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K -amenability for amalgamated free products of amenable discrete quantum groups

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Abstract

The basic notions and results of equivariant KK -theory concerning crossed products can be extended to the case of locally compact quantum groups. We recall these constructions and prove some useful properties of subgroups and amalgamated free products of discrete quantum groups. Using these properties and a quantum analogue of the Bass-Serre tree, we establish the K -amenability of amalgamated free products of amenable discrete quantum groups.

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The main goal of this paper is to prove the K -amenability of an amalgamated free product of amenable discrete quantum groups. The notion of K -amenability was first introduced by Cuntz [8] for discrete groups: the aim was to give a simpler proof to a result of Pimsner and Voiculescu [15] calculating the K -theory of the reduced C^* -algebra of a free group. Cuntz proves that the K -theory of the reduced and full C^* -algebras of the free groups are the same, and gives in [7] a very simple way to compute it in the full case.

Julg and Valette [11] extend the notion of K -amenability to the locally compact case and establish the K -amenability of locally compact groups acting on trees with amenable stabilizers. By a construction of Serre [16], this includes the case of

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amalgamated free products of amenable discrete groups. To prove the K -amenability of a locally compact group G , one has to construct an element $\alpha \in KK_G(\mathbb{C}, \mathbb{C})$ using representations of G that are weakly contained in the regular one, and then to prove that α is homotopic to the unit element of $KK_G(\mathbb{C}, \mathbb{C})$. In [11], both of the steps are carried out in a very geometrical way. Moreover, it turns out that α can be interpreted as the γ element used to prove the Baum–Connes conjecture in this context [12].

In fact, the K -amenability of the starting groups is often sufficient to get the K -amenability of their free product. This was already the case in the original result of Cuntz [8], which deals with the non-amalgamated and non-quantum case. This has been generalized to the amalgamated and non-quantum case by Pimsner [14] (in the wider framework of group actions on trees). Finally, the non-amalgamated and quantum case is covered by the work of Germain [9]. In this paper we treat the amalgamated and quantum case, but assume that both starting groups are amenable.

In the first section of this paper, we state some results that extend to the case of locally compact quantum groups classical tools of equivariant KK -theory related with crossed products, including the notion of K -amenability. The general framework is the equivariant KK -theory with respect to coactions of Hopf C^* -algebras [3] on the one hand, and the theory of locally compact quantum groups [4,13] on the other hand.

In the second section, we prove some useful results on subgroups and amalgamated free products of discrete quantum groups. Here we base ourselves on the definitions given by Wang [21], and extend some of the results therein. The main goal is to give an alternative form for the regular representation of an amalgamated free product of discrete quantum groups, which we then use for the K -theoretic construction of the last section.

Finally, we construct in the third section the Julg–Valette operator associated to the “quantum tree” of an amalgamated free product of amenable discrete quantum groups. We prove that it defines an element of KK -theory which is homotopic to the unit element, thus establishing the K -amenability of the amalgamated free product under consideration.

1. Equivariant KK -theory for locally compact quantum groups

In fact, most of the work of extending Kasparov’s equivariant theory to locally compact quantum groups was carried out by Baaj and Skandalis in [3], where they define equivariant KK -groups with respect to coactions of Hopf C^* -algebras, and prove the existence of the Kasparov product in this setting. When the Hopf C^* -algebra under consideration is associated to a locally compact quantum group, some additional features of the classical theory are expected to extend to the quantum case, especially in relation with duality and crossed products. It turns out that it is indeed what happens: we will give here a review of the results obtained in [18]. We refer the interested reader to the same reference for the proofs, which contain almost

no difficulty particular to the quantum case. Note that some of these results were already proved in [3] in the case of duals of locally compact groups.

The framework of our study will be the one of C^* -algebraic locally compact quantum groups, as defined by Kustermans and Vaes [13]. However, we mainly work with the Kac systems (H, V, U) associated to these quantum groups. Let us recall some notations and definitions from [4].

Let H be a Hilbert space, we will denote by Σ the flip operator on $H \otimes H$, and we will use the “leg numbering” notation for tensor products—for instance $\Sigma_{23} = \text{id} \otimes \Sigma \in L(H \otimes H \otimes H)$. A unitary operator $V \in L(H \otimes H)$ is said to be multiplicative if it verifies the pentagon equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$. We define two closed subalgebras of $L(H)$ associated to V in the following way:

$$S_{\text{red}} = \overline{\text{Lin}}\{(\omega \otimes \text{id})(V) \mid \omega \in L(H)_*\},$$

$$\hat{S}_{\text{red}} = \overline{\text{Lin}}\{(\text{id} \otimes \omega)(V) \mid \omega \in L(H)_*\}.$$

More generally, if X is a representation of V on some Hilbert B -module E , i.e. a unitary morphism $X \in L_B(E \otimes H)$ such that $X_{12}X_{13}V_{23} = V_{23}X_{12}$, we put $\hat{S}_X = \overline{\text{Lin}}\{(\text{id} \otimes \omega)(X) \mid \omega \in L(H)_*\} \subset L_B(E)$. Besides, if U is an involutive unitary on the same Hilbert space H , we put $\hat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ and $\tilde{V} = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma$. A Kac system in the (strong) sense of [4] is a triple (H, V, U) such that $(V(U \otimes 1)\Sigma)^3 = 1$, V and \hat{V} are multiplicative, and both fulfill an analytical condition called regularity.

The Kac systems (H, V, U) associated to locally compact quantum groups are not Kac systems in the sense of [4], however they verify the following useful properties [4,13,24]:

- (i) V and \hat{V} are multiplicative,
- (ii) $[S_{\text{red}}, US_{\text{red}}U] = [\hat{S}_{\text{red}}, U\hat{S}_{\text{red}}U] = 0$,
- (iii) S_{red} and \hat{S}_{red} are C^* -algebras,
- (iv) if X is a representation of V or \hat{V} , it belongs to $M(\hat{S}_X \otimes K(H))$.

In fact these are the only properties required for the validity of our results on equivariant KK -theory, which means that we do not make a direct use of the Haar weights on the locally compact quantum groups under consideration. Note that properties (iii) and (iv) are automatically verified when V and \hat{V} are regular [4], semi-regular [2] or manageable [24].

