THE RADIAL MASA IN FREE ORTHOGONAL QUANTUM GROUPS

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Abstract. We prove that the radial subalgebra in free orthogonal quantum group factors is maximal abelian and mixing, and we compute the associated bimodule. The proof relies on new properties of the Jones-Wenzl projections and on an estimate of certain scalar products of coefficients of irreducible representations.

1. Introduction

Discrete groups have been an important part of the theory of von Neumann algebras since its very beginning. Taking advantage of their algebraic or geometric properties, one can build interesting families of examples and counter-examples of von Neumann algebras, and get some insight into crucial structural properties like property (T) or approximation properties. In the last ten years, there has been an increasing number of results showing that discrete quantum groups can also produce interesting examples of von Neumann algebras. In this work, we continue this program by initiating the study of abelian subalgebras in von Neumann algebras of discrete quantum groups.

The importance of abelian subalgebras in the study of von Neumann algebras has been long known and, as already mentioned, group von Neumann algebras have played an important role in that history. For instance, the subalgebra generated by one of the generating copies of $\mathbb{Z}$ inside the von Neumann algebra of the free group $\mathbb{F}_2$ was proved by J. Dixmier to be maximal abelian [8], and by S. Popa to be maximal injective [15], thus answering a long-standing question of R.V. Kadison. The fact that the subalgebra comes from a group inclusion was crucial there.

Another example of abelian subalgebra in free group factors is the so-called *radial* (or *laplacian*) subalgebra, which is the one generated by the sum of the generators and their inverses. This subalgebra does not come from a subgroup, hence the aforementioned techniques do not apply. S. Radulescu introduced in [17] tools to prove that this subalgebra, which was already known to be maximal abelian by work of S. Pytlik [16], is singular. His techniques were later used again to prove that the radial subalgebra is maximal amenable [7]. For more background on maximal abelian subalgebras we refer to the book [20].

In this paper, we study the analogue of the radial subalgebra in free quantum group factors. More precisely, we consider the free orthogonal quantum group of Kac type O_N^+ and, inside its von Neumann algebra $L^\infty(O_N^+)$, the subalgebra generated by the characters of irreducible representations. Recall that O_N^+ is a compact quantum group introduced in [24], whose discrete dual is a quantum analogue of a free group. In particular $L^\infty(O_N^+)$ plays the role of a free group factor $\mathcal{L}(\mathbb{F}_N)$. This analogy, dating back to the seminal works of T. Banica [1], [2], has been supported since then by further work of several authors who proved that the von Neumann algebra $L^\infty(O_N^+)$, for $N \geqslant 3$, indeed shares many properties with free group factors:

- it is a full factor with the Akemann-Ostrand property [21],
- it has the Haagerup property [4] and the completely contractive approximation property [12],
- it is strongly solid [13] and has property strong HH [10],
- it satisfies the Connes embedding conjecture [5].

As far as the radial subalgebra is concerned, the techniques of S. Radulescu do not apply in the quantum case, because there is no clear way to mimic the construction of the so-called *Radulescu basis*. However, the properties of the radial algebra mentioned previously can all be proved using another tool which we briefly explain. Consider, for $n \in \mathbb{N}$, the element $w_n \in \mathcal{L}(\mathbb{F}_N)$ which is the sum of all words of length n. Then, if x, x' are two words of length k and y, y' are two other words of length n, we have

$$\langle (x - x')w_l, w_{l'}(y - y')\rangle \leqslant 2\min(k + 1, n + 1).$$

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This estimate can be proved by elementary counting arguments, similar to the ones in [19, Sec 4]. It can then be used to prove maximal abelianness and singularity in one shot. We will use the same strategy here.

Elements of the form $x - x'$ with x and x' of the same length k form a basis of the orthogonal of the radial subalgebra in $\mathcal{L}(\mathbb{F}_N)$, in the quantum case their role will be played by the coefficients $u^k_{\xi\eta}$ of an irreducible representation u^k with respect to vectors ξ, η such that ξ is orthogonal to η. As already mentioned, the role of w_l will be played by the character χ_l of the irreducible representation u^l – note however that $\|w_l\|^2 = 2N(2N-1)^{l-1}$ in $L^2(\mathbb{F}_N)$, whereas $\|\chi_l\|^2 = 1$ in $L^2(O_N^+)$. The estimate analogous to (1) that we will prove and use in the present article is then stated as follows (see Theorem 4.3):

$$\langle \chi_l u^k_{\xi',\eta'}, u^n_{\xi,\eta}\chi_{l'}\rangle \leqslant Kq^{\max(l,l')},$$

with $q \in]0,1[$. From this we will deduce all the results announced in the abstract.

Let us now outline the content of the paper. In Section 2, we recall some facts on compact quantum groups and in particular on free orthogonal quantum groups. Since the geometry of their representation theory will be crucial in the computations, we have to make some conventional choices and give the corresponding explicit formulæ for several related objects.

Section 3 and 4 form the core of the paper. There we prove the announced estimate for scalar products of coefficients and characters. The proof, presented in Section 4, is quite technical and relies on properties of the so-called Jones-Wenzl projections which are of independent interest and are established in Section 3.

Eventually, we prove in Section 5 all our structural results on the radial subalgebra, namely that it is maximal abelian, mixing and has spectral measure equivalent to the Lebesgue measure. The proofs here are very simple using the main estimate and the arguments are certainly well-known to experts in von Neumann algebras. Since however people interested in discrete quantum groups may not be so familiar with it, we give full proofs. The paper ends with some remarks on the results of this work.

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2. Preliminaries

In this section we give the basic definitions and results needed in the paper. All scalar products will be *left-linear* and we will denote by $\mathcal{B}(H)$ the algebra of all bounded operators on a Hilbert space H. When considering an operator $X \in \mathcal{B}(H_1 \otimes H_2)$, we will use the *leg-numbering notations*,

$$X_{12} := X \otimes 1, X_{23} := 1 \otimes X \text{ and } X_{13} := (\Sigma \otimes 1)(1 \otimes X)(\Sigma \otimes 1),$$

where $\Sigma : H_1 \otimes H_2 \to H_2 \otimes H_1$ is the flip map. For any two vectors $\xi, \eta \in H$, we define a linear form $\omega_{\xi\eta} : \mathcal{B}(H) \to \mathbb{C}$ by $\omega_{\xi\eta}(T) = \langle T(\xi), \eta\rangle$.

2.1. **Compact quantum groups.** We briefly review the theory of compact quantum groups as introduced by S.L. Woronowicz in [26]. In the sequel, all tensor products of C*-algebras are spatial and we denote $\overline{\otimes}$ the tensor product of von Neumann algebras.

Definition 2.1. A *compact quantum group* $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital C*-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$-homomorphism such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and the spaces $\mathrm{span}\{\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))\}$ and $\mathrm{span}\{\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)\}$ are both dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$.

According to [26, Thm 1.3], any compact quantum group $\mathbb{G}$ has a unique *Haar state* $h \in C(\mathbb{G})^*$, satisfying

$$(\mathrm{id} \otimes h) \circ \Delta(a) = h(a).1$$
$$(h \otimes \mathrm{id}) \circ \Delta(a) = h(a).1$$

for all $a \in C(\mathbb{G})$. Let $(L^2(\mathbb{G}), \pi_h, \Omega)$ be the associated GNS construction and let $C_{\mathrm{red}}(\mathbb{G})$ be the image of $C(\mathbb{G})$ under the GNS representation π_h. It is called the *reduced C*-algebra* of $\mathbb{G}$ and its bicommutant in $\mathcal{B}(L^2(\mathbb{G}))$ is the *von Neumann algebra of* $\mathbb{G}$, denoted by $L^\infty(\mathbb{G})$. To study this object, we will use representations of compact quantum groups.

Definition 2.2. A *representation* of a compact quantum group $\mathbb{G}$ on a Hilbert space H is an operator $u \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ such that $(\Delta\otimes\mathrm{id})(u) = u_{13}u_{23}$. It is said to be *unitary* if the operator u is unitary.

Definition 2.3. Let $\mathbb{G}$ be a compact quantum group and let u and v be two representations of $\mathbb{G}$ on Hilbert spaces H_u and H_v respectively. An *intertwiner* (or *morphism*) between u and v is a map $T \in \mathcal{B}(H_u, H_v)$ such that $v(\mathrm{id}\otimes T) = (\mathrm{id}\otimes T)u$. The set of intertwiners between u and v will be denoted by $\mathrm{Hom}(u, v)$.

A representation u is said to be *irreducible* if $\mathrm{Hom}(u, u) = \mathbb{C}.\mathrm{id}$ and it is said to be *contained* in v if there is an injective intertwiner between u and v. We will say that two representations are *equivalent* (resp. *unitarily equivalent*) if there is an intertwiner between them which is an isomorphism (resp. a unitary). Let us define two fundamental operations on representations.

Definition 2.4. Let $\mathbb{G}$ be a compact quantum group and let u and v be two representations of $\mathbb{G}$ on Hilbert spaces H_u and H_v respectively. The *direct sum* of u and v is the diagonal sum of the operators u and v seen as an element of $L^\infty(\mathbb{G})\otimes\mathcal{B}(H_u\oplus H_v)$. It is a representation denoted by $u\oplus v$. The *tensor product* of u and v is the element $u_{12}v_{13} \in L^\infty(\mathbb{G})\otimes\mathcal{B}(H_u\otimes H_v)$. It is a representation denoted by $u\otimes v$.

The theory of representations of compact groups can be generalized to this setting (see [26, Section 6]). If u is a representation of $\mathbb{G}$ on a Hilbert space H and if $\xi, \eta \in H$, then $u_{\xi\eta} = (\mathrm{id}\otimes\omega_{\xi\eta})(u) \in C(\mathbb{G})$ is called a *coefficient* of u.

Theorem 2.5 (Woronowicz)**.** *Every representation of a compact quantum group is equivalent to a unitary one. Every irreducible representation of a compact quantum group is finite-dimensional and every unitary representation is unitarily equivalent to a sum of irreducible ones. Moreover, the linear span of the coefficients of all irreducible representations is a dense Hopf $*$-subalgebra of $C(\mathbb{G})$ denoted by* $\mathrm{Pol}(\mathbb{G})$.

2.2. **Irreducible representations.** Let $\mathrm{Irr}(\mathbb{G})$ be the set of equivalence classes of irreducible unitary representations of $\mathbb{G}$. For $\alpha \in \mathrm{Irr}(\mathbb{G})$, we will denote by u^α a representative of the class α and by H_α the finite-dimensional Hilbert space on which u^α acts. The scalar product induced by the Haar state can be easily computed on coefficients of irreducible representations by [26, Eq. 6.7]:

$$\left\langle u^\alpha_{\xi\eta}, u^\beta_{\xi'\eta'}\right\rangle = \delta_{\alpha,\beta}\frac{\langle\xi,\xi'\rangle\langle\eta', Q_\alpha\eta\rangle}{d_\alpha}$$

where Q_α is a positive matrix determined by the representation α and $d_\alpha = \mathrm{Tr}(Q_\alpha) = \mathrm{Tr}(Q_\alpha^{-1}) > 0$ is called the *quantum dimension* of α. Note that in general, d_α is greater than $\dim(H_\alpha)$. However, it is easy to see that the two dimensions agree if and only if $Q_\alpha = \mathrm{id}$. When this is the case for all $\alpha \in \mathrm{Irr}(\mathbb{G})$ we say that $\mathbb{G}$ is of *Kac type.*

Because the coefficients of irreducible representations are dense in $C(\mathbb{G})$, it is enough to understand products of those coefficients to describe the whole C*-algebra structure of $C(\mathbb{G})$. For simplicity, we will assume from now on that for any two irreducible representations α and β, every irreducible subrepresentation appears with multiplicity one (this assumption will always be satisfied when considering free orthogonal quantum groups). For such a subrepresentation γ of $\alpha\otimes\beta$, let $v^{\alpha,\beta}_\gamma$ be an isometric intertwiner from H_γ to $H_\alpha\otimes H_\beta$. Then,

$$u^\alpha_{\xi\eta}u^\beta_{\xi'\eta'} = \sum_{\gamma\subset\alpha\otimes\beta} u^\gamma_{(v^{\alpha,\beta}_\gamma)^*(\xi\otimes\xi'),(v^{\alpha,\beta}_\gamma)^*(\eta\otimes\eta')}. \tag{1}$$

Note that even though $v^{\alpha,\beta}_\gamma$ is only defined up to a complex number of modulus one, the sesquilinearity of the scalar product ensures that the expression above is independent of this phase. We will also use the projection $P^{\alpha,\beta}_\gamma \in \mathcal{B}(H_\alpha\otimes H_\beta)$ onto the γ-homogeneous component, $P^{\alpha,\beta}_\gamma = v^{\alpha,\beta}_\gamma v^{\alpha,\beta*}_\gamma$, which is again independent of the choice of $v^{\alpha,\beta}_\gamma$.

