l| \leq |k - k'| + 2n$. This yields, using again Cauchy-Schwarz:

$$\begin{aligned} S_1 &= Cq^{\alpha(2m+l/2)} \sum_{\underline{i},\underline{j} \geq 0} q^{\frac{\alpha}{2}(\underline{i}+\underline{j})}\|x(\underline{i}+m,\underline{j}+m)\| \sum_{\underline{i}',\underline{j}' \geq 0} q^{\frac{\alpha}{2}(\underline{i}'+\underline{j}')}\|x'(\underline{i}'+m,\underline{j}'+m)\| \\ &\leq Cq^{\alpha(2m-\frac{1}{2}|k-k'|-n)}\|x\|_2\|x'\|_2 \textstyle\sum_{\underline{i},\underline{j}} q^{\alpha(\underline{i}+\underline{j})} \leq C'''Dq^{\alpha(m-2|k-k'|)}\|\zeta\|\|\zeta'\|, \end{aligned}$$

with C''' depending on q, n and y. □

Taking into account the finite propagation result established at the end of Section 2 we can finally prove the following global estimate.

Theorem 4.9. *Fix $n \in \mathbb{N}$ and $y \in p_n H^\circ$. Take the constant q_1 given by Theorem 3.10 and assume $q \leq q_1$. Then for any $m \geq 10n$ and $\zeta \in V_m$ we have $|(\zeta y \mid y\zeta)| \leq Cq^{\alpha m}\|\zeta\|^2$, where $\alpha > 0$ is a constant depending only on q, and C is a constant depending on q, n and y.*

Proof. We have the orthogonal decomposition $\zeta = \sum_{k \in \mathbb{N}^*} \zeta_k$ with $\zeta_k \in \overline{AW_kA}$. By Proposition 2.16 we have $\zeta_k y \perp y\zeta_{k'}$ if $|k - k'| > 2n$. Proposition 4.8 applies thank to Theorem 3.10 and the assumption on q. Thus we can write, using Cauchy-Schwarz:

$$\begin{aligned} |(\zeta y \mid y\zeta)| &\leq \textstyle\sum_{|k'-k| \leq 2n} |(\zeta_k y \mid y\zeta_{k'})| \leq Cq^{\alpha(m-2n)} \textstyle\sum_{|k'-k| \leq 2n} \|\zeta_k\|\|\zeta_{k'}\| \\ &\leq Cq^{\alpha(m-2n)} \textstyle\sum_{|k'-k| \leq 2n} \|\zeta_k\|^2 \leq Cq^{\alpha m}q^{-2\alpha n}(4n+1)\|\zeta\|^2. \end{aligned}$$

□

5. A Compactness Property

In this section we are interested in the following projections:

Notation 5.1. We denote $F_m \in B(H^\circ)$ the orthogonal projection onto the closed subspace generated by the vectors $x_{i,j}$ with $x \in H^{\circ\circ}$ and $i < m$.

Our aim is to establish a compactness property for these projections with respect to the left action of A. More precisely, for an arbitrary sequence of unitaries $u_i \in A$ converging weakly to 0, we shall prove that $F_m u_i F_m \in B(H)$ tends to 0 in norm. In the case of the generator MASA $A = a_1'' \subset \mathcal{L}(F_N)$ in a free group factor we have an isomorphism of bimodules $H^\circ \simeq L^2(A) \otimes H^{\circ\circ} \otimes L^2(A)$ where F_m corresponds to $f_m \otimes \mathrm{id} \otimes \mathrm{id}$ with f_m the finite rank projection onto $\mathrm{Span}\{a_1^k \mid |k| < m\}$, so that the result follows from usual compactness in $B(L^2(A))$.

We now fix m, $k \in \mathbb{N}$, $x = u_k(X) \in p_k H^{\circ\circ}$ with $X \in B(H_k)^{\circ\circ}$ an eigenvector of ρ, and we consider the associated vectors $x_{i,j} = p_{i+k+j}(\chi_i x \chi_j)$, $X_{i,j} = P_{i+k+j}(\mathrm{id}_i \otimes X \otimes \mathrm{id}_j)P_{i+k+j}$.

We first obtain an estimate for the scalar products $(x_{c,d} \mid \chi_i x_{a,b})$, where χ_i is the character of the irreducible corepresentation u_i. We denote $2s = a+b+i-c-d$ the "number of simplifications" in the product, with $0 \leq s \leq i$. The proof of the estimate is different in the "large s" and "small s" regimes, see the next Lemmata 5.3 and 5.4. The fist lemma below is used in both regimes.

Lemma 5.2. *There exist constants C, $\beta \in \left]0, \frac{1}{2}\right[$, depending only on q and m, such that for all $a \leq m$ and $l \leq a + k + b$ we have $\|(\mathrm{Tr}_l \otimes \mathrm{id}_{a+k+b-l})(X_{a,b})\|_2 \leq Cq^{-l/2+\beta l}q^{-(a+b)/2}\|X\|_2$.*

Proof. Note that the value $\beta = 0$ corresponds to the trivial estimate provided by Lemma 1.5. In this proof C denotes a "generic constant" depending only on q, that we will modify only a finite number of times. We first assume that $l \geq 3k$ and $l \geq 2m$.