If (H, V, U) is the Kac system associated to a locally compact quantum group, and more generally if it verifies properties (i)–(iv), the C^* -algebras S_{red} and \hat{S}_{red} are naturally endowed with coproducts:

$$\delta_{\text{red}}(s) = V(s \otimes 1)V^* = \hat{V}^*(1 \otimes s)\hat{V},$$

$$\hat{\delta}_{\text{red}}(\hat{s}) = V^*(1 \otimes \hat{s})V = \tilde{V}(\hat{s} \otimes 1)\tilde{V}^*.$$

Moreover one can check that $(S_{\text{red}}, \delta_{\text{red}})$ and $(\hat{S}_{\text{red}}, \hat{\delta}_{\text{red}})$ are then Hopf C^* -algebras, in particular δ_{red} is a non-degenerate homomorphism from S_{red} to $M(S_{\text{red}} \otimes S_{\text{red}})$ and verifies the coassociativity identity $(\text{id} \otimes \delta_{\text{red}}) \circ \delta_{\text{red}} = (\delta_{\text{red}} \otimes \text{id}) \circ \delta_{\text{red}}$.

Recall that a coaction of S_{red} on a C^* -algebra A is a non-degenerate homomorphism $\delta_A : A \rightarrow M(A \otimes S_{\text{red}})$ such that $\delta_A(A)(1 \otimes S_{\text{red}}) \subset (A \otimes S_{\text{red}})$ and $(\text{id} \otimes \delta_{\text{red}}) \circ \delta_A = (\delta_A \otimes \text{id}) \circ \delta_A$. We say that A is a S_{red} -algebra if δ_A is injective and $\delta_A(A)(1 \otimes S_{\text{red}})$ spans a dense subspace of $A \otimes S_{\text{red}}$. We say that $a \in A$ is invariant if $\delta_A(a) = a \otimes 1$, and we denote by $A^{S_{\text{red}}}$ the sub- C^* -algebra of invariant elements. There is also a notion of coaction on Hilbert C^* -modules: a coaction of S_{red} on a Hilbert A -module E is a linear map $\delta_E : E \rightarrow M(E \otimes S_{\text{red}})$ compatible with δ_{red} , δ_A and the Hilbertian structure—see [3] for the details. When δ_A is trivial, i.e. $A^{S_{\text{red}}} = A$, the coactions δ_E of S_{red} on E correspond in fact to the representations X of V on E , via the following formula: $\delta_E(\xi) = X(\xi \otimes 1)$.

Finally, let $\delta_A, \delta_B, \delta_E$ be coactions of S_{red} on two C^* -algebras A, B and a Hilbert B -module E . We say that a representation $\pi : A \rightarrow L_B(E)$ is covariant if

$$\forall a \in A, \xi \in E \quad \delta_E(\pi(a)\xi) = (\pi \otimes \text{id})\delta_A(a) \cdot \delta_E(\xi).$$

If the coaction on B is trivial and X is the representation of V associated to δ_E , the covariance of π can also be written $(\pi \otimes \text{id})\delta_A(a) = X(\pi(a) \otimes 1)X^*$.

Let A be a C^* -algebra endowed with a coaction of S_{red} . Let X be a representation of V on a Hilbert B -module E , where S_{red} coacts trivially on B . To each covariant representation $\pi : A \rightarrow L_B(E)$ one can associate a crossed product C^* -algebra in the following way:

$$A \rtimes_{\pi} \hat{S} = \overline{\text{Lin}}\{\pi(a)x \mid a \in A, x \in \hat{S}_X\}.$$

In particular, there is a universal covariant representation (π^A, X^A) giving a full crossed product $A \rtimes \hat{S}$, and a regular covariant representation $(\pi_{\text{red}}^A, X_{\text{red}}^A)$ giving a reduced crossed product $A \rtimes_{\text{red}} \hat{S}$. Let us call A_1 the C^* -algebra A endowed with the trivial coaction of S_{red} . The regular covariant representation of A is then defined to be $\delta_A : A \rightarrow M(A \otimes K(H))$, with $E = A \otimes H$ seen as a A_1 -module and equipped with the representation $X_{\text{red}}^A = 1 \otimes V$.

One important tool is the dual coaction of \hat{S}_{red} (or of its full version \hat{S}) on $A \rtimes_{\text{red}} \hat{S}$ (or $A \rtimes \hat{S}$). For instance we use the following notations for the coactions of \hat{S}_{red} :

$$\hat{\delta}_{A \rtimes \hat{S}} : A \rtimes \hat{S} \rightarrow M((A \rtimes \hat{S}) \otimes \hat{S}_{\text{red}}) \quad \text{and}$$

$$\hat{\delta}_{A \rtimes_{\text{red}} \hat{S}} : A \rtimes_{\text{red}} \hat{S} \rightarrow M((A \rtimes_{\text{red}} \hat{S}) \otimes \hat{S}_{\text{red}}).$$

These coactions are naturally related by the reduction homomorphism $\lambda_A : A \rtimes \hat{S} \rightarrow A \rtimes_{\text{red}} \hat{S}$, and they arise from the following weak inclusions of covariant

representations:

$$(\pi^A \otimes \text{id}, X_{13}^A V_{23}) \prec (\pi^A, X^A) \quad \text{and} \quad (\pi_{\text{red}}^A \otimes \text{id}, X_{\text{red},13}^A V_{23}) \prec (\pi_{\text{red}}^A, X_{\text{red}}^A).$$

There is another weak inclusion $(\pi^A \otimes \text{id}, X_{13}^A V_{23}) \prec (\pi_{\text{red}}^A, X_{\text{red}}^A)$ which induces a “false coaction” $\delta'_A : A \rtimes_{\text{red}} \hat{S} \rightarrow M((A \rtimes \hat{S}) \otimes \hat{S}_{\text{red}})$. The homomorphism δ'_A is one of the important tools for the proof of Theorem 1.4 on K -amenability.

To conclude on crossed products, let us present a simple characterization of $A \rtimes_{\text{red}} \hat{S}$ when S_{red} is unital (i.e. the quantum group is compact) and A is a S_{red} -algebra. We introduce a new coaction of S_{red} on $E = A \otimes H$, seen as an A -module: we put $\delta'_E(a \otimes \xi) = ((U \otimes 1)V(U \otimes 1)(\xi \otimes 1))_{23} \delta_A(a)_{13}$. This coaction induces a coaction of S_{red} on $K_A(E)$, and we have then $A \rtimes_{\text{red}} \hat{S} = K_A(E)^{S_{\text{red}}}$. This equality is the main tool for the proof of the “Green-Julg” Theorem 1.3.