For any $\alpha \in \mathrm{Irr}(\mathbb{G})$, there is a unique (up to unitary equivalence) irreducible representation, called the *contragredient representation* of α and denoted by $\overline{\alpha}$, such that $\mathrm{Hom}(\varepsilon, \alpha\otimes\overline{\alpha}) \neq \{0\} \neq \mathrm{Hom}(\varepsilon, \overline{\alpha}\otimes\alpha)$, ε denoting the trivial representation (i.e. the element $1\otimes 1 \in L^\infty(\mathbb{G})\otimes\mathbb{C}$). We choose morphisms $t_\alpha \in \mathrm{Hom}(\varepsilon, \alpha\otimes\overline{\alpha})$ and $s_\alpha \in \mathrm{Hom}(\varepsilon, \overline{\alpha}\otimes\alpha)$ connected by the conjugate equation

$$(\mathrm{id}_\alpha\otimes s^*_\alpha)(t_\alpha\otimes\mathrm{id}_\alpha) = \mathrm{id}_\alpha,$$

and normalized so that $\|s_\alpha\| = \|t_\alpha\| = \sqrt{d_\alpha}$. Then, t_α is unique up to a phase and s_α is determined by t_α. The morphism t_α induces a conjugate-linear isomorphism $j_\alpha : H_\alpha \to H_{\overline{\alpha}}$ such that, setting $j_\alpha(\xi) = \overline{\xi}$,

$$t_\alpha = \sum_{i=1}^{\dim(H_\alpha)} e_i \otimes \overline{e_i}$$

for any orthonormal basis $(e_i)_i$ of H_α. Note that j_α need not be a multiple of a conjugate-linear isometry in general – this is however the case if $\mathbb{G}$ is of Kac type. Let us also record the general fact that the map $\overline{v}_\gamma^{\alpha,\beta} : H_{\overline{\gamma}} \to H_{\overline{\beta}} \otimes H_{\overline{\alpha}}$ defined by

$$\overline{\xi} \mapsto \Sigma(v_\gamma^{\alpha,\beta}(\xi))^{\overline{\ }\otimes\overline{\ }}$$

is an isometric morphism from $\overline{\gamma}$ to $\overline{\beta} \otimes \overline{\alpha}$. In particular, when there is no multiplicity in the fusion rules $\overline{v}_\gamma^{\alpha,\beta}$ coincides with $v_{\overline{\gamma}}^{\overline{\beta},\overline{\alpha}}$ up to a complex number of modulus one.

2.3. **Free orthogonal quantum groups.** We will be concerned in the sequel with the free orthogonal quantum groups introduced by S. Wang and A. van Daele in [24] and [22]. This subsection is devoted to briefly recalling their definition and main properties.

Definition 2.6. For $N \in \mathbb{N}$, we denote by $C(O_N^+)$ the universal unital C*-algebra generated by N^2 *self-adjoint* elements $(u_{ij})_{1\leqslant i,j\leqslant N}$ such that the matrix $u = (u_{ij})$ is *unitary.* For $Q \in GL_N(\mathbb{C})$, we denote by $C(O^+(Q))$ the unital C*-algebra generated by N^2 elements $(u_{ij})_{1\leqslant i,j\leqslant N}$ such that the matrix $u = (u_{ij})$ is *unitary* and $Q\overline{u}Q^{-1} = u$, where $\overline{u} = (u_{ij}^*)$.

One can check that there is a unique $*$-homomorphism $\Delta : C(O^+(Q)) \to C(O^+(Q)) \otimes C(O^+(Q))$ such that for all i, j,

$$\Delta(u_{ij}) = \sum_{i,j=0}^{N} u_{ik} \otimes u_{kj}.$$

Definition 2.7. The pair $O_N^+ = (C(O_N^+), \Delta)$ is called the *free orthogonal quantum group* of size N.The pair $O^+(Q) = (C(O^+(Q)), \Delta)$ is called the free orthogonal quantum group of parameter Q.

One can show that the compact quantum group $O^+(Q)$ is of Kac type if and only if Q is a scalar multiple of a unitary matrix. Although all results of this article apply to general free orthogonal quantum groups of Kac type with $N \geqslant 3$, we will restrict for simplicity to the case of O_N^+ – see Section 5 for comments about the non-Kac type. The representation theory of free orthogonal quantum groups was computed by T. Banica in [1]:

Theorem 2.8 (Banica)**.** *The equivalence classes of irreducible representations of O_N^+ are indexed by the set of integers (u^0 being the trivial representation and $u^1 = u$ the fundamental one), each one is isomorphic to its contragredient and the tensor product is given inductively by*

$$u^1 \otimes u^n = u^{n+1} \oplus u^{n-1}.$$

If $N = 2$, then $d_n = n + 1$. Otherwise,

$$d_n = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}},$$

where $q + q^{-1} = N$ and $0 \leqslant q \leqslant 1$. Moreover, O_N^+ is of Kac type, hence $d_n = \dim(H_n)$.

Remark 2.9. There is an elementary estimate on d_n given by $q^{-n}(1 - q^2) \leqslant d_n \leqslant q^{-n}/(1 - q^2)$. We will use it several times in the sequel without refering to it explicitly.

To be able to do computations, we will use a particular set of representatives of the irreducible representations. More precisely, let $H_1 = \mathbb{C}^N$ be the carrier space of the fundamental representation $u = u^1$. Then, for each $n \in \mathbb{N}$, we let H_n be the unique subspace of $H_1^{\otimes n}$ on which the restriction of $u^{\otimes n}$ is equivalent to u^n. We denote by id_n the identity of H_n.

It is easy to check that the map $t_1 = \sum_{i=1}^N e_i \otimes e_i$ satisfies the requirements for the distinguished morphism $t_u \in \mathrm{Hom}(\varepsilon, u \otimes \overline{u})$ as defined in the previous subsection, with $\bar{u} = u$ and $s_1 = t_1$. We fix this choice in the rest of the article and we set

$$t_n = (P_n \otimes P_n)(t_1)_{1,2n}(t_1)_{2,2n-1} \dots (t_1)_{n,n+1} \in H_n \otimes H_n.$$

We then have $s_n = t_n$, $j_n \circ j_n = \mathrm{id}_n$, and j_n is a conjugate linear unitary. The standard trace on $\mathcal{B}(H_n)$ is given by

$$\mathrm{Tr}_n(f) = t_n^*(f \otimes \mathrm{id})t_n$$

and the normalized trace by $\mathrm{tr}_n(f) = d_n^{-1}\,\mathrm{Tr}_n(f)$. Moreover, writing again $\overline{\zeta} = j_n(\zeta)$ for $\zeta \in H_n$ we have

$$t_n^*(\zeta \otimes \mathrm{id}_n) = \overline{\zeta}^* \text{ and } t_n^*(\mathrm{id}_n \otimes \zeta) = s_n^*(\mathrm{id}_n \otimes \zeta) = \overline{\zeta}^*.$$

We will denote by P_n the orthogonal projection from $H_1^{\otimes n}$ onto H_n, sometimes called the *Jones-Wenzl projection*. Note that if $a + b = n$, then $P_n(P_a \otimes P_b) = P_n$, so that we may also see P_n as an element of $\mathcal{B}(H_a \otimes H_b)$. In other words we have, with the notation of the previous subsection, $P_n = P_n^{a,b}$ for any a, b such that $a+b = n$. The sequence of projections $(P_n)_{n\in\mathbb{N}}$ satisfies the so-called *Wenzl recursion relation* (see for instance [11, Eq 3.8] or [21, Eq 7.4]):

$$P_n = (P_{n-1} \otimes \mathrm{id}_1) + \sum_{l=1}^{n-1} (-1)^{n-l} \frac{d_{l-1}}{d_{n-1}} \left(\mathrm{id}_1^{\otimes(l-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes(n-l-1)} \otimes t_1^*\right)(P_{n-1} \otimes \mathrm{id}_1). \tag{2}$$

We also record the following obvious fact, which will be used frequently in the sequel without explicit reference: for any a, b we have $(\mathrm{id}_a \otimes t_1 \otimes \mathrm{id}_b)^* P_{a+b+2} = 0$. Indeed the image of $(\mathrm{id}_a \otimes t_1 \otimes \mathrm{id}_b)^*$ is contained in $H_a \otimes H_b$ which has no component equivalent to H_{a+b+2}. A first application is the following reduced form of the Wenzl relation above, which is actually the original relation presented in [25]:

$$P_n = (P_{n-1} \otimes \mathrm{id}_1) - \frac{d_{n-2}}{d_{n-1}}(P_{n-1} \otimes \mathrm{id}_1)\left(\mathrm{id}_1^{\otimes(n-1)} \otimes t_1 t_1^*\right)(P_{n-1} \otimes \mathrm{id}_1). \tag{3}$$

We also have a reflected version as follows:

$$P_n = (\mathrm{id}_1 \otimes P_{n-1}) - \frac{d_{n-2}}{d_{n-1}}(\mathrm{id}_1 \otimes P_{n-1})\left(t_1 t_1^* \otimes \mathrm{id}_1^{\otimes(n-1)}\right)(\mathrm{id}_1 \otimes P_{n-1}). \tag{4}$$

3. Manipulating the Jones-Wenzl projections

In this section we establish two results concerning the sequence of projections P_n in the representation category of O_N^+. The first one studies partial traces of these projections, while the second one is a kind of generalization of Wenzl's recursion relation.

3.1. **Partial traces of projections.**

The first result we need concerns projections onto irreducible representations that are cut down by a trace. To explain what is going on, let us first consider two integers $a, b \in \mathbb{N}$. Then, the operator

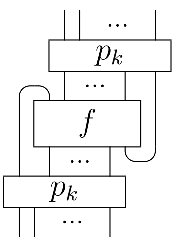
$$x_{a,b} = (\mathrm{id}_a \otimes \mathrm{tr}_b)(P_{a+b}) = d_b^{-1}(\mathrm{id}_a \otimes t_b^*)(P_{a+b} \otimes \mathrm{id}_b)(\mathrm{id}_a \otimes t_b) \in \mathcal{B}(H_\alpha)$$

is a scalar multiple of the identity because it is an intertwiner and u^a is irreducible. Of course, the same holds for $(\mathrm{tr}_b \otimes \mathrm{id}_c)(P_{b+c}) \in \mathcal{B}(H_c)$. However in general $x_{a,b,c} = (\mathrm{id}_a \otimes \mathrm{tr}_b \otimes \mathrm{id}_c)(P_{a+b+c})$ is not a scalar multiple of the identity. In fact, an easy explicit computation already shows that $x_{1,1,1} \in \mathcal{B}(H_1 \otimes H_1)$ is a non-trivial linear combination of the identity and the flip map, in particular it is not even an intertwiner. Proposition 3.2, which is the main result of this subsection, shows that when b tends to $+\infty$, the partially traced projection $x_{a,b,c}$ becomes asymptotically scalar.

To prove this, we need a lemma concerning the following construction: for a linear map $f \in \mathcal{B}(H_k)$, we define its *rotated version* $\rho(f)$ by

$$\rho(f) = (P_k \otimes t_1^*)(\mathrm{id}_1 \otimes f \otimes \mathrm{id}_1)(t_1 \otimes P_k) \in \mathcal{B}(H_k).$$

Diagrammatically, this transformation is represented as follows:

$\rho(f) =$ 

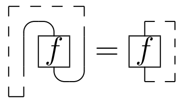
In the sequel, $\|.\|_{\mathrm{HS}}$ will denote the non-normalized Hilbert-Schmidt norm, i.e. $\|f\|_{HS}^2 = \mathrm{Tr}(f^*f)$.

Lemma 3.1. *For any $f \in \mathcal{B}(H_k)$, $\operatorname{Tr}(\rho(f)) = (-1)^{k-1}\operatorname{Tr}(f)/d_{k-1}$. Moreover, we have $\|\rho(f)\|_{\mathrm{HS}} = \|f\|_{\mathrm{HS}}$.*

Proof. For $k = 1$ we have

$$\begin{aligned}\operatorname{Tr}(\rho(f)) &= t_1^*(\mathrm{id}_1^{\otimes 2} \otimes t_1^*)(\mathrm{id}_1^{\otimes 2} \otimes f \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes t_1 \otimes \mathrm{id}_1)t_1 \\ &= t_1^*(f \otimes \mathrm{id}_1)(t_1^* \otimes \mathrm{id}_1^{\otimes 2})(\mathrm{id}_1 \otimes t_1 \otimes \mathrm{id}_1)t_1 \\ &= t_1^*(f \otimes \mathrm{id}_1)t_1 = \operatorname{Tr}(f).\end{aligned}$$

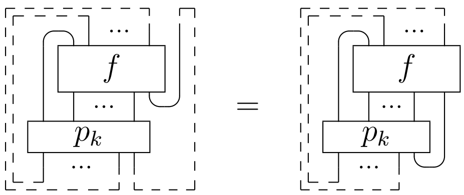
On diagrams, computing the trace correspond to connecting upper and lower points pairwise by non-crossing lines on the left or on the right. Representing this by dotted lines for clarity, the computation above can be pictured as follows:

$\operatorname{Tr}(\rho(f)) =$  $= \operatorname{Tr}(f)$

When $k \geqslant 2$, we first perform the transformation

$$\begin{aligned}\operatorname{Tr}(\rho(f)) &= \operatorname{Tr}((P_k \otimes t_1^*)(\mathrm{id}_1 \otimes f \otimes \mathrm{id}_1)(t_1 \otimes \mathrm{id}_1^{\otimes k})) \\ &= \operatorname{Tr}((\mathrm{id}_1^{\otimes k-1} \otimes t_1^*)(P_k \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes f)(t_1 \otimes \mathrm{id}_1^{\otimes k-1}))\end{aligned}$$

which can be diagrammatically represented as follows:

$\operatorname{Tr}(\rho(f)) =$ 

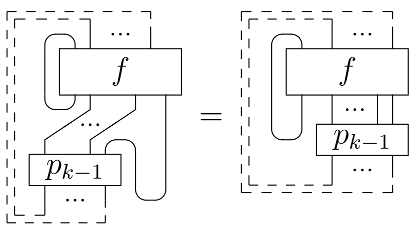
Then, we use the adjoint of Wenzl's formula (2). The term with $P_{k-1} \otimes \mathrm{id}_1$ yields

$$\operatorname{Tr}\left((\mathrm{id}_1^{\otimes(k-1)} \otimes t_1^*)(P_{k-1} \otimes \mathrm{id}_1^{\otimes 2})(\mathrm{id}_1 \otimes f)(t_1 \otimes \mathrm{id}_1^{\otimes(k-1)})\right) = \operatorname{Tr}\left((P_{k-1} \otimes t_1^*)(\mathrm{id}_1 \otimes f)(t_1 \otimes \mathrm{id}_1^{\otimes(k-1)})\right).$$

This vanishes because the range of f is contained in H_k and $\mathrm{id}_1^{\otimes(k-2)} \otimes t_1^*$ is an intertwiner to $H_1^{\otimes(k-2)}$, which contains no subrepresentation equivalent to H_k. The terms from (2) with $l > 1$ also vanish because $(\mathrm{id}_1^{\otimes(l-1)} \otimes t_1^* \otimes \mathrm{id}_1^{\otimes(k-l-1)} \otimes t_1 \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes f) = 0$ for the same reason as before. Hence, we are left with

$$\begin{aligned}\operatorname{Tr}(\rho(f)) &= \frac{(-1)^{k-1}}{d_{k-1}}\operatorname{Tr}\left((P_{k-1} \otimes t_1^*)(t_1^* \otimes \mathrm{id}_1^{\otimes(k-2)} \otimes t_1 \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes f)(t_1 \otimes \mathrm{id}_1^{\otimes(k-1)})\right) \\ &= \frac{(-1)^{k-1}}{d_{k-1}}\operatorname{Tr}\left(P_{k-1}(t_1^* \otimes \mathrm{id}_1^{\otimes(k-1)})(\mathrm{id}_1 \otimes f)(t_1 \otimes \mathrm{id}_1^{\otimes(k-1)})\right) = \frac{(-1)^{k-1}}{d_{k-1}}\operatorname{Tr}(f).\end{aligned}$$

Here is the diagrammatic computation:

 $= \operatorname{Tr}(f).$

For the Hilbert-Schmidt norm, we have

$$\begin{aligned}\operatorname{Tr}(\rho(f)^*\rho(f)) &= \operatorname{Tr}(t_1^* \otimes P_k)(\mathrm{id}_1 \otimes f^* \otimes \mathrm{id}_1)(P_k \otimes t_1t_1^*)(\mathrm{id}_1 \otimes f \otimes \mathrm{id}_1)(t_1 \otimes P_k) \\ &\leqslant \operatorname{Tr}\left((t_1^* \otimes \mathrm{id}_1^{\otimes k})(\mathrm{id}_1 \otimes f^* \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes k} \otimes t_1t_1^*)(\mathrm{id}_1 \otimes f \otimes \mathrm{id}_1)(t_1 \otimes \mathrm{id}_1^{\otimes k})\right) \\ &= \operatorname{Tr}\left((t_1^* \otimes \mathrm{id}_1^{\otimes k-1} \otimes t_1^*)(\mathrm{id}_1 \otimes f^* \otimes \mathrm{id}_1^{\otimes 2})(\mathrm{id}_1^{\otimes k} \otimes t_1t_1^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes f \otimes \mathrm{id}_1^{\otimes 2})(t_1 \otimes \mathrm{id}_1^{\otimes k-1} \otimes t_1)\right) \\ &= \operatorname{Tr}\left((t_1^* \otimes \mathrm{id}_1^{\otimes k-1})(\mathrm{id}_1 \otimes f^*)(\mathrm{id}_1 \otimes f)(t_1 \otimes \mathrm{id}_1^{\otimes k-1})\right) \\ &= \operatorname{Tr}(f^*f).\end{aligned}$$

□

Proposition 3.2. *Assume that $N > 2$. Let $a, b, c \in \mathbb{N}$ and consider the operator*

$$x_{a,b,c} = (\mathrm{id}_a \otimes \mathrm{tr}_b \otimes \mathrm{id}_c)(P_{a+b+c}) : H_a \otimes H_c \to H_a \otimes H_c.$$

Then, there exist two constants $\lambda_{a,c} > 0$ and $D_{a,c} > 0$ depending only on N, a and c such that

$$\|x_{a,b,c} - \lambda_{a,c}(\mathrm{id}_a \otimes \mathrm{id}_c)\| \leqslant D_{a,c} q^b.$$

In particular $x_{a,b,c} \to \lambda_{a,c}(\mathrm{id}_a \otimes \mathrm{id}_c)$ as $b \to \infty$.

Proof. For convenience, the proof will be done with the non-normalized trace, and hence we consider the non-normalized operator $X_{a,b,c} = (\mathrm{id}_a \otimes \mathrm{Tr}_b \otimes \mathrm{id}_c)(P_{a+b+c}) = d_b x_{a,b,c}$. We first observe that

$$(\mathrm{Tr}_a \otimes \mathrm{Tr}_c)(X_{a,b,c}) = \mathrm{Tr}(P_{a+b+c}) = d_{a+b+c} = d_b q^{-a-c} + O(q^b)$$

and accordingly set

$$\lambda_{a,c} = q^{-a-c}/d_a d_c \text{ and } X'_{a,b,c} = X_{a,b,c} - d_b \lambda_{a,c}(\mathrm{id}_a \otimes \mathrm{id}_c).$$

With this notation, we have $\mathrm{Tr}(X'_{a,b,c}) = O(q^b)$ and we want to show that $\|X'_{a,b,c}\| \leqslant D_{a,c}$. We will prove that

$$|(\mathrm{Tr}_a \otimes \mathrm{Tr}_c)(X'_{a,b,c} f)| \leqslant D_{a,c} \|f\|_{\mathrm{HS}}$$

for any $f \in \mathcal{B}(H_a \otimes H_c)$. Moreover any such f can be decomposed into a multiple of the identity and a map with zero trace, and since the estimate is satisfied for $f = \mathrm{id}$ by our choice of $\lambda_{a,c}$ we can assume $(\mathrm{Tr}_a \otimes \mathrm{Tr}_c)(f) = 0$. Eventually, we note that in this case $(\mathrm{Tr}_a \otimes \mathrm{Tr}_c)(X'_{a,b,c} f) = (\mathrm{Tr}_a \otimes \mathrm{Tr}_c)(X_{a,b,c} f)$.

Now we observe that $(\mathrm{Tr}_a \otimes \mathrm{Tr}_c)(X_{a,b,c} f) = \mathrm{Tr}(P_{a+b+c} f_{13})$ where Tr is the trace of $H_1^{\otimes(a+b+c)}$, and we use Wenzl's formula (2) to write

$$\begin{aligned}\mathrm{Tr}(X_{a,b,c} f) &= \mathrm{Tr}((P_{a+b+c-1} \otimes \mathrm{id}_1) f_{13}) \\ &\quad - \sum_{l=1}^{a+b+c-1} \frac{d_{l-1}}{d_{a+b+c-1}} \mathrm{Tr}\left((\mathrm{id}_1^{\otimes(l-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes(a+b+c-l-1)} \otimes t_1^*)(P_{a+b+c-1} \otimes \mathrm{id}_1) f_{13}\right).\end{aligned}$$

Moreover, one can factor $P_a \otimes P_b \otimes P_c$ out of the right side of $(P_{a+b+c-1} \otimes \mathrm{id}_1) f_{13}$. Since $P_k(\mathrm{id} \otimes t_1 \otimes \mathrm{id}) = 0$ on $H_1^{\otimes(k-2)}$, we see that $(P_a \otimes P_b \otimes P_c)(\mathrm{id}_1^{\otimes(l-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes(a+b+c-l-1)}) = 0$ if $l \neq a$ and $l \neq a+b$. Hence there are only three terms to bound in the expression above.

The first term is equal to

$$\mathrm{Tr}((P_{a+b+c-1} \otimes \mathrm{id}_1) f_{13}) = \mathrm{Tr}(X_{a,b,c-1} f^\flat),$$

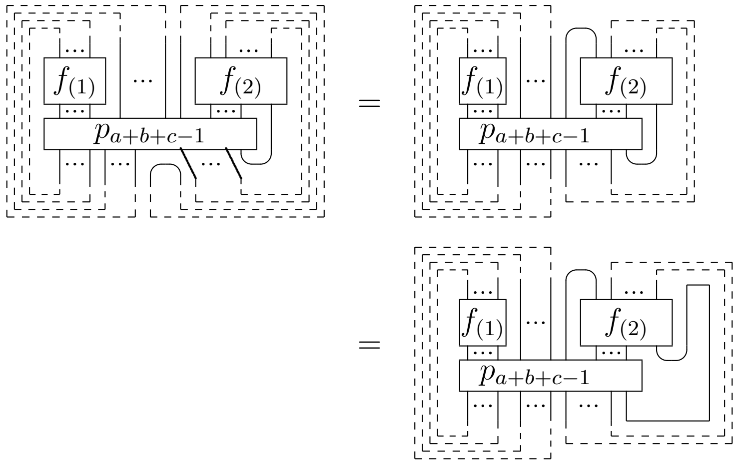
where $f^\flat = (\mathrm{id}_a \otimes \mathrm{id}_{c-1} \otimes \mathrm{Tr}_1)(f)$ satisfies $\mathrm{Tr}(f^\flat) = 0$ and $\|f^\flat\|_{\mathrm{HS}} \leqslant \sqrt{d_1} \|f\|_{\mathrm{HS}}$. For $l = a$, we use the trivial bound

$$\frac{d_{a-1}}{d_{a+b+c-1}} \times \|f\|_{\mathrm{HS}} \times \|t_1\|^2 \times \|P_{a+b+c-1} \otimes \mathrm{id}_1\|_{\mathrm{HS}} = \frac{d_1^{3/2} d_{a-1}}{\sqrt{d_{a+b+c-1}}} \|f\|_{\mathrm{HS}}.$$

For $l = a+b$, if we denote the term we are interested in by Y, we have, with $f = \sum f_{(1)} \otimes f_{(2)} \in \mathcal{B}(H_a) \otimes \mathcal{B}(H_c)$,

$$\begin{aligned} Y &= \mathrm{Tr}\left((\mathrm{id}_1^{\otimes(a+b-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes(c-1)} \otimes t_1^*)(P_{a+b+c-1} \otimes \mathrm{id}_1) f_{13}\right) \\ &= \mathrm{Tr}\left((\mathrm{id}_1^{\otimes(a+b+c-2)} \otimes t_1^*)(P_{a+b+c-1} \otimes \mathrm{id}_1) f_{13} (\mathrm{id}_1^{\otimes(a+b-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes(c-1)})\right) \\ &= \mathrm{Tr}\left((\mathrm{id}_1^{\otimes(a+b+c-2)} \otimes t_1^*)(P_{a+b+c-1} \otimes (t_1^* \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes t_1)) f_{13} (\mathrm{id}_1^{\otimes(a+b-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes(c-1)})\right) \\ &= \mathrm{Tr}\left((\mathrm{id}_1^{\otimes(a+b+c-2)} \otimes t_1^*)([(P_{a+b+c-1} \otimes t_1^*)(f_{13} \otimes \mathrm{id}_1)(\mathrm{id}_1^{(a+b-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes c})] \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes(a+b+c-2)} \otimes t_1)\right) \\ &= \sum \mathrm{Tr}\left((P_{a+b+c-1} \otimes t_1^*)(f_{(1)} \otimes \mathrm{id}_{b-1} \otimes f_{(2)} \otimes \mathrm{id}_1)(\mathrm{id}_1^{\otimes(a+b-1)} \otimes t_1 \otimes \mathrm{id}_1^{\otimes c})\right) \\ &= \mathrm{Tr}(P_{a+(b-1)+c} f_{13}^\sharp) = \mathrm{Tr}(X_{a,b-1,c} f^\sharp) \end{aligned}$$

where $f^\sharp = (\mathrm{id}_a \otimes \rho)(f)$ satisfies $\mathrm{Tr}(f^\sharp) = 0$ and $\|f^\sharp\|_{\mathrm{HS}} \leqslant \|f\|_{\mathrm{HS}}$ by Lemma 3.1. Here is the diagrammatic version of the previous computation,



We recognize indeed $\rho(f_{(2)})$ in the last diagram. The projections P_c included in the definition of $\rho(f_{(2)})$ do not appear on the diagram since they are absorbed by $P_{a+b+c-1}$ (through the trace for one of them), but they must be taken into account. Summing up, we have

$$|\,\mathrm{Tr}(X_{a,b,c}f)| \leqslant |\,\mathrm{Tr}(X_{a,b,c-1}f^\flat)| + \frac{d_{a+b-1}}{d_{a+b+c-1}}|\,\mathrm{Tr}(X_{a,b-1,c}f^\sharp)| + \frac{d_1^{3/2}d_{a-1}}{\sqrt{d_{a+b+c-1}}}\|f\|_{\mathrm{HS}}. \tag{5}$$

We will now proceed by induction on c with the following induction hypothesis

$H(c)$: "for all $a \in \mathbb{N}$ there exists a constant $D_{a,c}$ such that for all $b \in \mathbb{N}$ and all $f \in \mathcal{B}(H_a \otimes H_c)$ satisfying $\mathrm{Tr}(f) = 0$ we have $|\,\mathrm{Tr}(X_{a,b,c}f)| \leqslant D_{a,c}\|f\|_{\mathrm{HS}}$".

Recall that $H(0)$ holds with $D_{a,c} = 0$ because $X_{a,b,0}$ is an intertwiner, hence a multiple of the identity.