Recall that $X_{a,b} = P_{a+k+b}(\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)P_{a+k+b}$, and decompose $\mathrm{Tr}_l = \mathrm{Tr}_{l_1} \otimes \mathrm{Tr}_{l_2}$ on $B(H_l)$, with $l_1 = \lceil l/2 \rceil + k$ and $l_2 = \lfloor l/2 \rfloor - k$. We have $l_1 \geq a + k$ and we use the estimate $P_{a+k+b} \simeq (P_{l_1} \otimes \mathrm{id}_{a+k+b-l_1})(\mathrm{id}_{a+k} \otimes P_b) =: \tilde{P}$ from Lemma 1.3, with error $E := P_{a+k+b} - \tilde{P}$ in $B(H_{a+k} \otimes H_{l_1-a-k} \otimes H_{a+k+b-l_1})$ such that $\|E\| \leq Cq^{l_1-a-k}$. We can then write

$$\begin{aligned}(\mathrm{Tr}_l \otimes \mathrm{id}_{a+k+b-l})(X_{a,b}) &= (\mathrm{Tr}_{l_1} \otimes \mathrm{Tr}_{l_2} \otimes \mathrm{id})(P_{a+k+b}(\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)P_{a+k+b}) \\ &\simeq (\mathrm{Tr}_{l_1} \otimes \mathrm{Tr}_{l_2} \otimes \mathrm{id})((P_{l_1} \otimes \mathrm{id})(\mathrm{id}_a \otimes X \otimes P_b)(P_{l_1} \otimes \mathrm{id})) \\ &= (\mathrm{Tr}_{l_1} \otimes \mathrm{id})((P_{l_1} \otimes \mathrm{id})(\mathrm{id}_a \otimes X \otimes P'_b)(P_{l_1} \otimes \mathrm{id})) =: Z\end{aligned}$$

where $P'_b = (\mathrm{id}_{l_1-a-k} \otimes \mathrm{Tr}_{l_2} \otimes \mathrm{id}_{a+k+b-l})(P_b)$. Since $\mathrm{Tr}_{l_1} \otimes \mathrm{Tr}_{l_2} \otimes \mathrm{id}$ has operator norm $\sqrt{d_{l_1} d_{l_2}} \leq Cq^{-(l_1+l_2)/2} = Cq^{-l/2}$ with respect to the HS norms, the error is controlled in HS norm as follows:

$$\begin{aligned}\|(\mathrm{Tr}_l \otimes \mathrm{id})(X_{a,b}) - Z\|_2 &\leq Cq^{-l/2}\|P_{a+k+b}(\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)P_{a+k+b} - \tilde{P}(\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)\tilde{P}\|_2 \\ &\leq Cq^{-l/2}\|E(\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)P_{a+k+b} + \tilde{P}(\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)E\|_2 \\ &\leq 2Cq^{-l/2}q^{l_1-a-k}\sqrt{d_a d_b}\|X\|_2 \leq Cq^{-m}q^{-(a+b)/2}\|X\|_2.\end{aligned}$$

Now we use Lemma 1.6 which shows that $P'_b \simeq \lambda(\mathrm{id}_{l_1-a-k} \otimes \mathrm{id}_{a+k+b-l})$ up to $Cq^{\lfloor \alpha l_2 \rfloor} d_{l_2}$ in operator norm, where C, $\alpha \in]0,1[$ depend only on q. This yields

$$\begin{aligned}(\mathrm{Tr}_l \otimes \mathrm{id}_{a+k+b-l})(X_{a,b}) &\simeq \lambda(\mathrm{Tr}_{l_1} \otimes \mathrm{id})((P_{l_1} \otimes \mathrm{id})(\mathrm{id}_a \otimes X \otimes \mathrm{id}_{l_1-a-k} \otimes \mathrm{id}_{a+k+b-l})(P_{l_1} \otimes \mathrm{id})) \\ &= \lambda\,\mathrm{Tr}_{l_1}(P_{l_1}(\mathrm{id}_a \otimes X \otimes \mathrm{id}_{l_1-a-k})P_{l_1}) \otimes \mathrm{id}_{a+k+b-l} = 0,\end{aligned}$$

because for $X \in B(H_k)^{\circ\circ}$ we have $X_{a,l_1-a-k} \in B(H_{l_1})^{\circ}$. Since $l \geq 3k$ we have $l_2 \geq \frac{1}{6}l - 1$ and the new error term is controlled in HS norm as follows:

$$\begin{aligned}\|Z - 0\|_2 &\leq Cq^{\alpha l_2 - 1} d_{l_2}\sqrt{d_{l_1}}\|X\|_2\sqrt{d_a d_{l_1-a-k} d_{a+k+b-l}} \\ &\leq Cq^{\alpha l_2}q^{-l/2}q^{-(a+b)/2}\|X\|_2 \leq Cq^{-l/2+\alpha l/6}q^{-(a+b)/2}\|X\|_2.\end{aligned}$$

This has the right form, and the first approximation was better.