Now we turn to equivariant KK -theory. To simplify the statements we will assume that all the C^* -algebras we consider are separable. Let (H, V, U) be the Kac system associated to a locally compact quantum group, and let S_{red} be its reduced Hopf C^* -algebra. Let δ_B be a coaction of S_{red} on some C^* -algebra B . If δ_E is a coaction of S_{red} on a Hilbert B -module E , we provide the B -module $E \otimes H$ with the following coaction of $S_{\text{red}} : \delta_{E \otimes H}(\zeta \otimes \xi) = (V(\xi \otimes 1))_{23} \delta_E(\zeta)_{13}$. Let us call \mathcal{H}_B the Hilbertian direct sum of countably many copies of the Hilbert B -module B . We have then the following equivariant “Kasparov” stabilization theorem.

Theorem 1.1. *Let (H, V, U) be the Kac system associated to a locally compact quantum group, and let S_{red} be its reduced Hopf C^* -algebra. Let δ_B be a coaction of S_{red} on a C^* -algebra B .*

- (1) *Let E, F be two Hilbert B -modules equipped with coactions of S_{red} . If E and F are isomorphic as Hilbert B -modules, then there is an equivariant isomorphism between $E \otimes H$ and $F \otimes H$.*
- (2) *Let E be a countably generated Hilbert B -module endowed with a coaction of S_{red} . Then $(E \oplus \mathcal{H}_B) \otimes H$ is equivariantly isomorphic to $\mathcal{H}_B \otimes H$.*
- (3) *Assume moreover that S_{red} is unital. Then $E \oplus (\mathcal{H}_B \otimes H)$ is equivariantly isomorphic to $\mathcal{H}_B \otimes H$.*

Let us now recall the definition of $KK_{S_{\text{red}}}(A, B)$, where the C^* -algebras A and B are endowed with coactions of S_{red} : we denote by $E_{S_{\text{red}}}(A, B)$ the set of triples (E, π, F) where E is a countably generated, $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert B -module endowed with a coaction of S_{red} , π is a covariant, grading preserving representation of A on E , and $F \in L_B(E)$ is a morphism of degree 1 such that

$$[\pi(A), F], \pi(A)(F^2 - 1) \quad \text{and} \quad \pi(A)(F - F^*) \subset K_B(E),$$

$$\pi(A) \otimes S_{\text{red}}(F \otimes 1 - \delta_{K(E)}(F)) \subset K_{B \otimes S_{\text{red}}}(E \otimes S_{\text{red}}).$$

Then $KK_{S_{\text{red}}}(A, B)$ is the quotient of $E_{S_{\text{red}}}(A, B)$ with respect to the homotopy relation induced by $E_{S_{\text{red}}}(A, B[0, 1])$. Like in the classical case, every equivariant homomorphism $\phi : A \rightarrow B$ defines an element $[\phi] \in KK_{S_{\text{red}}}(A, B)$.

Using the fact that the Hopf C^* -algebras under consideration are associated to some locally compact quantum groups, one can define descent morphisms for these equivariant KK -groups. If E is a Hilbert B -module endowed with a coaction of S_{red} , the full and reduced crossed products of E by S_{red} are the relative tensor products $E \rtimes \hat{S} = E \otimes_B (B \rtimes \hat{S})$ and $E \rtimes_{\text{red}} \hat{S} = E \otimes_B (B \rtimes_{\text{red}} \hat{S})$. One can prove that the C^* -algebra $K(E \rtimes \hat{S})$ naturally identifies with the crossed product $K(E) \rtimes \hat{S}$, so that any C^* -representation $\pi : A \rightarrow M(K(E))$ gives rise to a homomorphism $\pi \rtimes \text{id} : A \rtimes \hat{S} \rightarrow M(K(E) \rtimes \hat{S})$. The same works for reduced crossed products, and this indeed defines descent morphisms:

Theorem 1.2 (cf. Baaj and Skandalis [4, rque 7.7b]). *Let S_{red} be the reduced Hopf C^* -algebra associated to a locally compact quantum group, and A, B two C^* -algebras equipped with coactions of S_{red} .*

- (1) *For any $(E, \pi, F) \in E_{S_{\text{red}}}(A, B)$, the triple $(E \rtimes \hat{S}, \pi \rtimes \text{id}, F \otimes_B \text{id})$ is an element of $E(A \rtimes \hat{S}, B \rtimes \hat{S})$, and this defines a group morphism $j : KK_{S_{\text{red}}}(A, B) \rightarrow KK(A \rtimes \hat{S}, B \rtimes \hat{S})$.*
- (2) *This morphism is compatible with the Kasparov product: for all $x \in KK_{S_{\text{red}}}(A, D)$, $y \in KK_{S_{\text{red}}}(D, B)$ one has $j(x \otimes y) = j(x) \otimes j(y)$.*
- (3) *In the same way, considering reduced crossed products one obtains a group morphism $j_{\text{red}} : KK_{S_{\text{red}}}(A, B) \rightarrow KK(A \rtimes_{\text{red}} \hat{S}, B \rtimes_{\text{red}} \hat{S})$ compatible with the Kasparov product.*

Finally we have a generalization of the Green–Julg theorem, when S_{red} is unital and the coaction δ_A is trivial. We will need the following notations to state it precisely:

- (1) Let $(B \rtimes_{\text{red}} \hat{S})_1$ be the C^* -algebra $B \rtimes_{\text{red}} \hat{S}$ endowed with the trivial coaction of S_{red} . Taking into account the fact that any B -module E can be fitted with the trivial coaction of S_{red} , we get a morphism $\psi : KK(A, B \rtimes_{\text{red}} \hat{S}) \rightarrow KK_{S_{\text{red}}}(A, (B \rtimes_{\text{red}} \hat{S})_1)$.
- (2) The inclusion $B \rtimes_{\text{red}} \hat{S} \subset K(E)^{S_{\text{red}}}$, where $E = B \otimes H$ is endowed with the coaction δ'_E , defines an element β of $KK_{S_{\text{red}}}((B \rtimes_{\text{red}} \hat{S})_1, B)$.
- (3) In the compact case, the trivial representation factorizes through \hat{S}_{red} and its central support p_0 is an element of \hat{S}_{red} . We call ϕ the homomorphism from A to $A \rtimes_{\text{red}} \hat{S} = A \otimes \hat{S}_{\text{red}}$ given by $\phi(a) = a \otimes p_0$.