Now we take $c > 0$, we assume that $H(c-1)$ holds and we apply it to the first term in the right-hand side of Equation (5). Since $\|f^\flat\|_{\mathrm{HS}} \leqslant \sqrt{d_1}\|f\|_{\mathrm{HS}}$ and $d_{a-1} \leqslant d_{a+b+c-1}$, this yields

$$|\,\mathrm{Tr}(X_{a,b,c}f)| \leqslant (\sqrt{d_1}D_{a,c-1} + d_1^{3/2}\sqrt{d_{a-1}})\|f\|_{\mathrm{HS}} + \frac{d_{a+b-1}}{d_{a+b+c-1}}|\,\mathrm{Tr}(X_{a,b-1,c}f^\sharp)|.$$

We set $D' = \max(\sqrt{d_1}D_{a,c-1} + d_1^{3/2}\sqrt{d_{a-1}}, \sqrt{d_{a+c}})$ and we iterate the inequality above over b. Noticing that $|\,\mathrm{Tr}(X_{a,0,c}f^{\sharp b})| \leqslant \sqrt{d_{a+c}}\|f^{\sharp b}\|_{\mathrm{HS}} \leqslant D'\|f^{\sharp b}\|_{\mathrm{HS}}$ this yields, with the convention that the product equals 1 for $l = 0$:

$$|\,\mathrm{Tr}(X_{a,b,c}f)| \leqslant D'\sum_{l=0}^{b}\|f^{\sharp l}\|\left(\prod_{t=b-l+1}^{b}\frac{d_{a+t-1}}{d_{a+t+c-1}}\right).$$

Using the inequality $\|f^\sharp\|_{\mathrm{HS}} \leqslant \|f\|_{\mathrm{HS}}$, as well as the estimate $d_x/d_y \leqslant q^{y-x}$ for $x < y$ and the fact that $|q| < 1$ if $N > 2$, we see that $H(c)$ holds :

$$|\,\mathrm{Tr}(X_{a,b,c}f)| \leqslant D'\|f\|_{\mathrm{HS}}\sum_{l=0}^{b}q^{lc} \leqslant D'\|f\|_{\mathrm{HS}}\sum_{l=0}^{\infty}q^{lc}.$$

□

It is clear from the beginning of the proof that the Proposition 3.2 has the following equivalent formulation, which we will use for the proof of Theorem 4.3:

Corollary 3.3. *Assume that $N > 2$. For any $a, b \in \mathbb{N}$ and any $f \in \mathcal{B}(H_a \otimes H_c)$ such that* $\mathrm{Tr}(f) = 0$*, there exists a constant $D_{a,c}$ such that we have, for any $b \in \mathbb{N}$:*

$$|\,\mathrm{Tr}(P_{a+b+c}f_{13})| \leqslant D_{a,c}\|f\|_{\mathrm{HS}}.$$

3.2. **A variation on Wenzl's recursion formula.** The second result can be called a "higher weight" version of Wenzl's recursion formula (4). As a matter of fact, let $\zeta = \sum \zeta^{(1)}\otimes\zeta^{(2)}$ be a vector in $H_2 \subset H_1\otimes H_1$. Then, the map $f = \zeta^{(2)}\overline{\zeta}^{(1)*} \in \mathcal{B}(H_1)$ has trace 0, so that applying $\mathrm{Tr}_1(f\,\cdot\,)\otimes \mathrm{id}_{n-1}$ to both sides of Equation (4) yields

$$\sum(\overline{\zeta}^*_{(1)}\otimes \mathrm{id}_{n-1})P_n(\zeta_{(2)}\otimes \mathrm{id}_{n-1}) = -\frac{d_{n-2}}{d_{n-1}}\sum P_{n-1}(\zeta_{(1)}\overline{\zeta}^*_{(2)}\otimes \mathrm{id}_{n-2})P_{n-1}.$$

What we are going to prove is a similar equality but with ζ being any highest weight vector, i.e. $\zeta \in H_{p+q} \subset H_p\otimes H_q$ for arbitrary p and q.

Lemma 3.4. *Let $\zeta \in H_{p+q}$ be decomposed as $\zeta = \sum\zeta^{(1)}\otimes\zeta^{(2)} \in H_p\otimes H_q$ and $\zeta = \sum\zeta_{(1)}\otimes\zeta_{(2)} \in H_q\otimes H_p$. For all $n\in\mathbb{N}$, there exist $\alpha^n_{p,q}\in\mathbb{C}$ such that*

$$\sum(\overline{\zeta}^{(1)*}\otimes \mathrm{id}_{n-p})P_n(\zeta^{(2)}\otimes \mathrm{id}_{n-q}) = \alpha^n_{p,q}\sum P_{n-p}(\zeta_{(1)}\overline{\zeta}^*_{(2)}\otimes \mathrm{id}_{n-p-q})P_{n-q} \tag{6}$$

$$\sum(\mathrm{id}_{n-p}\otimes\zeta^{(1)*})P_n(\mathrm{id}_{n-q}\otimes\overline{\zeta}^{(2)}) = \alpha^n_{p,q}\sum P_{n-p}(\mathrm{id}_{n-p-q}\otimes\overline{\zeta}_{(1)}\zeta^*_{(2)})P_{n-q}. \tag{7}$$

Moreover, there exist constants $C_{p,q}>0$ such that for all $n\in\mathbb{N}$, $C_{p,q}\leqslant|\alpha^n_{p,q}|\leqslant 1$.

Proof. Let us first note that the second equality follows from the first one by conjugation, hence we will only focus on the first one. If $p=0$, then

$$P_n(\zeta^{(2)}\otimes \mathrm{id}_{n-q}) = P_n(\zeta^{(2)}\otimes \mathrm{id}_{n-q})P_{n-q} = P_n(\zeta_{(1)}\otimes \mathrm{id}_{n-q})P_{n-q}$$

and the result is proved with $\alpha^n_{0,q}=1$ for all n. Similarly, the result holds for $q=0$ with $\alpha^n_{p,0}=1$. We will proceed by induction on p and q with the induction hypothesis

H_N : "For any p,q with $p+q\leqslant N$, there exists a constant $C_{p,q}>0$ such that for all $n\in\mathbb{N}$, there is a constant $\alpha^n_{p,q}$ such that Equations (6) and (7) hold and $C_{p,q}\leqslant|\alpha^n_{p,q}|\leqslant 1$."

As we have seen, H_0 and H_1 hold, so let us assume H_N and consider $p,q\geqslant 1$ such that $p+q=N+1$. In order to use the induction hypothesis, we refine the decompositions of ζ in the following way:

$$\begin{aligned}
\zeta^{(1)} &= \sum\zeta^{(11)}\otimes\zeta^{(12)} \in H_{p-1}\otimes H_1,\\
\zeta^{(1)} &= \sum\zeta^{(1)}_{(1)}\otimes\zeta^{(1)}_{(2)} \in H_1\otimes H_{p-1},\\
\zeta^{(2)} &= \sum\zeta^{(21)}\otimes\zeta^{(22)} \in H_1\otimes H_{q-1},\\
\zeta^{(2)} &= \sum\zeta^{(2)}_{(1)}\otimes\zeta^{(2)}_{(2)} \in H_{q-1}\otimes H_1.
\end{aligned}$$

Applying the map $\sum(\overline{\zeta}^{(1)*}\otimes \mathrm{id}_{n-p})(\,\cdot\,)(\zeta^{(2)}\otimes \mathrm{id}_{n-q})$ to Wenzl's formula (4), the first term on the right-hand side reads

$$\begin{aligned}
&\sum(\overline{\zeta}^{(12)*}\otimes\overline{\zeta}^{(11)*}\otimes \mathrm{id}_{n-p})(\mathrm{id}_1\otimes P_{n-1})(\zeta^{(21)}\otimes\zeta^{(22)}\otimes \mathrm{id}_{n-q})\\
&=\sum\overline{\zeta}^{(12)*}(\zeta^{(21)})(\overline{\zeta}^{(11)*}\otimes \mathrm{id}_{n-p})P_{n-1}(\zeta^{(22)}\otimes \mathrm{id}_{n-q}).
\end{aligned}$$

Consider the linear map $T: H_{p-1}\otimes H_{q-1}\to\mathcal{B}(H_{n-q},H_{n-p})$ defined by $T(x\otimes y) = (\overline{x}^*\otimes \mathrm{id}_{n-p})P_{n-1}(y\otimes \mathrm{id}_{n-q})$. Then, the term above equals

$$T\left(\sum\overline{\zeta}^{(12)*}(\zeta^{(21)})(\zeta^{(11)}\otimes\zeta^{(22)})\right) = T\left((\mathrm{id}_{p-1}\otimes t_1^*\otimes \mathrm{id}_{n-q})(\zeta)\right).$$

The argument of T in the right-hand side vanishes because ζ is a highest weight vector, so that the whole term vanishes. Coming back to (4) and setting $L = \sum(\overline{\zeta}^{(1)*}\otimes \mathrm{id}_{n-p})P_n(\zeta^{(2)}\otimes \mathrm{id}_{n-q})$, we thus have

$$\begin{aligned}
L &= -\frac{d_{n-2}}{d_{n-1}}\sum(\overline{\zeta}^{(1)*}\otimes \mathrm{id}_{n-p})(\mathrm{id}_1\otimes P_{n-1})(t_1t_1^*\otimes \mathrm{id}_{n-2})(\mathrm{id}_1\otimes P_{n-1})(\zeta^{(2)}\otimes \mathrm{id}_{n-q})\\
&= -\frac{d_{n-2}}{d_{n-1}}\sum(\overline{\zeta}^{(11)*}\otimes \mathrm{id}_{n-p})P_{n-1}(\overline{\zeta}^{(12)*}\otimes \mathrm{id}_{n-1})(t_1t_1^*\otimes \mathrm{id}_{n-2})(\zeta^{(21)}\otimes \mathrm{id}_{n-1})P_{n-1}(\zeta^{(22)}\otimes \mathrm{id}_{n-q})\\
&= -\frac{d_{n-2}}{d_{n-1}}\sum(\overline{\zeta}^{(11)*}\otimes \mathrm{id}_{n-p})P_{n-1}(\zeta^{(12)}\overline{\zeta}^{(21)*}\otimes \mathrm{id}_{n-2})P_{n-1}(\zeta^{(22)}\otimes \mathrm{id}_{n-q}).
\end{aligned}$$

Now we apply H_N to $\zeta^{(1)}$ (with $p' = p-1$, $q' = 1$) and to $\zeta^{(2)}$ (with $p' = 1$, $q' = q-1$) to get

$$L = -\frac{d_{n-2}}{d_{n-1}}\alpha^{n-1}_{p-1,1}\alpha^{n-1}_{1,q-1}\sum\left(P_{n-p}(\zeta^{(1)}_{(1)}\overline{\zeta}^{(1)*}_{(2)}\otimes \mathrm{id}_{n-p-1})P_{n-2}\right)\left(P_{n-2}(\zeta^{(2)}_{(1)}\overline{\zeta}^{(2)*}_{(2)}\otimes \mathrm{id}_{n-q-1})P_{n-q}\right).$$

The last step is to apply again the induction hypothesis. To do this, we need to refine once more our decomposition by setting

$$\begin{aligned}\zeta &= \sum \eta^{(1)}\otimes\eta^{(2)}\otimes\eta^{(3)} \in H_1\otimes H_{p+q-2}\otimes H_1\\ \eta^{(2)} &= \sum \eta^{(21)}\otimes\eta^{(22)} \in H_{p-1}\otimes H_{q-1}\\ \eta^{(2)} &= \sum \eta^{(2)}_{(1)}\otimes\eta^{(2)}_{(2)} \in H_{q-1}\otimes H_{p-1}.\end{aligned}$$

Note that in the above computations we can replace everywhere $\zeta^{(1)}_{(1)}$, $\zeta^{(1)}_{(2)}$, $\zeta^{(2)}_{(1)}$ and $\zeta^{(2)}_{(2)}$ respectively by $\eta^{(1)}$, $\eta^{(21)}$, $\eta^{(22)}$ and $\eta^{(3)}$. Thus, applying H_N to $\eta^{(2)}$ (with $p' = p-1$, $q' = q-1$) yields

$$\begin{aligned}L = &-\frac{d_{n-2}}{d_{n-1}}\alpha^{n-1}_{p-1,1}\alpha^{n-1}_{1,q-1}\alpha^{n-2}_{p-1,q-1}\sum P_{n-p}(\eta^{(1)}\otimes \mathrm{id}_{n-p-1})\\ &P_{n-p-1}(\eta^{(2)}_{(1)}\overline{\eta}^{(2)*}_{(2)}\otimes \mathrm{id}_{n-q-1})P_{n-q-1}(\overline{\eta}^{(3)*}\otimes \mathrm{id}_{n-q-1})P_{n-q}\\ = &-\frac{d_{n-2}}{d_{n-1}}\alpha^{n-1}_{p-1,1}\alpha^{n-1}_{1,q-1}\alpha^{n-2}_{p-1,q-1}\sum P_{n-p}\left((\eta^{(1)}\otimes\eta^{(2)}_{(1)})(\overline{\eta}^{(3)*}\otimes\overline{\eta}^{(2)*}_{(2)})\otimes \mathrm{id}_{n-p-q}\right)P_{n-q}\\ = &-\frac{d_{n-2}}{d_{n-1}}\alpha^{n-1}_{p-1,1}\alpha^{n-1}_{1,q-1}\alpha^{n-2}_{p-1,q-1}\sum P_{n-p}(\zeta_{(1)}\overline{\zeta}^{*}_{(2)}\otimes \mathrm{id}_{n-p-q})P_{n-q}.\end{aligned}$$

This proves Equation (6) for p and q and as mentioned in the beginning of the proof, Equation (7) follows by conjugation. Moreover, we see that

$$|\alpha^n_{p,q}| \geqslant \frac{d_{n-2}}{d_{n-1}}C_{p-1,1}C_{1,q-1}C_{p-1,q-1} \geqslant \frac{1}{d_1}C_{p-1,1}C_{1,q-1}C_{p-1,q-1} > 0$$

hence H_{N+1} holds and the proof is complete. □

4. The key estimate

We now turn to the main technical result of this article, Theorem 4.3, which concerns the behaviour of the scalar product $\langle\chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\rangle$ as l, l' tend to $+\infty$. Its proof will span the whole of this section.