The case when $l \leq 2m$ follow from the trivial estimate by adjusting the constant C (which is allowed to depend on m). To conclude we deal with the case when $m + 1 \leq l \leq 3k$. In this situation we decompose the trace differently by writing $\mathrm{Tr}_l = \mathrm{Tr}_{a+1} \otimes \mathrm{Tr}_{l-a-1}$ on $B(H_l)$. We use again Lemma 1.3 to write $P_{a+k+b} \simeq (P_{a+k} \otimes \mathrm{id}_b)(\mathrm{id}_{a+1} \otimes P_{b+k-1})$ with error $E \in B(H_{a+1} \otimes H_{k-1} \otimes H_b)$ such that $\|E\| \leq Cq^{k-1}$. This yields

$$\begin{aligned}(\mathrm{Tr}_l \otimes \mathrm{id}_{a+k+b-l})(X_{a,b}) &= (\mathrm{Tr}_{a+1} \otimes \mathrm{Tr}_{l-a-1} \otimes \mathrm{id})(P_{a+k+b}(\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)P_{a+k+b}) \\ &\simeq (\mathrm{Tr}_{a+1} \otimes \mathrm{Tr}_{l-a-1} \otimes \mathrm{id})((\mathrm{id}_{a+1} \otimes P_{b+k-1})(P_{a+k} \otimes \mathrm{id}_b) \\ &\qquad (\mathrm{id}_a \otimes X \otimes \mathrm{id}_b)(P_{a+k} \otimes \mathrm{id}_b)(\mathrm{id}_{a+1} \otimes P_{b+k-1})).\end{aligned}$$

We claim that by Lemma 2.17 the last expression vanishes: indeed it can be written in the form $(\mathrm{Tr}_{l-a-1} \otimes \mathrm{id})(P_{b+k-1}(X' \otimes \mathrm{id}_b)P_{b+k-1})$ where $X' = (\mathrm{Tr}_{a+1} \otimes \mathrm{id}_{k-1})(P_{a+k}(\mathrm{id}_a \otimes X)P_{a+k}) = 0$.

The error term above is bounded in HS norm by

$$2Cq^{k-1}\sqrt{d_{a+1}d_{l-a-1}}\sqrt{d_a d_b}\|X\|_2 \leq 2Cq^{-l/2+k}q^{-(a+b)/2}\|X\|_2$$

which has the right form since $k \geq l/3$. □

Lemma 5.3. *There exists constants C, $\beta > 0$, depending only on q and m, such that we have $|(x_{c,d} \mid \chi_i x_{a,b})| \leq Cq^{\beta i}\|x\|_2^2$ as soon as $a, c \leq m$ and $i/2 \leq s \leq i$, where $2s = a + b + i - c - d$.*

Proof. In this proof C denotes a "generic constant" depending only on q and m, that we will modify only a finite number of times. We have $p_{c+k+d}(\chi_i x_{a,b}) = (\kappa^{i,a+k+b}_{c+k+d})^2 u_{c+k+d}(Y)$, where

$$Y = P_{c+k+d}(\mathrm{id}_{i-s} \otimes t_s^* \otimes \mathrm{id}_{a+k+b-s})(P_i \otimes X_{a,b})(\mathrm{id}_{i-s} \otimes t_s \otimes \mathrm{id}_{a+k+b-s})P_{c+k+d}.$$

Lemma 1.3 gives the estimate $P_i \simeq (\mathrm{id}_{i-s} \otimes P_s)(P_{i-\lfloor s/2\rfloor} \otimes \mathrm{id}_{\lfloor s/2\rfloor})$, with an error $E \in B(H_{i-s} \otimes H_{\lceil s/2\rceil} \otimes H_{\lfloor s/2\rfloor})$ such that $\|E\| \leq Cq^{s/2}$, hence $\|E\|_2 \leq Cq^{s/2}\sqrt{d_{i-s}d_{\lceil s/2\rceil}d_{\lfloor s/2\rfloor}}$. Since P_s is absorbed by t_s^* this yields:

$$\begin{aligned}Y &\simeq P_{c+k+d}(\mathrm{id}_{i-s} \otimes t_s^* \otimes \mathrm{id}_{a+k+b-s})(P_{i-\lfloor s/2\rfloor} \otimes \mathrm{id}_{\lfloor s/2\rfloor} \otimes X_{a,b})(\mathrm{id}_{i-s} \otimes t_s \otimes \mathrm{id}_{a+k+b-s})P_{c+k+d} \\ &= P_{c+k+d}(\mathrm{id}_{i-s} \otimes t^*_{\lceil s/2\rceil} \otimes \mathrm{id}_{a+k+b-s})(P_{i-\lfloor s/2\rfloor} \otimes X'_{a,b})(\mathrm{id}_{i-s} \otimes t_{\lceil s/2\rceil} \otimes \mathrm{id}_{a+k+b-s})P_{c+k+d} =: Z,\end{aligned}$$

where $X'_{a,b} = (\mathrm{Tr}_{\lfloor s/2\rfloor} \otimes \mathrm{id}_{a+k+b-\lfloor s/2\rfloor})(X_{a,b})$, using the fact that $t_s^* = t^*_{\lceil s/2\rceil}(\mathrm{id}_{\lceil s/2\rceil} \otimes t^*_{\lfloor s/2\rfloor} \otimes \mathrm{id}_{\lceil s/2\rceil})$ on $H_{\lceil s/2\rceil} \otimes H_{\lfloor s/2\rfloor} \otimes H_s$. Applying Lemma 1.5 to $E \otimes X_{a,b}$ we see that

$$\|Y - Z\|_2 \leq Cq^{s/2}\sqrt{d_{i-s}d_{\lfloor s/2\rfloor}d_{\lceil s/2\rceil}}\|X_{a,b}\|_2 \leq Cq^{s/2}q^{-(a+b)/2}q^{-i/2}\|X\|_2.$$