Theorem 1.3. *Let S_{red} be the reduced Hopf C^* -algebra associated to a compact quantum group (S_{red} is unital). Assume that S_{red} coacts trivially on A , and that B is a S_{red} -algebra. Then the morphism*

$$\Phi_1 : KK_{S_{\text{red}}}(A, B) \xrightarrow{j_{\text{red}}} KK(A \rtimes_{\text{red}} \hat{S}, B \rtimes_{\text{red}} \hat{S}) \xrightarrow{\phi^*} KK(A, B \rtimes_{\text{red}} \hat{S})$$

is an isomorphism, and its inverse is given by

$$\Phi_2 : KK(A, B \rtimes_{\text{red}} \hat{S}) \xrightarrow{\psi} KK_{S_{\text{red}}}(A, (B \rtimes_{\text{red}} \hat{S})_1) \xrightarrow{\cdot \otimes \beta} KK_{S_{\text{red}}}(A, B).$$

As in the classical case, there is a “dual statement” of the Green–Julg theorem, which is much easier to prove: if S_{red} is unital and if \hat{S}_{red} coacts trivially on B , there is a canonical isomorphism $\Psi : KK_{\hat{S}_{\text{red}}}(A, B) \rightarrow KK(A \rtimes S, B)$.

Let us close this section with the notion of K -amenability, which was introduced in [8] for discrete groups and in [11] for locally compact groups. We say that a locally compact quantum group $(\hat{S}_{\text{red}}, \hat{\delta}_{\text{red}})$ is K -amenable if the unit element of $KK_{\hat{S}_{\text{red}}}(\mathbb{C}, \mathbb{C})$ can be represented by a triple (E, π, F) such that the representation of the quantum group on E is weakly contained in its regular representation. Using the tools introduced in this section, most notably the descent morphisms and the homomorphism δ'_A , one can give other characterizations of K -amenability, at least in the discrete case:

Theorem 1.4. *Let S and S_{red} be the full and reduced dual Hopf C^* -algebras of a locally compact quantum group $(\hat{S}_{\text{red}}, \hat{\delta}_{\text{red}})$. Let $\varepsilon : S \rightarrow \mathbb{C}$ be the trivial representation of S . Then we have (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) and, if S_{red} is unital, (iv) \Rightarrow (i):*

- (i) $(\hat{S}_{\text{red}}, \hat{\delta}_{\text{red}})$ is K -amenable.
- (ii) For every C^* -algebra A endowed with a coaction of \hat{S}_{red} , $[\lambda_A] \in KK(A \rtimes S, A \rtimes_{\text{red}} S)$ is invertible.
- (iii) $[\lambda] \in KK(S, S_{\text{red}})$ is invertible.
- (iv) There exists $\alpha \in KK(S_{\text{red}}, \mathbb{C})$ such that $\lambda^*(\alpha) = [\varepsilon] \in KK(S, \mathbb{C})$.

2. Complements on discrete quantum groups

We start with some useful results on subgroups and amalgamated free products of discrete quantum groups. In particular, Proposition 2.4 generalizes to the amalgamated case the identity $h = h_1 * h_2$ for the Haar state of a free product [21]. Together with Proposition 2.2, it shows that the reduced Hopf C^* -algebra of an amalgamated free product $S_1 *_T S_2$ can be identified with the reduced amalgamated free product of the reduced Hopf C^* -algebras.

In this section discrete quantum groups will be given by (one of) their C^* -algebra(s), i.e. by a Woronowicz C^* -algebra (S, δ) . Let us recall that a Woronowicz C^* -algebra is a unital Hopf C^* -algebra such that the sub spaces $(1 \otimes S)\delta(S)$ and $(S \otimes 1)\delta(S)$ are dense in $S \otimes S$. These conditions ensure the existence of a unique Haar state on S , i.e. a state $h \in S'$ such that $(h \otimes \text{id}) \circ \delta = (\text{id} \otimes h) \circ \delta = 1_S h$. The GNS representation of h is called the regular representation, and its image S_{red} is the reduced Woronowicz C^* -algebra of the discrete quantum group. We say that S is reduced if its regular representation is faithful, and in this case the Haar state h is also faithful (cf. the proof of Proposition 2.2).

The general theory of Tannaka–Krein duality [23] shows that the discrete quantum group associated to (S, δ) is fully characterized by the category \mathcal{C} of the finite-dimensional unitary corepresentations of (S, δ) . We will use the notation $\text{Irr } \mathcal{C}$ for a complete system of irreducible corepresentations in \mathcal{C} .

As a direct consequence of the general theory, we can see the subgroups of a discrete quantum group in both pictures.

Lemma 2.1. *Let (S, δ) be a Woronowicz C^* -algebra and \mathcal{C} the category of its finite-dimensional unitary corepresentations. There is then a natural bijection between:*

- (1) Woronowicz sub- C^* -algebras $T \subset S$ (with $1_T = 1_S$),
- (2) full subcategories $\mathcal{D} \subset \mathcal{C}$ such that $1_{\mathcal{C}} \in \mathcal{D}$, $\mathcal{D} \otimes \mathcal{D} \subset \mathcal{D}$ and $\bar{\mathcal{D}} = \mathcal{D}$.

Given such a subcategory $\mathcal{D} \subset \mathcal{C}$, the corresponding Woronowicz sub- C^* -algebra T is the closed subspace of S generated by the coefficients of the corepresentations $r \in \mathcal{D}$.

Proposition 2.2. *Let (S, δ) be a reduced Woronowicz C^* -algebra, and T a Woronowicz sub- C^* -algebra of S . Then:*

- (1) the Haar state h_T is the restriction of h_S , and T is reduced,
- (2) there exists a unique conditional expectation $P: S \rightarrow T$ such that $h_S = h_T \circ P$,
- (3) this conditional expectation is also characterized by the invariance property $(\text{id} \otimes P) \circ \delta_S = (P \otimes \text{id}) \circ \delta_S = \delta_T \circ P$.