We start by recalling two technical lemmata from the literature on free orthogonal quantum groups. The first one gives a norm estimate for some explicit intertwiners in tensor products of irreducible representations. For any four integers l, k, m and a such that $k+l = m+2a$, the map

$$\left(V^{l,k}_m\right)^* = P_m(\mathrm{id}_{l-a}\otimes t^*_a\otimes \mathrm{id}_{k-a})$$

is an intertwiner from $H_l\otimes H_k$ to H_m, hence there is a scalar $\kappa^{l,k}_m$ such that $v^{l,k}_m = \kappa^{l,k}_m V^{l,k}_m$ is an isometric intertwiner. The scalar $\kappa^{l,k}_m$ can be explicitly computed, see [23]. However, we will only need the following consequence of this computation.

Lemma 4.1. *There exists a constant B_a, depending only on a and N, such that for all k, l and $m = k+l-2a$ we have $\left|\kappa^{l,k}_m\right| \leqslant B_a$.*

Proof. This is a consequence of the estimates given in [23, Lem 4.8], see also [10]. The sequence $(B_a)_a$ diverges exponentially as $q^{-a/2}$. □

We will also need the following estimates which were already used in [21] and [10].

Lemma 4.2. *Let x, y and z be integers and let $\mu \neq x+y+z$ be a subrepresentation of both $x\otimes(y+z)$ and $(x+y)\otimes z$. Then, there exists a constant $A > 0$ depending only on N such that*

$$\|(\mathrm{id}_x\otimes P_{y+z})(P_{x+y}\otimes \mathrm{id}_z) - P_{x+y+z}\| \leqslant Aq^y \text{ and } \|P^{x,y+z}_\mu P^{x+y,z}_\mu\| \leqslant Aq^y.$$

Proof. The first inequation is [21, Lem A.4]. For the second one, note that $P_\mu^{x,y+z}P_{x+y+z} = 0 = P_{x+y+z}P_\mu^{x+y,z}$ because μ is not the highest weight. Thus, we have

$$\begin{aligned}\|P_\mu^{x,y+z}P_\mu^{x+y,z}\| &= \|P_\mu^{x,y+z}\left((\mathrm{id}_x \otimes P_{y+z})(P_{x+y}\otimes \mathrm{id}_z) - P_{x+y+z}\right)P_\mu^{x+y,z}\| \\ &\leqslant \|P_\mu^{x,y+z}\|\|(\mathrm{id}_x \otimes P_{y+z})(P_{x+y}\otimes \mathrm{id}_z) - P_{x+y+z}\|\|P_\mu^{x+y,z}\| \\ &\leqslant \|(\mathrm{id}_x \otimes P_{y+z})(P_{x+y}\otimes \mathrm{id}_z) - P_{x+y+z}\| \\ &\leqslant Aq^y.\end{aligned}$$

□

We now state and prove an estimate, as l, l' tend to $+\infty$, about the scalar product between products of the characters χ_l, $\chi_{l'}$ with coefficients of fixed representations. Since χ_l, $\chi_{l'}$ have norm 1 in the GNS space $L^2(\mathbb{G})$, it is clear that these scalar products are bounded when l, l' tend to $+\infty$. However one can do much better:

Theorem 4.3. *Assume that $N > 2$. Let k, n be integers, let $\xi, \eta \in H_n$ be* orthogonal *unit vectors and let $\xi', \eta' \in H_k$ be arbitrary unit vectors. Then, there exists $K > 0$ such that we have, for all integers l, l':*

$$\left|\left\langle \chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\right\rangle\right| \leqslant Kq^{\max(l,l')}.$$

In particular $\left|\left\langle \chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\right\rangle\right| \to 0$ *when l or l' tends to $+\infty$.*

Proof. The proof will consist in the following steps:

1. computation of the scalar product as a sum $S = \sum S_m$ in the category of representations,
2. simplification of S_m into T_m,
3. expression of T_m as a trace,
4. application of Lemma 3.4 to reduce the trace,
5. application of Proposition 3.2 to estimate the trace,
6. backtracking of all approximations.

Step 1. We compute the products and the scalar product using the formulæ given in Subsection 2.2:

$$\begin{aligned} S = \left\langle \chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\right\rangle &= \sum_{i=1}^{d_l}\sum_{j=1}^{d_{l'}} \left\langle u^l_{e_ie_i}u^k_{\xi'\eta'}, u^n_{\xi\eta}u^{l'}_{e_je_j}\right\rangle \\ &= \sum_{i=1}^{d_l}\sum_{j=1}^{d_{l'}}\sum_{m=0}^{+\infty} \left\langle u^m_{v_m^{l,k*}(e_i\otimes\xi'),v_m^{l,k*}(e_i\otimes\eta')}, u^m_{v_m^{n,l'*}(\xi\otimes e_j),v_m^{n,l'*}(\eta\otimes e_j)}\right\rangle \\ &= \sum_{i=1}^{d_l}\sum_{j=1}^{d_{l'}}\sum_{m=0}^{+\infty} \frac{1}{d_m}\left\langle v_m^{l,k*}(e_i\otimes\xi'), v_m^{n,l'*}(\xi\otimes e_j)\right\rangle\left\langle v_m^{n,l'*}(\eta\otimes e_j), v_m^{l,k*}(e_i\otimes\eta')\right\rangle \\ &= \sum_{i=1}^{d_l}\sum_{j=1}^{d_{l'}}\sum_{m=0}^{+\infty} \frac{1}{d_m}\left\langle v_m^{l,k*}(e_i\otimes\xi'), v_m^{n,l'*}(\xi\otimes e_j)\right\rangle\left\langle v_m^{k,l*}(\overline{\eta}'\otimes \overline{e}_i), v_m^{l',n*}(\overline{e}_j\otimes\overline{\eta})\right\rangle \end{aligned}$$

$$(8)\qquad = \sum_{m=0}^{+\infty}\frac{1}{d_m}\left\langle\left(v_m^{l,k}\otimes v_m^{k,l}\right)^*\circ(\Sigma\otimes\Sigma)\left(\xi'\otimes t_l\otimes\overline{\eta}'\right), \left(v_m^{n,l'}\otimes v_m^{l',n}\right)^*\left(\xi\otimes t_l\otimes\overline{\eta}\right)\right\rangle.$$

Let us denote by S^m the m-th term in brackets in (8) and note that it can only be non-zero if u^m is a subrepresentation of both $u^k\otimes u^l$ and $u^n\otimes u^{l'}$. This means that there are integers a and b such that

$$l + k = m + 2a \text{ and } n + l' = m + 2b.$$

Note that $l-n+b-a=l'-k+a-b$ and let us denote by c this number. To estimate S^m, we will use the explicit formula for the intertwiners given in Subsection 2.3:

$$\left(v_m^{l,k}\right)^* = \kappa_m^{kl} P_m(\mathrm{id}_{l-a} \otimes t_a^* \otimes \mathrm{id}_{k-a}),\ \left(v_m^{k,l}\right)^* = \kappa_m^{kl} P_m(\mathrm{id}_{k-a} \otimes t_a^* \otimes \mathrm{id}_{l-a}),$$
$$\left(v_m^{l',n}\right)^* = \kappa_m^{nl'} P_m(\mathrm{id}_{l'-b} \otimes t_b^* \otimes \mathrm{id}_{n-b}),\ \left(v_m^{n,l'}\right)^* = \kappa_m^{nl'} P_m(\mathrm{id}_{n-b} \otimes t_b^* \otimes \mathrm{id}_{l'-b}).$$

so that (8) becomes:

$$S^m = \left(\kappa_m^{kl}\right)^2 \left(\kappa_m^{nl'}\right)^2 \left\langle (P_m \otimes P_m) \left(\mathrm{id}_{l-a} \otimes t_a^* \otimes \mathrm{id}_{k-a}^{\otimes 2} \otimes t_a^* \otimes \mathrm{id}_{l-a}\right) (\Sigma \otimes \Sigma)(\xi' \otimes t_l \otimes \overline{\eta}') \right.,$$
$$\left. (P_m \otimes P_m) \left(\mathrm{id}_{n-b} \otimes t_b^* \otimes \mathrm{id}_{l'-b}^{\otimes 2} \otimes t_b^* \otimes \mathrm{id}_{n-b}\right) (\xi \otimes t_{l'} \otimes \overline{\eta}) \right\rangle.$$

Step 2. Let us set, for $0 \leqslant \mu, \mu' \leqslant m$,

$$S_{\mu,\mu'}^m = \left\langle (P_\mu^{l-a,k-a} \otimes P_{\mu'}^{k-a,l-a}) \left(\mathrm{id}_{l-a} \otimes t_a^* \otimes \mathrm{id}_{k-a}^{\otimes 2} \otimes t_a^* \otimes \mathrm{id}_{l-a}\right) (\Sigma \otimes \Sigma)(\xi' \otimes t_l \otimes \overline{\eta}') \right.,$$
$$\left. (P_\mu^{n-b,l'-b} \otimes P_{\mu'}^{l'-b,n-b}) \left(\mathrm{id}_{n-b} \otimes t_b^* \otimes \mathrm{id}_{l'-b}^{\otimes 2} \otimes t_b^* \otimes \mathrm{id}_{n-b}\right) (\xi \otimes t_{l'} \otimes \overline{\eta}) \right\rangle$$

so that $S^m = S_{m,m}^m$. If μ or μ' is strictly less than m, then we know by Lemma 4.2 that there is a constant A depending only on N such that either

$$\left\| P_\mu^{l-a,k-a} P_\mu^{n-b,l'-b} \right\| \leqslant A q^{l-a-(n-b)} \text{ or } \left\| P_{\mu'}^{k-a,l-a} P_{\mu'}^{l'-b,n-b} \right\| \leqslant A q^{l-a-(n-b)}.$$

This gives the bound $|S_{\mu,\mu'}^m| \leqslant A\|t_l\|\|t_{l'}\| q^c = A\sqrt{d_l}\sqrt{d_{l'}} q^c$ which will be used in the end to estimate S^m. Let us expand back the vectors $t_l = \sum e_t^l \otimes \overline{e}_t^l$ and $t_{l'} = \sum e_s^{l'} \otimes \overline{e}_s^{l'}$, and we introduce

$$T^m = \sum_{t=1}^{d_l} \sum_{s=1}^{d_{l'}} \left\langle \left(\mathrm{id}_{l-a} \otimes t_a^* \otimes \mathrm{id}_{k-a}^{\otimes 2} \otimes t_a^* \otimes \mathrm{id}_{l-a}\right) (e_t^l \otimes \xi' \otimes \overline{\eta}' \otimes \overline{e}_t^l) \right.,$$
$$\left. \left(\mathrm{id}_{n-b} \otimes t_b^* \otimes \mathrm{id}_{l'-b}^{\otimes 2} \otimes t_a^* \otimes \mathrm{id}_{n-b})\right) (\xi \otimes e_s^{l'} \otimes \overline{e}_s^{l'} \otimes \overline{\eta}) \right\rangle$$

so that $S^m = (\kappa_m^{kl})^2 (\kappa_m^{nl'})^2 (T^m - \sum S_{\mu,\mu'}^m)$, where the sum rune over all $(\mu, \mu') \neq (m, m)$.