On the other hand the HS norm of the result of our approximation is controlled by Lemma 5.2. Applying again Lemma 1.5, to $P_{i-\lfloor s/2\rfloor} \otimes X'_{a,b}$, we have

$$\begin{aligned}\|Z\|_2 &\leq \sqrt{d_{i-\lfloor s/2\rfloor}}\|X'_{a,b}\|_2 \leq \sqrt{d_{i-\lfloor s/2\rfloor}}Cq^{-s/4+\beta s/2}q^{-(a+b)/2}\|X\|_2 \\ &\leq Cq^{\beta s/2}q^{-(a+b)/2}q^{-i/2}\|X\|_2,\end{aligned}$$

where C is still a constant depending only on q.

Remember the identity $c + k + d + 2s = i + a + k + b$ and the estimate $\left(\kappa^{i,a+k+b}_{c+k+d}\right)^2 \leq Cd_s^{-1}$ from Lemma 1.4. Altogether we get the estimate

$$\begin{aligned}\|p_{c+k+d}(\chi_i x_{a,b})\|_2 &\leq Cd_s^{-1}d_{c+k+d}^{-1/2}\|Y\|_2 \leq Cq^{(a+k+b+i)/2}\|Y\|_2 \\ &\leq Cq^{(a+k+b+i)/2}q^{\beta s/2}q^{-(a+b+i)/2}\|X\|_2 \\ &\leq Cq^{\beta i/4}q^{k/2}\|X\|_2 \leq Cq^{\beta i/4}\|x\|_2,\end{aligned}$$

since we are in the case $s \geq i/2$. On the other hand we have $\|x_{c,d}\|_2 \leq C\|x\|_2$ by Lemma 3.3, so that the result follows. ☐

Lemma 5.4. *There exists constants C, $\beta > 0$, depending only on q and m, such that we have $|(x_{c,d} \mid \chi_i x_{a,b})| \leq Cq^{\beta i}\|x\|_2^2$ as soon as $a, c \leq m$ and $s \leq i/2$, where $2s = a + b + i - c - d$.*

Proof. Again in this proof C denotes a "generic constant" depending only on q and m, that we will modify only a finite number of times. For the computation of $\chi_i x_{a,b}$ we use the same element $Y \in B(H_{c+k+d})$ as in the proof of the previous lemma. We have more precisely, using the Hilbert-Schmidt scalar product:

$$(x_{c,d} \mid \chi_i x_{a,b}) = \left(\kappa^{i,a+k+b}_{c+k+d}\right)^2 d^{-1}_{c+k+d}(X_{c,d} \mid Y).$$

Our strategy will be to apply Lemma 4.4 to this scalar product.

First we analyze Y, using Lemma 1.3 "on the other side" than in the previous lemma. Putting $r = \lfloor i/8\rfloor$ we have $P_i \simeq (P_{4r} \otimes \mathrm{id}_{i-4r})(\mathrm{id}_{2r} \otimes P_{i-2r})$ with error term $E \in B(H_{2r} \otimes H_{2r} \otimes H_{i-4r})$ dominated by $Cq^{2r}d_{2r}\sqrt{d_{i-4r}}$ in HS norm. Since $4r \leq i/2 \leq i - s$ we can write $(\mathrm{id}_{i-s} \otimes t_s^*)(P_{4r} \otimes \mathrm{id}_{i-4r} \otimes \mathrm{id}_s) = (P_{4r} \otimes \mathrm{id}_{i-s-4r})(\mathrm{id}_{i-s} \otimes t_s^*)$ and since P_{4r} is then absorbed by P_{c+k+d} we obtain

$$\begin{aligned}Y &\simeq P_{c+k+d}(\mathrm{id}_{i-s} \otimes t_s^* \otimes \mathrm{id}_{a+k+b-s})(\mathrm{id}_{2r} \otimes P_{i-2r} \otimes X_{a,b})(\mathrm{id}_{i-s} \otimes t_s \otimes \mathrm{id}_{a+k+b-s})P_{c+k+d} \\ &= P_{c+k+d}(\mathrm{id}_{2r} \otimes Z)P_{c+k+d},\end{aligned}$$

where $Z = P_{c+k+d-2r}(\mathrm{id}_{i-s-2r} \otimes t_s^* \otimes \mathrm{id}_{a+k+b-s})(P_{i-2r} \otimes X_{a,b})(\mathrm{id}_{i-s-2r} \otimes t_s \otimes \mathrm{id}_{a+k+b-s})P_{c+k+d-2r}$. By Lemma 1.5 the error is controlled in HS norm as follows:

$$\|Y - P_{c+k+d}(\mathrm{id}_{2r} \otimes Z)P_{c+k+d}\|_2 \leq Cq^{2r}d_{2r}\sqrt{d_{i-4r}}\|X_{a,b}\|_2 \leq Cq^{2r}q^{-(a+b+i)/2}\|X\|_2.$$

Moreover by Lemma 1.5 we have $\|Z\|_2 \leq \|P_{i-2r}\|_2\|X_{a,b}\|_2 \leq Cq^r q^{-(a+b+i)/2}\|X\|_2$.