Proof. (1) The fact that h_T is the restriction of h_S to T follows immediately from the uniqueness of h_T . Let λ_S, λ_T be the respective regular representations of S and T . Because Haar states of quantum groups are KMS, we have

$$\begin{aligned} \text{Ker } \lambda_T &= \{y \in T \mid h_T(y^*y) = 0\} \\ &= \{y \in T \mid h_S(y^*y) = 0\} = \text{Ker } \lambda_S \cap T = \{0\}. \end{aligned}$$

(2) Let $\mathcal{D} \subset \mathcal{C}$ be the categories of finite-dimensional unitary corepresentations of T and S , respectively. In this paragraph we will repeatedly use the structure theorem of Woronowicz [22] giving the Hopf C^* -algebra structure of S in terms of the category \mathcal{C} . Let \mathcal{S}° (resp. \mathcal{T}) be the linear span in S of the coefficients of the corepresentations $r \in \text{Irr } \mathcal{C} \setminus \text{Irr } \mathcal{D}$ (resp. $r \in \text{Irr } \mathcal{D}$): the direct sum $\mathcal{S} = \mathcal{S}^\circ \oplus \mathcal{T}$ is dense in S . As a consequence of the stability properties of \mathcal{D} , we have the inclusions $\mathcal{S}^\circ \mathcal{T} \subset \mathcal{S}^\circ$, $\mathcal{T} \mathcal{S}^\circ \subset \mathcal{S}^\circ$ and $\mathcal{S}^{\circ*} \subset \mathcal{S}^\circ$. For instance if $s \in \text{Irr } \mathcal{C} \setminus \text{Irr } \mathcal{D}$, $t \in \text{Irr } \mathcal{D}$ and $r \subset s \otimes t$ then $s \subset r \otimes \bar{t}$, so that r cannot be in \mathcal{D} . When passing to coefficients one obtains the first inclusion. In particular, these inclusions imply that $A_h(\mathcal{S}^\circ)$ and $A_h(\mathcal{T})$ are orthogonal subspaces in the GNS construction of h_S , because h_S maps the coefficients of all irreducible corepresentations to 0, except the ones of the trivial corepresentation.

Now we identify S and T with their images in the regular representation of S . Let p be the orthogonal projection onto the closure of $A_h(\mathcal{T})$. The results of the previous paragraph (and the fact that T is reduced) show that $Tp = pTp \simeq T$ and $p\mathcal{S}^\circ p = \{0\}$. Hence the compression by p defines a conditional expectation $P : S \rightarrow pTp \simeq T$. The identity $h_S = h_{T \circ P}$ follows from the fact that the Haar states are the vectorial states associated to $A_h(1)$. Conversely, if $P' : S \rightarrow T$ is a conditional expectation such that $h_S = h_{T \circ P'}$, for every $x \in S$ we have $P'(x) = P(x)$ because T is reduced and

$$\forall y \in T \quad h_T(P'(x)y) = h_T(P'(xy)) = h_S(xy) = h_T(P(xy)) = h_T(P(x)y).$$

(3) Finally, the invariance property of P can be easily verified on \mathcal{T} and \mathcal{S}° separately: we have $\delta(\mathcal{T}) \subset \mathcal{T} \otimes \mathcal{T}$ and $\delta(\mathcal{S}^\circ) \subset \mathcal{S}^\circ \otimes \mathcal{S}^\circ$, but P coincides with id on T , and with 0 on \mathcal{S}° . Conversely, if $P' : S \rightarrow T$ is a conditional expectation verifying the invariance property, we have

$$1_T h_S = (\text{id} \otimes h_S)(P' \otimes \text{id})\delta_S = (\text{id} \otimes h_T)\delta_T P' = 1_T h_{T \circ P'}$$

so that $h_S = h_{T \circ P'}$ and $P' = P$. \square

Lemma 2.3. *Let (S, δ) be a reduced Woronowicz C^* -algebra, and T a Woronowicz sub- C^* -algebra of S . Let E be the Hilbert T -module of the GNS representation of S associated to the conditional expectation P introduced in Proposition 2.2.*

- (1) *The relation on $\text{Irr } \mathcal{C}$ defined by $r \sim r' \Leftrightarrow (\exists t \in \mathcal{D} \quad t \subset \bar{r} \otimes r')$ is an equivalence relation, let $(\text{Irr } \mathcal{C})/\mathcal{D}$ be the corresponding quotient set.*
- (2) *Let \mathcal{S}^r be the subspace of \mathcal{S} generated by the coefficients of a corepresentation $r \in \mathcal{C}$. For $\alpha \in (\text{Irr } \mathcal{C})/\mathcal{D}$ we put $\mathcal{S}^\alpha = \sum_{r \in \alpha} \mathcal{S}^r$, let S° be the closure of $\mathcal{S}^\circ = \sum_{\alpha \neq 1} \mathcal{S}^\alpha$. Then we have $\text{Ker } P = S^\circ$.*
- (3) *We denote by E^α and E^r the respective closures of $A_P(\mathcal{S}^\alpha)$ and $A_P(\mathcal{S}^r)$ in E . For $r \in \alpha$, the closed submodule of E generated by E^r is E^α . Moreover E is the orthogonal direct sum of the E^α .*

Proof. (1) The first point essentially results from the following equivalence, that we already used in the previous proof: $t \subset \bar{r} \otimes r' \Leftrightarrow r' \subset r \otimes t$. We therefore have $r \sim r' \Leftrightarrow r' \subset r \otimes \mathcal{D}$, and the transitivity of \sim follows from the hypothesis $\mathcal{D} \otimes \mathcal{D} \subset \mathcal{D}$. The other conditions are obvious on the definition (and result from the hypothesis $\tilde{\mathcal{D}} = \mathcal{D}, 1_{\mathcal{C}} \in \mathcal{D}$).

(2) It is clear that the class of $1_{\mathcal{C}}$ in $(\text{Irr } \mathcal{C})/\mathcal{D}$ is $\text{Irr } \mathcal{D}$. In particular the subspace \mathcal{S}° coincides with the one we have used in the proof of Proposition 2.2, and we have already seen that it is included in $\text{Ker } P$. Now if we write $x \in \text{Ker } P$ as a limit of sums $y_n + z_n$ with $y_n \in \mathcal{T}$ and $z_n \in \mathcal{S}^\circ$, we have $P(y_n + z_n) = y_n \rightarrow 0$, so that x is in the closure of \mathcal{S}° .