Step 3. The problem is now to estimate T^m, using the following tensor decomposition of the vectors ξ, η, ξ' and η' in Sweedler's notation:

$$\xi = \sum \xi_{(1)} \otimes \xi_{(2)} \in H_{n-b} \otimes H_b,$$
$$\eta = \sum \eta_{(1)} \otimes \eta_{(2)} \in H_{n-b} \otimes H_b,$$
$$\xi' = \sum \xi'_{(1)} \otimes \xi'_{(2)} \in H_a \otimes H_{k-a},$$
$$\eta' = \sum \eta'_{(1)} \otimes \eta'_{(2)} \in H_a \otimes H_{k-a}.$$

Because $t_a^*(x\otimes\overline{y}) = y^*(x)$, we get

$$T^m = \sum_{t=1}^{d_l}\sum_{s=1}^{d_{l'}}\sum \left\langle \left(\mathrm{id}_{l-a}\otimes\overline{\xi}'^*_{(1)}\right)(e_t^l)\otimes\xi'_{(2)}\otimes\overline{\eta}'_{(2)}\otimes\left(\eta'^*_{(1)}\otimes\mathrm{id}_{l-a}\right)(\overline{e}_t^l)\ ,\right.$$
$$\left.\xi_{(1)}\otimes\left(\overline{\xi}^*_{(2)}\otimes\mathrm{id}_{l'-b}\right)(e_s^{l'})\otimes\left(\mathrm{id}_{l'-b}\otimes\eta^*_{(2)}\right)(\overline{e}_s^{l'})\otimes\overline{\eta}_{(1)}\right\rangle$$
$$= \sum_{t=1}^{d_l}\sum_{s=1}^{d_{l'}}\sum \left\langle \left(\xi^*_{(1)}\otimes\mathrm{id}_{l-a-(n-b)}\otimes\overline{\xi}'^*_{(1)}\right)(e_t^l)\otimes\left(\eta'^*_{(1)}\otimes\mathrm{id}_{l-a-(n-b)}\otimes\overline{\eta}^*_{(1)}\right)(\overline{e}_t^l)\ ,\right.$$
$$\left.\left(\overline{\xi}^*_{(2)}\otimes\mathrm{id}_{l'-b-(k-a)}\otimes\xi'^*_{(2)}\right)(e_s^{l'})\otimes\left(\overline{\eta}'^*_{(2)}\otimes\mathrm{id}_{l'-b-(k-a)}\otimes\eta^*_{(2)}\right)(\overline{e}_s^{l'})\right\rangle$$
$$= \sum_{t=1}^{d_l}\sum_{s=1}^{d_{l'}}\sum \left\langle \left(\xi^*_{(1)}\otimes\mathrm{id}_c\otimes\overline{\xi}'^*_{(1)}\right)(e_t^l), \left(\overline{\xi}^*_{(2)}\otimes\mathrm{id}_c\otimes\xi'^*_{(2)}\right)(e_s^{l'})\right\rangle$$
$$\times\left\langle \left(\eta'^*_{(1)}\otimes\mathrm{id}_c\otimes\overline{\eta}^*_{(1)}\right)(\overline{e}_t^l), \left(\overline{\eta}'^*_{(2)}\otimes\mathrm{id}_c\otimes\eta^*_{(2)}\right)(\overline{e}_s^{l'})\right\rangle .$$

The properties of conjugate vectors imply that

$$\left\langle \left(\eta'^*_{(1)}\otimes\mathrm{id}_c\otimes\overline{\eta}^*_{(1)}\right)(\overline{e}_t^l), \left(\overline{\eta}'^*_{(2)}\otimes\mathrm{id}_c\otimes\eta^*_{(2)}\right)(\overline{e}_s^{l'})\right\rangle = \left\langle \left(\overline{\eta}^*_{(2)}\otimes\mathrm{id}_c\otimes\eta'^*_{(2)}\right)(e_s^{l'}), \left(\eta^*_{(1)}\otimes\mathrm{id}_c\otimes\overline{\eta}'^*_{(1)}\right)(e_t^l)\right\rangle .$$

Making this change in the last expression of T^m and using the fact that $\sum\langle x, Se_s^{l'}\rangle\langle Te_s^{l'}, y\rangle = \langle x, SP_{l'}T^*y\rangle$ enables to simplify the sum over s, yielding

$$T^m = \sum\sum_{t=1}^{d_l}\left\langle \left(\xi^*_{(1)}\otimes\mathrm{id}_c\otimes\overline{\xi}'^*_{(1)}\right)(e_t^l),\ \left(\overline{\xi}^*_{(2)}\otimes\mathrm{id}_c\otimes\xi'^*_{(2)}\right)P_{l'}\left(\overline{\eta}_{(2)}\otimes\mathrm{id}_c\otimes\eta'_{(2)}\right)\left(\eta^*_{(1)}\otimes\mathrm{id}_c\otimes\overline{\eta}'^*_{(1)}\right)(e_t^l)\right\rangle$$
$$= \sum \mathrm{Tr}_{\otimes l}\left[P_l\left(\xi_{(1)}\overline{\xi}^*_{(2)}\otimes\mathrm{id}_c\otimes\overline{\xi}'_{(1)}\xi'^*_{(2)}\right)P_{l'}\left(\overline{\eta}_{(2)}\eta^*_{(1)}\otimes\mathrm{id}_c\otimes\eta'_{(2)}\overline{\eta}'^*_{(1)}\right)\right],$$

where $\mathrm{Tr}_{\otimes l}$ denotes the non-normalized trace on $H_1^{\otimes l}$.

Step 4. We cannot apply Corollary 3.3 to T^m because there are two highest weight projections instead of one. We will therefore use Lemma 3.4 to reduce the problem to a case where Corollary 3.3 applies. Let us first simplify the notation by setting

$$f = \sum \xi_{(1)}\overline{\xi}^*_{(2)} : H_b\to H_{n-b},$$
$$g = \sum \overline{\eta}_{(2)}\eta^*_{(1)} : H_{n-b}\to H_b,$$
$$f' = \sum \overline{\xi}'_{(1)}\xi'^*_{(2)} : H_{k-a}\to H_a,$$
$$g' = \sum \eta'_{(2)}\overline{\eta}'^*_{(1)} : H_a\to H_{k-a}.$$

By Lemma 4.2, $\|(\mathrm{id}_b\otimes P_{l'-b})(P_{l'-k+a}\otimes\mathrm{id}_{k-a}) - P_{l'}\| \leqslant Aq^c$ and $\|(P_{l-a}\otimes\mathrm{id}_a)(\mathrm{id}_{n-b}\otimes P_{l-n+b}) - P_l\| \leqslant Aq^c$, so that it is enough to study

$$Y^m = \mathrm{Tr}_{\otimes l}\left[(P_{l-a}\otimes\mathrm{id}_a)(\mathrm{id}_{n-b}\otimes P_{l-n+b})(f\otimes\mathrm{id}_c\otimes f')\right.$$
$$\left.(\mathrm{id}_b\otimes P_{l'-b})(P_{l'-k+a}\otimes\mathrm{id}_{k-a})(g\otimes\mathrm{id}_c\otimes g')\right]$$
$$= \mathrm{Tr}_{\otimes l}\left[(P_{l-a}\otimes\mathrm{id}_a)(f\otimes\mathrm{id}_{l-n+b})(\mathrm{id}_b\otimes P_{l-n+b})(\mathrm{id}_{l-n+2b-a}\otimes f')\right.$$
$$\left.(\mathrm{id}_b\otimes P_{l'-b})(\mathrm{id}_{l'-k+a}\otimes g')(P_{l'-k+a}\otimes\mathrm{id}_a)(g\otimes\mathrm{id}_{l-n+b})\right]$$
$$= \mathrm{Tr}_{\otimes l}\left[(\mathrm{id}_b\otimes P_{l-n+b})(\mathrm{id}_{l-n+2b-a}\otimes f')(\mathrm{id}_b\otimes P_{l'-b})(\mathrm{id}_{l'-k+a}\otimes g')\right.$$
$$\left.(P_{l'-k+a}\otimes\mathrm{id}_a)(g\otimes\mathrm{id}_{l-n+b})(P_{l-a}\otimes\mathrm{id}_a)(f\otimes\mathrm{id}_{l-n+b})\right].$$

We now apply Lemma 3.4 to f' (with $p = k-a$ and $q = a$) and g (with $p = n-b$ and $q = b$):

$$P_{l-n+b}(\mathrm{id}_c\otimes f')P_{l'-b} = \sum(\alpha^{l'+a-b}_{k-a,a})^{-1}(\mathrm{id}_{l-n+b}\otimes\xi'^{(1)*})P_{l'+a-b}(\mathrm{id}_{l'-b}\otimes\overline{\xi}'^{(2)})$$
$$P_{l'-k+a}(g\otimes\mathrm{id}_c)P_{l-a} = \sum(\alpha^{l-a+b}_{n-b,b})^{-1}(\eta^{(2)*}\otimes\mathrm{id}_{l'-k+a})P_{l+b-a}(\overline{\eta}^{(1)}\otimes\mathrm{id}_{l-a})$$

where $\xi' = \sum \xi'^{(1)} \otimes \xi'^{(2)} \in H_{k-a} \otimes H_a$ and $\eta = \sum \eta^{(1)} \otimes \eta^{(2)} \in H_b \otimes H_{n-b}$. This yields

$$\begin{aligned} Y^m &= \beta \sum \mathrm{Tr}_{\otimes l} \Big[(\mathrm{id}_{l-n+2b} \otimes \xi'^{(1)*})(\mathrm{id}_b \otimes P_{l'+a-b})(\mathrm{id}_{l'} \otimes \overline{\xi}'^{(2)})(\mathrm{id}_{l'-k+a} \otimes g') \\ &\qquad (\eta^{(2)*} \otimes \mathrm{id}_{l'-k+2a})(P_{l+b-a} \otimes \mathrm{id}_a)(\overline{\eta}^{(1)} \otimes \mathrm{id}_l)(f \otimes \mathrm{id}_{l-n+b}) \Big] \\ &= \beta \sum \mathrm{Tr}_{\otimes l} \Big[(\eta^{(2)*} \otimes \mathrm{id}_{l'-k+2a} \otimes \xi'^{(1)*})(\mathrm{id}_n \otimes P_{l'+a-b}) \\ &\qquad (P_{l+b-a} \otimes \mathrm{id}_k)(\overline{\eta}^{(1)} \otimes f \otimes \mathrm{id}_{l-n+b} \otimes g' \otimes \overline{\xi}'^{(2)}) \Big] \\ &= \beta \,\mathrm{Tr}_{\otimes l} \Big[(\mathrm{id}_n \otimes P_{l'+a-b})(P_{l+b-a} \otimes \mathrm{id}_k)(\tilde{g} \otimes f \otimes \mathrm{id}_c \otimes g' \otimes \tilde{f}') \Big] \end{aligned}$$

where $\beta = (\alpha_{k-a,a}^{l'+a-b} \alpha_{n-b,b}^{l-a+b})^{-1}$ and

$$\begin{aligned} \tilde{f}' &= \sum \overline{\xi}'^{(2)} \xi'^{(1)*} : H_{k-a} \to H_a, \\ \tilde{g} &= \sum \overline{\eta}^{(1)} \eta^{(2)*} : H_{n-b} \to H_b. \end{aligned}$$

To conclude the computation, we use again Lemma 4.2 to get the following bound:

$$\|(\mathrm{id}_n \otimes P_{l'+a-b})(P_{l+b-a} \otimes \mathrm{id}_k) - P_{l+k+b-a}\| \leqslant A q^c,$$

enabling us to eventually reduce the problem to the study of

$$Z^m = \beta \,\mathrm{Tr}_{\otimes l} \Big[P_{l+k+b-a} (\tilde{g} \otimes f \otimes \mathrm{id}_c \otimes g' \otimes \tilde{f}') \Big].$$

Step 5. We are now in the setting of Corollary 3.3. Note that

$$\mathrm{Tr}(P_n(\tilde{g} \otimes f)) = t_n^* \left[P_n((\mathrm{id}_b \otimes \eta^*)(t_b \otimes \mathrm{id}_{n-b}) \otimes (\mathrm{id}_{n-b} \otimes t_b^*)(\xi \otimes \mathrm{id}_b)) \otimes \mathrm{id}_n \right] t_n$$

so that the map $\eta^* \otimes \xi \mapsto \mathrm{Tr}(P_n(\tilde{g} \otimes f))$ is an intertwiner from $\overline{H}_n \otimes H_n \simeq H_n^* \otimes H_n$ to $\mathbb{C}$. Since the only intertwiners between these spaces are multiples of the scalar product and since we have assumed ξ and η to be orthogonal, we see that the trace vanishes. Besides, we have

$$\|P_n(\tilde{g} \otimes f) P_n\|_{\mathrm{HS}} \leqslant \|\tilde{g} \otimes f\|_{\mathrm{HS}} = \|\xi\| \|\eta\| = 1$$

and similarly $\|P_k(g' \otimes \tilde{f}')P_k\|_{\mathrm{HS}} \leqslant 1$. Thus, Corollary 3.3 applied to $F = P_n(\tilde{g} \otimes f)P_n \otimes P_k(g' \otimes \tilde{f}')P_k$ yields $|Z^m| \leqslant \beta D_{n,k}$.

Step 6. Now we can rewind the successive approximations to bound S^m. In the remainder of this proof, the symbols K_i will denote numbers possibly depending on n and k, but not on m, l and l'. Recall that a, b, c are defined in terms of m, l and l'. To bound $T^m - Z^m$, we use the rough estimate $|\mathrm{Tr}_H(X)| \leqslant \dim(H) \|X\|$ which holds for any Hilbert space H and any $X \in \mathcal{B}(H)$. Let us note that the operator norms of f, g, f', g' are dominated by their Hilbert-Schmidt norms, which are equal to 1. However, the space over which we take the trace is $H_1^{\otimes l}$, which is too big. We therefore take advantage of the projections inside the trace to restrict to $H_b \otimes H_{l'-b}$ and $H_{l-a} \otimes H_a$ when passing from T^m to Y^m and to $H_n \otimes H_{l'+a-b}$ when passing from Y^m to Z^m. This yields:

$$|T^m| \leqslant |T^m - Y^m| + |Y^m - Z^m| + |Z^m| \leqslant A(d_b d_{l'-b} + d_a d_{l-a} + d_n d_{l'+a-b}) q^c + \beta D_{n,k}.$$

By the second part of Lemma 3.4, $\beta D_{n,k}$ is bounded by $C_{k-a,a}^{-1} C_{n-b,b}^{-1} D_{n,k}$. Because $a \leqslant k$ and $b \leqslant n$ take only a finite number of values when n and k are fixed, all these constants can be bounded by a constant K_0. We can also bound the coefficient of q^c by

$$A(d_n d_{l'} + d_k d_l + d_n d_{l'+k}) \leqslant K_1 q^{-\max(l,l')}.$$

Secondly, we have to consider the sum of the $|S_{\mu,\mu'}^m|$'s for $(\mu,\mu') \neq (m,m)$. Note that this term is non-zero only if μ and μ' are subrepresentations respectively of $(l-a) \otimes (k-a)$ and $(n-b) \otimes (l'-b)$. Thus, there are at most $\min(k-a, n-b) \leqslant \min(k,n)$ such terms and each of them is bounded by $A\sqrt{d_l}\sqrt{d_{l'}} q^c$, as explained in the beginning of the proof. Also recall from Lemma 4.1 that κ_m^{kl} and $\kappa_m^{nl'}$ are respectively bounded by B_a

and B_b, and since a, b take only a finite number of values (determined by k and n), they are bounded by a constant K_2. Summing up, we have

$$\begin{aligned}|S^m| &\leqslant K_2^4|T^m| + \min(k,n)A\sqrt{d_l}\sqrt{d_{l'}}q^c \\ &\leqslant K_2^4K_1q^{c-\max(l,l')} + K_2^4K_0 + K_3q^{c-\max(l,l')}.\end{aligned}$$

Let $t = \min(n+a-b, k+b-a)$. Then, $c \geqslant \max(l,l') - t$ and thus we have proved that $|S^m|$ is bounded by a constant K_4 independent of m, l and l'.