On the other hand Lemma 5.2 gives the estimate

$$\|(\mathrm{Tr}_r \otimes \mathrm{id})(X_{c,d})\|_2 \leq Cq^{-r/2+\beta r}q^{-(c+d)/2}\|X\|_2.$$

We apply now Lemma 4.4, or rather its left-right flipped version, to obtain

$$\begin{aligned}|(Y \mid X_{c,d})| &\leq Cq^{2r}q^{-(a+b+i)/2}\|X\|_2\|X_{c,d}\|_2 + \sqrt{d_r}\|(\mathrm{Tr}_r \otimes \mathrm{id})(X_{c,d})\|_2\|Z\|_2 + C\|Z\|_2\|X_{c,d}\|_2 \\ &\leq q^{-(a+b+i+c+d)/2} \times [\ Cq^{2r}\|X\|_2^2 + Cq^{\beta r}\|X\|_2^2 + Cq^r\|X\|_2^2\ ].\end{aligned}$$

Finally we still have $\big(\kappa^{i,a+k+b}_{c+k+d}\big)^2 d^{-1}_{c+k+d} \leq Cd_s^{-1}d^{-1}_{c+k+d} \leq Cq^{c+k+d+s}$ by Lemma 1.4. Observing that $c + d + s = (a + b + i + c + d)/2$ we arrive at

$$\begin{aligned}|(x_{c,d} \mid \chi_i x_{a,b})| &\leq Cq^{2r}q^k\|X\|_2^2 + Cq^{\beta r}q^k\|X\|_2^2 + Cq^r q^k\|X\|_2^2 \\ &\leq Cq^{\beta r}q^k\|X\|_2^2 \leq Cq^{\beta i/8}\|x\|_2^2.\end{aligned}$$

□

Now we assume that x belongs to the basis W introduced at Notation 2.11, so that in particular $\overline{AxA}$ is spanned by the vectors $x_{a,b}$, a, $b \in \mathbb{N}$. Recall also from Notation 3.1 that we denote $G(x)$ the Gram matrix of this family of vectors. Observe finally that, by definition of F_m, the operator $F_m a F_m$, for any $a \in A$, stabilizes $H(x)$.

Proposition 5.5. *There exist constants C, $\beta > 0$, depending only on q and m, such that the restriction to $H(x)$ of $F_m\chi_i F_m$ has operator norm less than $Cq^{\beta i} \times \|G(x)^{-1}\|\|x\|_2^2$, assuming that $x \in W$ and the Gram matrix $G(x)$ is invertible.*

Proof. Put $I = \{0, \dots, m-1\} \times \mathbb{N}$ and consider the operator $T : \ell^2(I) \to H(x)$, $(a, b) \mapsto x_{a,b}$. The operator T^*T is a corner of $G(x)$, hence we have $\|T\|^2 \leq \|G(x)\|$ and $\|T^{-1}\|^2 \leq \|G(x)^{-1}\|$. In particular the operator norm of the restriction to $H(x)$ of $F_m\chi_i F_m$ is dominated by $\|G(x)^{-1}\|\|M^{(i)}\|$, where $M^{(i)} \in B(\ell^2(I))$ has matrix entries $(x_{c,d} \mid \chi_i x_{a,b})$ for (a, b), $(c, d) \in I$.

Observe moreover that if $(x_{c,d} \mid \chi_i x_{a,b}) \neq 0$ we must have $c+d \leq a+b+i$, hence $i \geq d-b-m$. We can then write, using the estimates of Lemmata 5.3 and 5.4 and assuming $d - b \geq 0$:

$$\big|M^{(i)}_{c,d;a,b}\big| \leq Cq^{\beta i}\|x\|_2^2 \leq Cq^{\beta i/2}q^{-\beta m/2}\|x\|_2^2 q^{\beta|d-b|/2}.$$

This estimates also holds if $d-b \leq 0$ since $M^{(i)}$ is symmetric. It is then a standard fact — already used in the proof of Theorem 3.10 — that this entrywise estimate implies $\|M^{(i)}\| \leq Cq^{\beta i/2}\|x\|_2^2$, where C is a constant depending only on q and m. Indeed for $\lambda \in \ell^2(I)$ we have, by Cauchy-Schwarz and symmetry:

$$\textstyle\sum_{a,b,c,d}|\bar\lambda_{c,d}q^{\beta|d-b|/2}\lambda_{a,b}| \leq \sum_{a,b,c,d}|\lambda_{a,b}|^2 q^{\beta|d-b|/2} \leq \sum_{a,b}|\lambda_{a,b}|^2\dfrac{2m}{1-q^{\beta/2}}.$$

□

Theorem 5.6. *There exists $q_1 \in\,]0, 1[$ such that, assuming $q \leq q_1$, $F_m u_j F_m$ converges to 0 in operator norm as $j \to \infty$ for any $m \in \mathbb{N}$ and any sequence of unitaries $u_j \in A$ which converges weakly to 0.*