(3) The linear span of $\mathcal{S}^r \mathcal{T}$ coincides with the subspace generated by the coefficients of the corepresentations $r \otimes t, t \in \mathcal{D}$, which is exactly \mathcal{S}^α : recall that $r' \sim r \Leftrightarrow r' \subset r \otimes \mathcal{D}$. So the closed submodule of E generated by E^r is E^α . Now let α

and β be two distinct elements of $(\text{Irr } \mathcal{C})/\mathcal{D}$, and take $x \in \mathcal{S}^\alpha, y \in \mathcal{S}^\beta$. We can assume that x and y are respective coefficients of two corepresentations $r \in \alpha, s \in \beta$. The subobjects of $\bar{r} \otimes s$ are not in \mathcal{D} because α and β are distinct, so that $x^*y \in \mathcal{S}^\circ$ and

$$\langle A_P(x), A_P(y) \rangle = P(x^*y) = 0.$$

This establishes the fact that E^α and E^β are orthogonal, and it is clear that the sum of the subspaces E^α is dense in E . \square

We will use the following notations about reduced amalgamated free products [1,19,20]. Let S_1, S_2 be unital C^* -algebras, and T a common sub- C^* -algebra of S_1 and S_2 containing their unit element. Let P_1, P_2 be two conditional expectations of S_1 and S_2 onto T , and $(E_1, \eta_1), (E_2, \eta_2)$ the corresponding GNS constructions. Like in Lemma 2.3, we denote by S_i° the kernel of P_i , and by E_i° the orthogonal of η_i in E_i —i.e. the closure of $A_{P_i}(S_i^\circ)$. Following [19], we put

$$E_1 *_T E_2 = \eta T \oplus \bigoplus_{n=1}^\infty \bigoplus_{(i_j) \in I_n} (E_{i_1}^\circ \otimes_T \dots \otimes_T E_{i_n}^\circ) \quad \text{where} \tag{*}$$

$$I_n = \{(i_1, \dots, i_n) \in \{1, 2\}^n \mid \forall k \ i_k \neq i_{k+1}\}.$$

The Hilbert T -module $E_1 *_T E_2$ carries then a natural representation of the (full) free product with amalgamation $S_1 *_T S_2$, whose image is called the reduced free product with amalgamation $(S_1, P_1) *_T (S_2, P_2)$. The vacuum vector η defines a conditional expectation $P_1 *_T P_2$ from $S_1 *_T S_2$ onto T . Finally, we will denote by $E(r, i)$ the closed submodule of E associated to the n -tuples $(i_j) \in I_n$ in $(*)$ such that $i_n \neq i$.

In [21], it is shown that the free product of two Woronowicz C^* -algebras with amalgamation over a Woronowicz sub- C^* -algebra carries again a natural structure of Woronowicz C^* -algebra. Moreover the Haar state and the category of finite-dimensional unitary corepresentations of that free product are investigated in the non-amalgamated case. It is not hard to generalize the result on the Haar state to the amalgamated case.

Proposition 2.4. *Let S'_1, S'_2 be two Woronowicz C^* -algebras, and T' be a common Woronowicz sub- C^* -algebra of S'_1 and S'_2 . Let S_1, S_2 and T be their respective reduced Woronowicz C^* -algebras. We consider the free product with amalgamation $S = S_1 *_T S_2$ and denote by $\lambda : S \rightarrow S_{\text{red}}$ the reduction homomorphism of (S, δ) .*

- (1) *If P, P_1 and P_2 are the canonical conditional expectations of S_{red}, S_1 and S_2 onto T , we have $P_1 *_T P_2 = P \circ \lambda$.*
- (2) *The reduced Woronowicz C^* -algebra of $S'_1 *_T S'_2$ is isomorphic to the reduced amalgamated free product of (S_1, P_1) and (S_2, P_2) .*

Proof. (1) We put $P' = P_1 *_T P_2$, and we prove in the next paragraph that $h_{T \circ P'}$ is the Haar state of S . This will prove that P' factors through S_{red} and, by Proposition 2.2, that its factorization equals P . Thanks to the uniqueness of the Haar state, it is enough to establish the invariance of the state $h_{T \circ P'}$.

The C^* -algebra S is a quotient of the free product $S_1 *_\mathbb{C} S_2$, so that the elements $s_1 \cdots s_n$ with $s_k \in \mathcal{S}_{i_k}$ and $(i_j) \in I_n$ generate the normed vector space S . Moreover it is easily seen by induction, using the decompositions $\mathcal{S}_i = \mathcal{T} \oplus \mathcal{S}_i^\circ$, that it is enough to consider the elements of \mathcal{T} and the elements $s_1 \cdots s_n$ with $s_k \in \mathcal{S}_{i_k}^\circ$ and $(i_j) \in I_n$. On the subspace \mathcal{T} , the expectation P' coincides with the identity, so that the invariance property is evident.

Therefore, we consider an element $s = s_1 \cdots s_n$ with $s_k \in \mathcal{S}_{i_k}^\circ$ and $(i_j) \in I_n$. For such an element we have $P'(s) = 0$ because S_1, S_2 are free in (S, P') and $\mathcal{S}_i^\circ \subset \text{Ker } P_i$. On the other hand, each $\delta_{i_k}(s_k)$ can be written as a sum $\sum_l s_{k,l} \otimes s'_{k,l}$ with $s_{k,l}, s'_{k,l} \in \mathcal{S}_{i_k}^\circ$, and one obtains for $\delta(s) = \delta_{i_1}(s_1) \cdots \delta_{i_n}(s_n)$ an expression of the form $\sum_l s_l \otimes s'_l$, where the s_l and s'_l are all in $\text{Ker } P'$ —using again the freeness of S_1 and S_2 . Therefore we also have $(P' \otimes \text{id}) \circ \delta(s) = 0$ and $(\text{id} \otimes P') \circ \delta(s) = 0$.

(2) The Hopf C^* -algebras $S_1 *_T S_2$ and $S'_1 *_T S'_2$ having the same dense Hopf sub- C^* -algebras of coefficients, they have the same reduced Hopf C^* -algebras. Now the first point shows that the Haar state of $S_1 *_T S_2$ is $h_{T \circ (P_1 *_T P_2)}$. But T is reduced, so that the reduced Hopf C^* -algebra of $S_1 *_T S_2$ is exactly $(S_1, P_1) *_T (S_2, P_2)$. \square

Remark. In the non-amalgamated case, it is quite easy to compute the category of finite-dimensional unitary corepresentations of S : the corepresentations $v_1 \otimes \cdots \otimes v_n$, where $v_k \in \text{Irr } \mathcal{C}_{i_k} \setminus \{1\}$ and $(i_k) \in I_n$, form a complete system of irreducible corepresentations of S , and the fusion rules can be naturally deduced from the ones of \mathcal{C}_1 and \mathcal{C}_2 [21]. This is very similar to the structure theorem for the free product of two discrete groups.