To obtain our estimate for S, we now have to sum the S^m's. Note that for S^m to be non-zero, $m = k+l-2a = n+l'-2b$ must be a subrepresentation of both $l\otimes k$ and $n\otimes l'$. There are at most $\min(k,n)$ such m's and they moreover satisfy $m \geqslant \max(l-k, l'-n)$, so that $d_m \geqslant K_5q^{-\max(l,l')}$ and we can write

$$|S| \leqslant \sum_{m=0}^{+\infty}\frac{1}{d_m}|S^m| \leqslant \min(k,n)K_5q^{\max(l,l')}K_4.$$

□

5. The radial subalgebra

We are now ready to prove the announced results on the radial subalgebra. Before going into the proofs, we recall the definition of this subalgebra as well as some of its basic properties.

Definition 5.1. For any representation v of a compact quantum group $\mathbb{G}$, the *character* of v is the element $\chi_v = (\mathrm{id}\otimes\mathrm{Tr})(v) \in C(\mathbb{G})$. This element depends only on the equivalence class of v.

The *radial subalgebra* $A \subset L^\infty(O_N^+)$ is the von Neumann subalgebra generated by the fundamental character $\chi_1 = \chi_u$, where u is the matrix of generators.

Note that the radial subalgebra was also used as a sub-C*-algebra A_f of the full C*-algebra $C(O_N^+)$ by M. Brannan in [4]. The spectrum of χ_1 in $C(O_N^+)$ is $[-N,N]$, whereas it is $[-2,2]$ in $C_{\mathrm{red}}(O_N^+)$ and $L^\infty(O_N^+)$. In the full case, the evaluation functionals $\mathrm{ev}_t : A_f \to \mathbb{C}$ at $t \in [-N,N]$ induce completely positive maps $T_t : L^\infty(O_N^+) \to L^\infty(O_N^+)$ which approximate the identity as $t \to N$. This allowed M. Brannan to prove that $L^\infty(O_N^+)$ has the Haagerup approximation property.

The terminology is justified by the following analogy with the "classical case" of the free group factors $\mathcal{L}(\mathbb{F}_N)$. More precisely, denote the standard generators of $\mathbb{F}_N$ by a_i and consider

$$u = \mathrm{diag}(a_1,\dots,a_N,a_1^{-1},\dots,a_N^{-1}) \in \mathcal{L}(\mathbb{F}_N)\otimes\mathcal{B}(\mathbb{C}^{2N}).$$

This is indeed a representation of the compact quantum group dual to $\mathbb{F}_N$, we put $\chi_1 = \chi_u = \sum_{i=1}^N(a_i+a_i^{-1}) \in \mathcal{L}(\mathbb{F}_N)$ and we define the radial subalgebra $A \subset \mathcal{L}(\mathbb{F}_N)$ as the von Neumann subalgebra generated by χ_1. If we consider, for $x \in \mathcal{L}(\mathbb{F}_N)$ and $g \in \mathbb{F}_N$, the coefficient $x_g = \langle x, g\rangle = \tau(g^*x)$ with respect to the standard trace τ, then x belongs to A if and only if the function $(g \mapsto x_g)$ is *radial*, i.e. x_g only depends on the word length of g.

The fusion rules of O_N^+ imply that $\chi_1\chi_n = \chi_n\chi_1 = \chi_{n+1} + \delta_{n>0}\chi_{n-1}$, so that the radial subalgebra is abelian and generated as a weakly closed subspace by the characters $(\chi_n)_{n\in\mathbb{N}}$. Moreover, it was proved in [1] that the spectrum of χ_1 in $L^\infty(O_N^+)$ is $[-2,2]$ and that the restriction of the Haar state is the semi-circle law. More precisely, one can identify A with $L^\infty([-2,2])$ via the functional calculus $f \mapsto f(\chi_1)$ and the scalar product induced by the Haar state is computed via

$$\langle f(\chi_1), g(\chi_1)\rangle = \frac{1}{2\pi}\int_{-2}^{2} f(s)\overline{g(s)}\sqrt{4-s^2}ds.$$

In particular, the radial subalgebra is diffuse. The characters χ_n correspond to dilated Chebyshev polynomials of the second kind: $\chi_n(X) = T_n(X) = U_n(X/2)$ where $T_0 = 1$, $T_1 = X$ and $T_1T_n = T_{n+1} + T_{n-1}$ if $n \geqslant 1$. It is known from the classical theory of Chebyshev polynomials that when restricted to $[-2,2]$, $\|T_n\|_\infty = \|U_n\|_\infty = n+1$. Since $L^\infty(O_N^+)$ is a finite von Neumann algebra, there is a unique h-preserving conditional expectation $\mathbb{E} : M \to A$, which is explicitly given by

$$\mathbb{E}(u^n_{\xi\eta}) = \frac{\langle\xi,\eta\rangle}{d_n}\chi_n. \tag{9}$$

We shall denote by $A^\perp = \{z \in M, \mathbb{E}(z) = 0\}$, which by Equation (9) is the weak closure of the linear span of coefficients $u^n_{\xi\eta}$ with $\langle \xi, \eta\rangle = 0$.

As mentioned in the preliminaries, all the results of this article apply in fact to general free orthogonal quantum groups $O^+(Q)$ *of Kac type*, i.e. such that Q is a scalar multiple of a unitary matrix. The situation for non-Kac type free orthogonal quantum groups is however quite different. First recall that $L^\infty(O^+(Q))$ is in that case a type III factor, at least for some values of the parameter Q (see [21]). More precisely the Haar state has then a non-trivial modular group, which is given on the generating matrix $u \in L^\infty(\mathbb{G}) \otimes \mathcal{B}(\mathbb{C}^N)$ by

$$(\sigma_t \otimes \mathrm{id})(u) = (\mathrm{id} \otimes {}^t(Q^*Q)^{-it})u(\mathrm{id} \otimes {}^t(Q^*Q)^{-it}),$$

where we assume Q to be normalized so that $\mathrm{Tr}(Q^*Q) = \mathrm{Tr}((Q^*Q)^{-1})$. In particular, it is clear that $\sigma_t(\chi_1)$ does not belong to A for all t unless $Q^*Q \in \mathbb{C}I_N$, and this implies that there exists no h-invariant conditional expectation onto A in the non-Kac case. It might even be that there exist no normal conditional expectation onto A at all. On the other hand, as far as we know all the available tools for the study of abelian subalgebras require the presence of a conditional expectation. Let us also comment on the $N = 2$ case, where the tools developed in the previous section break down. If we restrict to Kac type free orthogonal quantum groups, there are only two examples at $N = 2$ up to isomorphism, namely $SU(2)$ and $SU_{-1}(2)$. In the first case $C(SU(2))$ is commutative so that A is clearly not maximal abelian, and in fact A is not maximal abelian either in the second case – this is easily seen by embedding $C(SU_{-1}(2))$ into $C(S^3, M_2(\mathbb{C}))$ by [27].

With the estimate of Theorem 4.3, we can investigate the structure of the radial subalgebra. In fact, all the proofs are quite straightforward using techniques which are well-known to experts in von Neumann algebras. We however chose to give detailed proof both for convenience of the reader and for the sake of completeness. From now on, we will write $M = L^\infty(O_N^+)$ and $A = \{\chi_1\}''$.

5.1. **Maximal abelianness.** We first prove that A is maximal abelian. This will follow from the following lemma concerning unitary sequences in A, which relies itself on Theorem 4.3. In fact here we only use the fact that $|\langle \chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\rangle| \to 0$ as l, $l' \to \infty$ if ξ is orthogonal to η.

Lemma 5.2. *Let $N \geqslant 3$. Let $(u_i)_i$ be a sequence of unitaries in A weakly converging to 0 and let $z \in A^\perp$. Then, $u_i z u_i^*$ converges $*$-weakly to 0.*

Proof. For any i, let us decompose u_i as $u_i = \sum_{l=0}^{+\infty} a^i_l \chi_l$ and note that by unitarity, $\|(a^i_l)_l\|_2 = 1$. Assume for the moment that z is of the form $u^n_{\xi\eta}$ for some integer n and two orthogonal unit vectors $\xi, \eta \in H_n$. Considering another integer k and two arbitrary unit vectors $\xi', \eta' \in H_k$, we will first prove that

$$S_i = |\langle u^k_{\xi'\eta'}, u_i u^n_{\xi\eta} u_i^*\rangle| = \left| \sum_{l,l'=0}^{+\infty} a^i_l \overline{a}^i_{l'} \langle u^k_{\xi'\eta'}, \chi_l u^n_{\xi\eta}\chi_{l'}\rangle \right| \underset{i\to+\infty}{\longrightarrow} 0.$$

Let $\epsilon > 0$ and note that $\langle u^k_{\xi'\eta'}, \chi_l u^n_{\xi\eta}\chi_{l'}\rangle = \langle \chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\rangle$. By Theorem 4.3, there exists $L \in \mathbb{N}$ such that $|\langle \chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\rangle| \leqslant \epsilon/2$ as soon as $l, l' > L$. Thus,

$$\begin{aligned} S_i &\leqslant \sum_{l,l'=0}^{L} |a^i_l \overline{a}^i_{l'} \langle u^k_{\xi'\eta'}, \chi_l u^n_{\xi\eta}\chi_{l'}\rangle| + \frac{\epsilon}{2} \sum_{l,l'=L+1}^{+\infty} |a^i_l \overline{a}^i_{l'}| \\ &\leqslant \sum_{l,l'=0}^{L} |a^i_l \overline{a}^i_{l'} \langle u^k_{\xi'\eta'}, \chi_l u^n_{\xi\eta}\chi_{l'}\rangle| + \frac{\epsilon}{2} \|(a^i_l)_l\|_2^2 \end{aligned}$$

Now, because $u_i \to 0$ in the weak topology, $a^i_l = h(\chi_l u_i) \to 0$ for all fixed $l \in \mathbb{N}$ as $i \to +\infty$. In particular, there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$ and all $l, l' \leqslant L$,

$$|a^i_l \overline{a}^i_{l'}| \leqslant \frac{\epsilon}{2} \left(\sum_{l,l'=0}^{L} \langle u^k_{\xi'\eta'}, \chi_l u^n_{\xi\eta}\chi_{l'}\rangle \right)^{-1}.$$

Thus, for $i > i_0$, $|\langle u^k_{\xi'\eta'}, u_i u^n_{\xi\eta} u_i^*\rangle| \leqslant \epsilon$ and $S_i \to 0$.

Making finite linear combinations in the left-hand side, we see that $\langle t, u_i u^n_{\xi\eta} u_i^*\rangle$ tends to 0 as $i \to \infty$ for any $t \in \mathrm{Pol}(O_N^+)$. Since $\mathrm{Pol}(O_N^+)$ is dense in $L^2(O_N^+)$ and $(u_i u^n_{\xi\eta} u_i^*)_i$ is bounded $L^2(O_N^+)$, this is also true for any $t \in L^\infty(O_N^+) \subset L^2(O_N^+)$. Then, we can write $\langle t, u_i u^n_{\xi\eta} u_i^* \rangle = \langle u_i^* t u_i, u^n_{\xi\eta} \rangle$ and use similarly the density of $A^\perp \cap \mathrm{Pol}(O_N^+)$ in $A^\perp$ for the norm of $L^2(O_N^+)$. This shows that $\langle t, u_i z u_i^* \rangle = \langle u_i^* t u_i, z \rangle \to 0$ as $i \to \infty$ for any $t \in M$ and $z \in A^\perp$. Since h is a faithful trace and $(u_i z u_i^*)_i$ is bounded in $L^\infty(O_N^+)$, this shows the stated $*$-weak convergence. □

Theorem 5.3. *Let $N \geqslant 3$. Then, the radial subalgebra A is maximal abelian in M.*

Proof. Let $x \in A' \cap M$ and consider the decomposition $x = y + z$ with $y \in A$ and $z \in A^\perp$. Note that

$$x = u_i x u_i^* = u_i y u_i^* + u_i z u_i^* = y + u_i z u_i^*,$$

so that Lemma 5.2 yields $x = y + \lim_i u_i z u_i^* = y$. □

The argument above also proves that the C*-algebra generated by χ_1 is maximal abelian in the reduced C*-algebra $C_{\mathrm{red}}(O_N^+)$. From the theorem, following the strategy of [18], one can also recover the factoriality of $L^\infty(O_N^+)$ established in [21] and also in [10] (as a byproduct of non-inner amenability).