Proof. This follows from Proposition 5.5 by a standard argument. Let us give the details nevertheless. The value q_1 is given by Theorem 3.10, we have then $\|G(x)^{-1}\| \leq D\|x\|_2^{-2}$ for all $x \in W$. Write $u_j = \sum_i \mu_{j,i}\chi_i$ in $L^2(A)$. For any integer p the norm of the restriction of $F_m u_i F_m$ to $H(x)$ satisfies, thank to Proposition 5.5:

$$\|F_m u_j F_m\|_{B(H(x))} \leq \sum_{i\leq p}|\mu_{j,i}|\|\chi_i\| + CD\sum_{i>p}|\mu_{j,i}|q^{\beta i} \leq \sum_{i\leq p}|\mu_{j,i}|\|\chi_i\| + \frac{CDq^{\beta p}}{\sqrt{1-q^{2\beta}}},$$

using Cauchy-Schwartz and the fact that $\sum_i |\mu_{j,i}|^2 = \|u_j\|_2^2 = 1$. This upper bound is independent of $x \in W$, hence it is also an upper bound for the operator norm of $F_m u_j F_m$ on the whole of H. Fix $\epsilon > 0$ and choose p large enough such that the second term above is less than $\epsilon/2$. Since $u_j \to 0$ weakly we have $\mu_{j,i} = (\chi_i \mid u_j) \to_j 0$ for each i, so that for j large enough the first term above is less than $\epsilon/2$ as well. □

6. Proof of the Main Theorem

We will need the following general Lemma, which is certainly well-known.

Lemma 6.1. *Let H be a Hilbert space and $(\zeta_i)_{i\in I}$ be a family of vectors in H. Assume that the corresponding Gram matrix $G = (\zeta_i \mid \zeta_j)_{i,j}$ defines an isomorphism of $\ell^2(I)$. Fix I_1, $I_2 \subset I$ and denote $I'_p = I \setminus I_p$. Consider the orthogonal projection F_p onto $\overline{\mathrm{Span}}\{\zeta_i \mid i \in I_p\}$ and denote $F = F_1 \vee F_2$.*

1. *We have $\|F\xi\| \leq (\|G\|\|G^{-1}\|)^{3/2}(\|F_1\xi\| + \|F_2\xi\|)$ for all $\xi \in H$.*
2. *Assume $I_1 \cap I_2 = \emptyset$. We have $\|F_1F_2\| \leq \|G^{-1}\|\|G_{12}\|$, where M_{12} denotes the block of M corresponding to $\ell^2(I_1)$, $\ell^2(I_2) \subset \ell^2(I)$.*
3. *Assume $I_1 \cup I_2 = I$. We have $\|F_1^\perp F_2^\perp\| \leq \|G\|\|(G^{-1})_{1'2'}\|$, where $M_{1'2'}$ denotes the block of M corresponding to $\ell^2(I'_1)$, $\ell^2(I'_2)$.*

Proof. Denote $T : \ell^2(I) \to H$, $\delta_i \mapsto \zeta_i$, so that $G = T^*T$. By assumption T is an isomorphism.

1. Since $F_1F = F_1$, $F_2F = F_2$ one can assume $F = \mathrm{id}$. If $F' = F_1 \wedge F_2$ we have $\|\xi\| \leq \|F'\xi\| + \|F'^\perp\xi\| \leq \|F_1\xi\| + \|F'^\perp\xi\|$ hence one can assume $F' = 0$. We have then $I = I_1 \sqcup I_2$.

Denote $\tilde F_p = T^{-1}F_pT \in B(\ell^2(I))$. In the decomposition $\ell^2(I) = \ell^2(I_1) \oplus \ell^2(I_2)$ we denote

$$\tilde F_1 = \begin{pmatrix} 1 & F'_1 \\ 0 & 0 \end{pmatrix}, \qquad \tilde F_2 = \begin{pmatrix} 0 & 0 \\ F'_2 & 1 \end{pmatrix}.$$

Since $F_p^* = F_p$ we have $\tilde F_p^* G = G\tilde F_p$, which is equivalent to $F'_1 = G_{11}^{-1}G_{12}$ and $F'_2 = G_{22}^{-1}G_{21}$.

For $\eta = (\eta_1, \eta_2) \in \ell^2(I_1) \oplus \ell^2(I_2)$ we can then write

$$\begin{aligned}\|\tilde F_1\eta\|^2 + \|\tilde F_2\eta\|^2 &= \|\eta_1 + G_{11}^{-1}G_{12}\eta_2\|^2 + \|G_{22}^{-1}G_{21}\eta_1 + \eta_2\|^2 \\ &\geq \|G_{11}\|^{-2}\|G_{11}\eta_1 + G_{12}\eta_2\|^2 + \|G_{22}\|^{-2}\|G_{21}\eta_1 + G_{22}\eta_2\|^2 \\ &\geq C^{-2}\|G\eta\|^2 \geq C^{-2}\|G^{-1}\|^{-2}\|\eta\|^2,\end{aligned}$$

where $C = \max(\|G_{11}\|, \|G_{22}\|) \leq \|G\|$. Putting $\eta = T^{-1}\xi$ we get

$$\begin{aligned}\|G\|^{-1/2}\|\xi\| \leq \|T^{-1}\xi\| &\leq \|G\|\|G^{-1}\|(\|T^{-1}F_1\xi\| + \|T^{-1}F_2\xi\|) \\ &\leq \|G\|\|G^{-1}\|^{3/2}(\|F_1\xi\| + \|F_2\xi\|).\end{aligned}$$