There is no such “simple” generalization of the classical theory in the amalgamated case. More precisely, if v_1 and v_2 are respective irreducible corepresentations of S_1 and S_2 , but not of T , the corepresentation $v_1 \otimes v_2$ does not need to be irreducible for S : see the example hereafter. However, looking at the amalgamated free product modulo T , one recovers a part of the classical structure: this is the meaning of the decomposition of $E_1 *_T E_2$ into orthogonal T -modules given by Lemma 2.3 and (*).

Example. Consider two copies S_1, S_2 of $A_o(Q)$, the Woronowicz C^* -algebra of an orthogonal free quantum group [21,17,5]. Let us recall that $\text{Irr } \mathcal{C}_1$ and $\text{Irr } \mathcal{C}_2$ can be indexed by \mathbb{N} , with the same fusion rules as the ones of $SU(2)$: we will denote by $v_{i,k}$ the corresponding irreducible corepresentations, with $i \in \{1, 2\}$ and $k \in \mathbb{N}$. The discrete quantum group corresponding to $A_o(Q)$ has a unique non-trivial subgroup, associated to the subcategory generated by $\text{Irr } \mathcal{D} = \{v_{2k} | k \in \mathbb{N}\}$. Its Woronowicz C^* -algebra T is the sub- C^* -algebra of even elements for the natural $\mathbb{Z}/2\mathbb{Z}$ -grading of $A_o(Q)$ —when $Q = I_n$, T is also the Woronowicz C^* -algebra $A_{\text{aut}}(M_n)$ of [6].

The tensor product of the fundamental corepresentations of S_1 and S_2 , $v_{1,1} \otimes v_{2,1}$, defines a corepresentation of the amalgamated free product $S = S_1 *_T S_2$, and it is not hard to see that it admits a strict sub-corepresentation a of dimension 1. More generally the sub-corepresentations of $v_{i_1, k_1} \otimes \cdots \otimes v_{i_n, k_n}$, with $(i_l) \in \{1, 2\}^n$ and $(k_l) \in \mathbb{N}^n$, have the same dimensions and multiplicities as the ones of

$v_{1,k_1} \otimes \cdots \otimes v_{1,k_n}$. Moreover one can prove, using the freeness of S_1 and S_2 in S , that a generates a copy of $C^*(\mathbb{Z})$ in S . Finally the corepresentations $a^k \otimes v_{1,l}$, with $k \in \mathbb{Z}$ and $l \in \mathbb{N}$, form a complete system of irreducible corepresentations of S . The fusions rules are induced by the ones of $C^*(\mathbb{Z})$ and $A_o(Q)$, plus the relation $v_{1,1} \otimes a \simeq v_{2,1} \simeq a^{-1} \otimes v_{1,1}$.

3. The Bass-Serre tree and K -amenability

The aim of this section is to prove the K -amenability of amalgamated free products of amenable discrete quantum groups. This extends to the quantum case a result of Julg and Valette [11, Corollary 4.1], which was obtained as a corollary of their more general result about locally compact groups acting on trees. Let us note however that the homotopy we construct in Theorem 3.3 is closer to the one used by Cuntz for the case of free products $G * \mathbb{Z}$ [8]. Indeed the homotopy of Julg and Valette relies on the construction of negative type functions on trees, which seems more complicated to extend to the quantum case.

Let $S = S_1 *_T S_2$ be an amalgamated free product of reduced Woronowicz C^* -algebras. Let P, R_i be the canonical conditional expectations of S_{red} onto T, S_i , respectively, and E, F_i the associated GNS constructions. For the definition of the Julg–Valette operators we will need a slight reinforcement of the well-known isomorphisms $E(r, i) \otimes_T E_i \simeq E$:

Lemma 3.1. *The map $\Psi_i : A_P(x) \otimes y \mapsto A_{R_i}(xy)$ defines an isomorphism between $E(r, i) \otimes_T S_i$ and F_i .*

Proof. We take $i = 1$ for the proof. In the definition of Ψ_1 one can take $x \in \mathcal{T}$ or $x = y_1 \cdots y_n, y_k \in \mathcal{S}_{i_k}^\circ, (i_k) \in I_n$ and $i_n = 2$. In particular the surjectivity of Ψ_1 will follow immediately if we prove that it is an isometry. So we have to show that $y^* P(x^* x) y = R_1(y^* x^* x y)$ for $y \in S_1$ and x like above. By S_1 -linearity of R_1 we can assume that $y = 1$, and then the equality $P(x^* x) = R_1(x^* x)$ amounts to the fact that $R_1(x^* x) \in \mathcal{T}$, which is clear when $x \in \mathcal{T}$.

So we take $x = y_1 \cdots y_n$ with $y_k \in \mathcal{S}_{i_k}^\circ, (i_k) \in I_n, i_n = 2$, and we prove that $R_1(x^* x)$ is orthogonal to \mathcal{S}_1° in the GNS construction of P . For any $x_1 \in \mathcal{S}_1^\circ$, we have

$$P(x_1 R_1(x^* x)) = P(R_1(x_1 x^* x)) = P(x_1 x^* x)$$

by S_1 -linearity of R_1 . To conclude we proceed by induction on n in the expression $x = y_1 \cdots y_n$ —we put $n = 0$ when $x \in \mathcal{T}$. By Proposition 2.4, S_1 and S_2 are free in S_{red} with respect to P . Therefore

$$\begin{aligned} P(x_1 x^* x) &= P(x_1 y_n^* \cdots y_2^* y_1^* y_1 y_2 \cdots y_n) \\ &= P(x_1 y_n^* \cdots y_2^* P(y_1^* y_1) y_2 \cdots y_n) \\ &= P(x_1 y_n^* \cdots y_2^* y_2' \cdots y_n) \end{aligned}$$

with $y_2' = P(y_1^* y_1)^{1/2} y_2$, and the last expression equals 0 by induction. \square

We now assume that S_1 and S_2 are amenable Woronowicz C^* -algebras. This means that they are reduced and admit co-units (also called trivial representations in our context), i.e. continuous characters $\varepsilon : S_i \rightarrow \mathbb{C}$ such that $(\text{id} \otimes \varepsilon) \circ \delta = (\varepsilon \otimes \text{id}) \circ \delta = \text{id}$. In the following definition, we use the isomorphism $F_i \otimes_{\varepsilon} \mathbb{C} \simeq E(r, i) \otimes_{\varepsilon} \mathbb{C}$ induced by Ψ_i .