Corollary 5.4. *For $N \geqslant 2$, the von Neumann algebra $L^\infty(O_N^+)$ is a factor.*

Proof. We exploit the natural action of the classical group O_N on M given by the following formula, for $g \in O_N$ and $x \in C_{\mathrm{red}}(O_N^+)$:

$$\alpha_g(x) = (\mathrm{ev}_g \pi \otimes \mathrm{id})\Delta'(x),$$

where $\pi : C(O_N^+) \to C(O_N)$ is the canonical quotient map, $\mathrm{ev}_g : C(O_N) \to \mathbb{C}$ is the evaluation map at g, and $\Delta' : C_{\mathrm{red}}(O_N^+) \to C(O_N^+) \otimes C_{\mathrm{red}}(O_N^+)$ is induced from the coproduct of $C(O_N^+)$ thanks to Fell's absorption principle. The $*$-automorphism of $C_{\mathrm{red}}(O_N^+)$ defined in this way leaves the Haar state h invariant, and thus it extends to M. The action of α_g on coefficients of an irreducible representation u^n of O_N^+ is given by the following expression, where $v^n = (\pi \otimes \mathrm{id})(u^n)$ is the restriction of u^n to O_N:

$$(\alpha_g \otimes \mathrm{id})(u^n) = (\mathrm{ev}_g \otimes \mathrm{id} \otimes \mathrm{id})(v^n_{13} u^n_{23}) = (1 \otimes v^n(g)) u^n.$$

In particular we have $\alpha_g(\chi_n) = \sum_{rs} v^n(g)_{rs} u^n_{sr}$ where r, s are indices corresponding to an orthonormal basis of H_n. Note that α_g leaves the subspace of coefficients of any fixed representation of O_N^+ invariant.

Since A is maximal abelian in M, $\alpha_g(A)$ is maximal abelian in M for every $g \in O_N$, and so the center of M is contained in $\alpha_g(A)$ for every $g \in O_N$. Hence it suffices to show that the intersection of the subalgebras $\alpha_g(A)$ reduces to $\mathbb{C}1$. Equivalently, we take $c \in A$ such that $\alpha_g(c) \in A$ for all $g \in O_N$, and we want to prove that $c = \lambda 1$. For this we write $c = \sum c_n \chi_n$ in $L^2(O_N^+)$. The orthogonal projection of $\alpha_g(c)$ onto the subspace generated by the coefficients of u^n is $c_n \alpha_g(\chi_n)$, whereas the projection of A is $\mathbb{C}\chi_n$. Hence, if $c_n \neq 0$ then we must have $\alpha_g(\chi_n) \in \mathbb{C}\chi_n$ for all $g \in O_N$. By the computation above and the fact that the coefficients u^n_{rs} are linearly independent, this happens if and only if $v^n(g)$ is scalar for all g, i.e. v^n is a multiple of a one-dimensional representation. But then $v^{2n} \subset v^n \otimes v^n$ would be trivial, and if $n > 0$ this would imply that O_N has only finitely many irreducible representations up to equivalence, since any of them is contained in one of the v^k and $v^{k+1} \subset v^k \otimes v^1$. Hence $c_n = 0$ for all $n > 0$. □

5.2. **Singularity and the mixing property.** Now that we know that the radial subalgebra is a MASA, we can investigate further properties. By [13], we now that A cannot be a regular MASA (also called Cartan subalgebra) because M is strongly solid. In view of this result and of the case of the radial MASA in free group factors treated in [17], it is natural to conjecture that A is *singular*. Recall that for a von Neumann algebra N, we denote by $\mathcal{U}(N)$ the group of unitary elements of N.

Definition 5.5. A MASA $A \subset M$ is said to be singular if $\{u \in \mathcal{U}(M), uAu^* \subset A\} = \mathcal{U}(A)$.

There are several ways of proving that a MASA is singular. One way goes through a von Neumann algebraic analogue of the mixing property for group actions, called weak mixing, which eventually turns out to be equivalent to singularity. In our case, we can prove a stronger statement than singularity, namely that A is *mixing* in the following sense:

Definition 5.6. A subalgebra A of a von Neumann algebra M is said to be mixing if for any sequence $(u_n)_n$ of unitaries in A converging weakly to 0 and any elements $x, y \in A^\perp$,

$$\|\mathbb{E}_A(x u_n y)\|_2 \longrightarrow 0.$$

Again, the proof is an easy application of Theorem 4.3.

Theorem 5.7. *The radial MASA is mixing.*

Proof. Let $k, n \in \mathbb{N}$ and consider two pairs of orthogonal unit vectors $\xi, \eta \in H_n$ and $\xi', \eta' \in H_k$. We first estimate, for $l \in \mathbb{N}$,

$$X_{n,k}(i) = \|\mathbb{E}(u^{n*}_{\xi\eta} u_i u^k_{\xi'\eta'})\|_2^2.$$

To compute the square norm, we can use the orthonormal basis given by the characters to get

$$\begin{aligned} X_{n,k}(i) &= \sum_{l'=0}^{+\infty} |\langle \mathbb{E}(u^{n*}_{\xi\eta} u_i u^k_{\xi'\eta'}), \chi_{l'}\rangle|^2 \\ &= \sum_{l'=0}^{+\infty} |\langle u^{n*}_{\xi\eta} u_i u^k_{\xi'\eta'}, \chi_{l'}\rangle|^2 \\ &= \sum_{l'=0}^{+\infty} |\langle u_i u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\rangle|^2. \end{aligned}$$

We can now decompose the unitaries u_i according to the basis of characters: $u_i = \sum_{i=0}^{+\infty} a^i_l \chi_l$. We have $\|(a^i_l)_l\|_2 = 1$ for all i, and in particular $|a^i_l| \leqslant 1$ for all i and l. Moreover since $u_i \to 0$ weakly we have $a^i_l = h(\chi_l u_i) \to 0$ as $i \to \infty$ for each l. Then, Theorem 4.3 yields

$$\begin{aligned} X_{n,k}(i) &= \sum_{l'=0}^{+\infty}\sum_{l=0}^{+\infty} |a^i_l|^2 |\langle \chi_l u^k_{\xi'\eta'}, u^n_{\xi\eta}\chi_{l'}\rangle|^2 \\ &\leqslant K \sum_{l'=0}^{+\infty}\sum_{l=0}^{+\infty} |a^i_l|^2 q^{\max(l,l')}. \end{aligned}$$

We have $\sum_{l,l'} q^{\max(l,l')} < +\infty$, hence the dominated convergence theorem applies and we have $X_{n,k}(i) \to 0$ as $i \to \infty$. Since elements of the form $u^k_{\xi'\eta'}$ (resp. $u^{n*}_{\xi\eta}$) with $\xi' \perp \eta'$ (resp. $\xi \perp \eta$) span a dense subspace of $A^\perp \subset L^2(O_N^+)$, the proof is complete. □

Corollary 5.8. *The radial MASA is singular.*

Proof. This is a direct consequence of [6, Thm 4.1]. □

5.3. **The spectral measure.** Another very natural problem for a given MASA is to study the A-A-bimodule structure of $H = L^2(M) \ominus L^2(A)$. This can be done through the associated *spectral measure*. Because the representations of A on H on the left and on the right commute, their images generate an abelian von Neumann subalgebra of $\mathcal{B}(H)$ isomorphic to $L^\infty([-2,2]\times[-2,2])$. Thus, disintegrating H with respect to this subalgebra yields a measure class $[\nu]$ on $[-2,2]\times[-2,2]$ which encapsulates some properties of the bimodule.

Theorem 5.9. *The measure ν is Lebesgue equivalent to $\lambda\otimes\lambda$, where λ denotes the Lebesgue measure on $[-2,2]$.*

Proof. We will follow the strategy of [9]. Let us look at some "projections" of ν in the following sense: for two integers k and n and two pairs of orthogonal unit vectors $\xi, \eta \in H_n$ and $\xi', \eta' \in H_k$, there exists a measure μ on $[-2,2]\times[-2,2]$ such that for any $a, b \in A$,

$$\langle a u^k_{\xi'\eta'} b, u^n_{\xi,\eta}\rangle = \iint_{[-2,2]\times[-2,2]} a(s)b(t)d\mu(s,t).$$

We will compute the Radon-Nikodym derivative of μ with respect to $\lambda\otimes\lambda$. To do this, let us set

$$D_{l,l'} = \langle \chi_l u^k_{\xi'\eta'}\chi_{l'}, u^n_{\xi,\eta}\rangle$$
$$A_l(t) = \sum_{l'=0}^{+\infty} T_{l'}(t) D_{l,l'}.$$

Recall that if k, n and l are fixed, then there are at most $\max(k,n)$ integers l' such that $D_{l,l'}$ is non-zero and that they are all smaller than $l+k+n$. Thus, by Theorem 4.3 and the fact that $\|\chi_{l'}\| = \|T_{l'}\|_\infty = l'+1$ we have

$$\|A_l\|_\infty \leqslant \max(k,n)(l+k+n+1)Kq^{\max(l,l+k+n)} \leqslant K'lq^l$$

for some constant K' depending only on N, k and n. This implies that the series of functions

$$f(s,t) = \sum_{l=0}^{+\infty} T_l(s)A_l(t)$$

is normally convergent and since all summands are polynomials, f is analytic in s and t. This function is linked to the measure μ by the following computation:

$$\begin{aligned}\langle au^k_{\xi'\eta'}b, u^n_{\xi,\eta}\rangle &= \sum_{l,l'=0}^{+\infty} \langle a,\chi_l\rangle\langle b,\chi_{l'}\rangle\langle \chi_l u^k_{\xi'\eta'}\chi_{l'}, u^n_{\xi,\eta}\rangle = \sum_{l,l'=0}^{+\infty}\langle a,\chi_l\rangle\langle b,\chi_{l'}\rangle D_{l,l'}\\ &= \frac{1}{4\pi^2}\sum_{l,l'=0}^{+\infty} D_{l,l'}\left(\int_{-2}^{2} a(s)T_l(s)\sqrt{4-s^2}ds\right)\left(\int_{-2}^{2} b(t)T_l(t)\sqrt{4-t^2}dt\right)\\ &= \frac{1}{4\pi^2}\iint_{[-2,2]\times[-2,2]} a(s)b(t)f(s,t)\sqrt{4-s^2}\sqrt{4-t^2}d(\lambda\otimes\lambda)(s,t).\end{aligned}$$

Hence, $f(s,t)\sqrt{4-s^2}\sqrt{4-t^2}$ is the Radon-Nikodym derivative of μ with respect to $\lambda\otimes\lambda$. Since f is analytic and obviously not 0, its zeros are contained in a set of Lebesgue measure 0, hence μ is equivalent to $\lambda\otimes\lambda$.

Consider now an arbitrary element ζ in $\mathrm{Pol}(O_N^+)\cap A^\perp$. It can be written as a finite linear combination of coefficients corresponding to orthogonal vectors, hence the measure μ_ζ defined by

$$\langle a\zeta b,\zeta\rangle = \iint_{[-2,2]\times[-2,2]} a(s)b(t)d\mu_\zeta(s,t).$$

is equivalent to $\lambda\otimes\lambda$. Because $\mathrm{Pol}(O_N^+)\cap A^\perp$ is dense in $L^2(M)\ominus L^2(A)$, this implies that $[\nu]=[\lambda\otimes\lambda]$. □

Note that as a consequence, the $A-A$-bimodule $L^2(M)\ominus L^2(A)$ is contained in a multiple of the coarse bimodule, see [14, Section 2]. Since the coarse bimodule is mixing, we can also recover Theorem 5.7 in this way.

5.4. **Concluding remarks.** We would like to briefly discuss some possible extensions of this work. First consider the quantum automorphism group $\mathbb{G}(M_N(\mathbb{C}),\mathrm{tr})$ of $M_N(\mathbb{C})$ endowed with the canonical trace. It is known that the von Neumann algebra $L^\infty(\mathbb{G}(M_N(\mathbb{C}),\mathrm{tr}))$ of this quantum group embeds into $L^\infty(O_N^+)$ as the subalgebra generated by all $u^{2n}_{\xi,\eta}$ for $n\in\mathbb{N}$ and $\xi,\eta\in H_{2n}$. Let us set $v^n = u^{2n}$. Then, the v^n's form a complete family of representatives of irreducible representations of $\mathbb{G}(M_N(\mathbb{C}),\mathrm{tr})$ with corresponding characters $\psi_n=\chi_{2n}$. In particular, for any orthogonal unit vectors $\xi,\eta\in H_{2n}$ and $\xi',\eta'\in H_{2k}$,

$$\langle \psi_l v^k_{\xi',\eta'}, v^n_{\xi,\eta}\psi_{l'}\rangle \leqslant Kq^{\max(2l,2l')}$$

by Theorem 4.3. From this we see that the radial subalgebra in $L^\infty(\mathbb{G}(M_N(\mathbb{C}),\mathrm{tr}))$ is maximal abelian and mixing and that its associated bimodule is a direct sum of coarse bimodules. This is an interesting example because $\mathbb{G}(M_N(\mathbb{C}),\mathrm{tr})$ has $SO(3)$-type fusion rules, like another important family of discrete quantum groups called the quantum permutation groups S_N^+. This of course suggests that our result extends to S_N^+. One way to prove this may be through monoidal equivalence [3].

Another possible extension of our work would be to the non-Kac case. It is possible that the estimate of Theorem 4.3 still holds with appropriate modification for an arbitrary free orthogonal quantum group. However, the proofs of Section 5 all break down if the von Neumann algebra is type III, because the radial MASA has no h-invariant conditional expectation in that case. There is therefore an additional von Neumann algebraic problem to solve in that case, but this could yield very explicit examples of singular MASAs in type III factors.

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