2. Denote T_p the restriction of T to $\ell^2(I_p)$. Then $S_p = T_p(G_{pp})^{-1/2}$ is an isometry from $\ell^2(I_p)$ onto $\mathrm{Im}(F_p)$ and we have

$$\begin{aligned}\|F_1F_2\| = \|S_1S_1^*S_2S_2^*\| = \|S_1^*S_2\| &= \|(G_{11})^{-1/2}T_1^*T_2(G_{22})^{-1/2}\| \\ &\leq \|(G_{11})^{-1/2}\|\|G_{12}\|\|(G_{22})^{-1/2}\|.\end{aligned}$$

Finally we have $\|(G_{pp})^{-1}\|^{-1} = \min \mathrm{Sp}(G_{pp}) \geq \|G^{-1}\|^{-1}$ and the result follows.

3. Let $(\zeta'_i)_i$ be the dual basis of $(\zeta_i)_i$, so that $(\zeta'_i \mid \zeta_j) = \delta_{i,j}$. Then $F'_p = F_p^\perp$ is the orthogonal projection onto $\overline{\mathrm{Span}}\{\zeta'_i \mid i \in I'_p\}$. Moreover the Gram matrix of $(\zeta'_i)_i$ is $G' = G^{-1}$. We can then apply the previous point to F'_1, F'_2 and we obtain:

$$\|F_1^\perp F_2^\perp\| \leq \|G'^{-1}\|\|G'_{1'2'}\|. \qquad \square$$

Let us now proceed to the proof of Theorem A. Thanks to the results obtained in the previous sections, it will be possible to follow very closely the general strategy discovered by Popa [Pop83], but we include the details for the convenience of the reader. We assume that $q \leq q_1$, where the value $q_1 \in]0,1[$ is given by Theorem 3.10 ; this is equivalent to assuming $N \geq N_1 = \lceil q_1 + q_1^{-1} \rceil$.

Let $(z_r)_r$ a sequence of elements of $A^\perp \cap M$ such that $\|z_r\| \leq 1$ for all r and $\|[b, z_r]\|_2 \to_\omega 0$ for all $b \in B$, where $B \subset A$ is diffuse. We also take $y \in A^\perp \cap M$ and we want to show that $(yz_r \mid z_ry) \to_\omega 0$. By Kaplansky's theorem and linearity we can assume that $y \in p_nH^\circ$ and $\|y\| \leq 1$.

We first apply the orthogonality property established in Section 4. Denote E_m the orthogonal projection onto the subspace V_m introduced at Notation 4.1, and $E_m^\perp = 1 - E_m$. We have

$$|(yz_r \mid z_ry)| \leq |(yE_m(z_r) \mid E_m(z_r)y)| + 2\|E_m^\perp(z_r)\|_2. \tag{6.1}$$

We can apply Theorem 4.9 to $\zeta = E_m(z_r)$. Since $\|\zeta\| \leq \|z_r\|_2 \leq 1$ we see that the scalar product above tends to 0 as $m \to \infty$, uniformly with respect to r.

Recall the orthogonal projection F_m from Notation 5.1. Denote similarly F'_m the orthogonal projection onto the closed subspace generated by the vectors $x_{i,j}$ with $j < m$, and put $F_m^\vee = F_m \vee F'_m$, which is the orthogonal projection onto $\overline{\text{Span}}\{x_{i,j} \mid x \in H^{\infty} \text{ and } (i < m \text{ or } j < m)\}$. Note that the projections $F_m^\vee$, $E_m^\perp$ do not coincide since $\{x_{i,j}\}$ is not orthogonal, and this is were we need the Riesz basis property established in Section 3. We first write

$$\|E_m^\perp(z_r)\|_2 \le \|E_m^\perp F_{2m}^{\vee\perp}\| + \|F_{2m}^\vee z_r\|_2 \le \|E_m^\perp F_{2m}^{\vee\perp}\| + C(\|F_{2m} z_r\|_2 + \|F'_{2m} z_r\|_2), \tag{6.2}$$

using the first point of Lemma 6.1, with C depending only on q.

Note that we can indeed apply Lemma 6.1 to the basis $(x_{i,j})_{i,j}$ of $H(x)$, provided $q \in]0, q_1]$ where $q_1 \in]0,1[$ is given by Theorem 3.10. We moreover estimate the norm of $E_m^\perp F_{2m}^{\vee\perp}$ using the third point of the lemma. More precisely we work in each subspace $H(x)$, $x \in W$, and we consider the subsets $I_1 = \{(i,j) \in I \mid \min(i,j) \ge m\}$ and $I_2 = \{(i,j) \in I \mid \min(i,j) < 2m\}$ of $I = \mathbb{N} \times \mathbb{N}$. We obtain $\|E_m^\perp F_{2m}^{\vee\perp}\| \le \|G(x)\| \|H_{1'2'}\|$ where $H = G(x)^{-1}$ and the block $1'2'$ refers to indices (i,j) such that $\min(i,j) < m$ on the one hand, and indices (p,q) such that $\min(p,q) \ge 2m$ on the other hand.

It follows easily from the third point of Theorem 3.10 that the operator norm of this block tends to 0 as $m \to \infty$, uniformly over $x \in W$. Consider indeed vectors $\lambda \in \ell^2(I'_1)$, $\mu \in \ell^2(I'_2)$. As in Section 3, use $(n;i)$ as index in place of (i,j), where $i + j + k = n$, and write $\mu_n = (\mu_{n;i})_i$. Since G is block diagonal for the decomposition with respect to n, it is also the case of H. We have then using Cauchy-Schwarz:

$$\begin{aligned}
|(\lambda \mid H\mu)| &\le D \sum_{n=k}^{\infty} \sum_{p=2m}^{n-k-2m} \left(\textstyle\sum_{i=0}^{m-1} + \sum_{i=n-k-m+1}^{n-k}\right) |\lambda_{n;i}\mu_{n;p}| q^{\beta|i-p|} \\
&\le D \sum_{n=k}^{\infty} \|\mu_n\|_2 \left(\sum_{p=2m}^{n-k-2m} \left[\left(\textstyle\sum_{i=0}^{m-1} + \sum_{i=n-k-m+1}^{n-k}\right) |\lambda_{n;i}| q^{\beta|i-p|} \right]^2 \right)^{1/2} \\
&\le D \sum_{n=k}^{\infty} \|\mu_n\|_2 \|\lambda_n\|_2 \left(\sum_{p=2m}^{n-k-2m} \left(\textstyle\sum_{i=0}^{m-1} + \sum_{i=n-k-m+1}^{n-k}\right) q^{2\beta|i-p|} \right)^{1/2} \\
&\le D \sum_{n=k}^{\infty} \|\mu_n\|_2 \|\lambda_n\|_2 \left(2 \sum_{p=2m}^{\infty} \sum_{i=0}^{m-1} q^{2\beta(p-i)} \right)^{1/2} \\
&\le D \|\mu\|_2 \|\lambda\|_2 \left(2 \frac{q^{4\beta m}}{1-q} \frac{q^{-2\beta m} - 1}{q^{-1} - 1} \right)^{1/2} \le D' q^{\beta m} \|\mu\|_2 \|\lambda\|_2,
\end{aligned}$$

where D', $\beta > 0$ depend only on q.

This shows indeed that $\lim_{m\to\infty} \|E_m^\perp F_{2m}^{\vee\perp}\| = 0$. Taking into account (6.1) and (6.2), it remains to show that $\|F_m z_r\|_2 \to 0$ and $\|F'_m z_r\|_2 \to 0$ as $r \to \omega$, for m fixed. The case of F'_m follows from the one of F_m by symmetry.

To prove $\|F_m z_r\|_2 \to_\omega 0$ we shall find unitaries u_1, ..., $u_p \in B$ such that $\|F_m^* u_i^* u_j F_m\| < 1/p^2$ for all $i \ne j$, i.e. the projections $u_i F_m u_i^*$ are almost pairwise orthogonal. Then it is easy to show that $\|\sum_1^p u_i F_m u_i^*\| \le 2$:

$$\begin{aligned}
\|\textstyle\sum_{i=1}^p u_i F_m u_i^*\|^2 &= \|\textstyle\sum_{i,j} u_i F_m^* u_i^* u_j F_m u_j^*\| \\
&\le \textstyle\sum_{i\ne j} \frac{1}{p^2} + \|\sum_i u_i F_m u_i^*\| \le 2\|\sum_i u_i F_m u_i^*\|,
\end{aligned}$$

since $\|\sum_i u_i F_m u_i^*\| \ge 1$ because we have a sum of non-zero projections. In particular we get

$$\textstyle\sum_1^p \|u_i F_m u_i^* z_r\|_2^2 = (z_r \mid \sum_1^p u_i F_m u_i^* z_r) \le 2\|z_r\|_2^2 \le 2.$$

On the other hand we have assumed that $\|[b, z_r]\|_2 \to_\omega 0$ for all $b \in B$. Putting $b = u_i^*$ we obtain

$$\|u_i F_m u_i^* z_r\|_2^2 \ge \|u_i F_m z_r u_i^*\|_2^2 - 1/p = \|F_m z_r\|_2^2 - 1/p$$

for ω-almost all r's, and comparing with the previous estimate we obtain $\|F_m z_r\|_2 \leq \sqrt{3/p}$ for these r's. This holds for every p, so that we have indeed $\|F_m z_r\|_2 \to_\omega 0$.

We finally construct the required unitaries u_i by induction, using the compactness property established in Section 5. Since B is diffuse, we can find a sequence of unitaries $v_l \in B$ which converges weakly to 0. We start with $u_1 = 1$ and, assuming that u_1, ..., u_i are already constructed, we observe that $(u_j^* v_l)_l$ still converges weakly to 0 as $l \to \infty$, for all $1 \leq j \leq i$. According to Theorem 5.6 this implies $\|F_m u_j^* v_l F_m\| \to_l 0$ for $1 \leq j \leq i$ and we can indeed find $u_{i+1} = v_l$ such that $\|F_m^* u_j^* u_{i+1} F_m\| < 1/p^2$ for $1 \leq j \leq i$. This concludes the proof.

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