Definition 3.2. Let $S = S_1 *_T S_2$ be an amalgamated free product of two amenable Woronowicz C^* -algebras. Let $(E, \eta), (F_i, \eta_i)$ be the GNS constructions associated to the canonical conditional expectations of S onto T, S_i , respectively. Let ε denote the co-unit of S_1 and S_2 .

- (1) The Hilbert space $K_g = E \otimes_{\varepsilon} \mathbb{C}$ (resp. $H = F_1 \otimes_{\varepsilon} \mathbb{C} \oplus F_2 \otimes_{\varepsilon} \mathbb{C}$) is called the ℓ^2 -space of geometric edges (resp. of vertices) of the quantum graph associated to (S_1, S_2, T) . They carry the GNS representations of S .
- (2) For that quantum graph, we define two Julg–Valette operators $\Phi_i, i \in \{1, 2\}$, given by the following formulas (where $\{i, j\} = \{1, 2\}$):

$$\Phi_i : \begin{cases} H \rightarrow K_g, \\ \eta_j \otimes_{\varepsilon} 1 \mapsto \eta \otimes_{\varepsilon} 1, & \eta_i \otimes_{\varepsilon} 1 \mapsto 0, \\ F_k^{\circ} \otimes_{\varepsilon} \mathbb{C} \xrightarrow{\sim} E(r, k)^{\circ} \otimes_{\varepsilon} \mathbb{C} & \text{for } k = 1, 2. \end{cases}$$

Example. When $S_1 = C_{\text{red}}^*(G_1), S_2 = C_{\text{red}}^*(G_2)$ and $T = C_{\text{red}}^*(H)$, the C^* -algebra S is some C^* -completion of $\ell^1(G)$, with $G = G_1 *_H G_2$. Then $K_g \simeq \ell^2(G/H)$ and $H \simeq \ell^2(G/G_1 \sqcup G/G_2)$ are the ℓ^2 -spaces of edges and vertices of the Bass-Serre tree associated to (G_1, G_2, H) [16]. Note that the isomorphism $F_i \otimes_{\varepsilon} \mathbb{C} \simeq E(r, i) \otimes_{\varepsilon} \mathbb{C}$ in this case is just the ℓ^2 -formulation of the canonical bijection $G(r, i)/H \xrightarrow{\sim} G/G_i$, where $G(r, i) \subset G$ is the subset of alternate products $g_i \cdots g_{i_k}$ that “do not end in G_i ” (including H).

In the Bass-Serre tree, there are two evident ways of choosing one endpoint for each edge. They are given by the following maps from G/H to $G/G_1 \sqcup G/G_2$, that determine the graph structure:

$$xH \mapsto xG_1 \quad \text{and} \quad xH \mapsto xG_2.$$

However, the corresponding “target operators” on the ℓ^2 -spaces do not need to be bounded. If we choose $G_i \in G/G_i$ as the origin of the tree, there is another natural, more geometrical way of selecting one endpoint for each edge: namely the furthest one from the origin. The adjoint of the corresponding target operator is then precisely the Julg–Valette operator Φ_i of Definition 3.2.

Theorem 3.3. *Let $S = S_1 *_T S_2$ be an amalgamated free product of two amenable Woronowicz C^* -algebras. Take $i \in \{1, 2\}$.*

- (1) *The GNS representations of S on the Hilbert spaces H and K_g introduced in Definition 3.2 are weakly contained in the regular representation.*
- (2) *The Julg–Valette operator $\Phi_i : H \rightarrow K_g$ is a Fredholm operator and commutes to the representations of S modulo compact operators. Let α be the associated element of $KK(S_{\text{red}}, \mathbb{C})$.*
- (3) *The element $\lambda^*(\alpha) \in KK(S, \mathbb{C})$ is equal to the element $[\varepsilon] \in KK(S, \mathbb{C})$ associated to the co-unit of S . Hence the discrete quantum group associated to S is K -amenable.*

Proof. (1) The Woronowicz C^* -algebras S_1 and S_2 being amenable, so is T , and this means that its trivial representation is weakly contained in the regular one. As a consequence, the GNS representation of $\varepsilon \circ P$ is weakly contained in the GNS representation of $h \circ P$. But these representations are, respectively, the GNS representation on $E \otimes_{\varepsilon} \mathbb{C}$ and the regular representation, by Proposition 2.2. The same argument works for F_1 and F_2 .

(2) It is clear on the definition, and thanks to Lemma 3.1, that the Julg–Valette operators Φ_i are surjective and have a kernel of dimension one. Hence they are Fredholm operators. We will now prove that Φ_1 commutes to the action of an element $s \in S_1$. For symmetry reasons, this will imply that Φ_2 commutes to S_2 , and because Φ_1 and Φ_2 are equal modulo a compact operator, they will both commute to $S_1 \cup S_2$, hence to S , modulo compact operators.

The main point is the following one: if ζ is a vector of $E(r, k)$ such that $s\zeta$ is again in $E(r, k)$, we have evidently

$$\Psi_k(s\zeta \otimes_T x) = s\Psi_k(\zeta \otimes_T x)$$

for every $x \in S_k$. But it is easy to check that $sE(r, 1)^\circ \subset E(r, 1)^\circ$ and $sE(r, 2) \subset E(r, 2)$ for $s \in S_1$. As Φ_1 coincides, respectively, on $E(r, 1)^\circ$ and $E(r, 2)$ with the inverses of $\Psi_1 \otimes_{\varepsilon} \text{id}$ and $\Psi_2 \otimes_{\varepsilon} \text{id}$, it commutes with the action of s on $E(r, 1)^\circ \oplus E(r, 2) = (\eta_1 \otimes_{\varepsilon} 1)^\perp$. Finally, S_1 acts trivially on the line $\eta_1 \otimes_{\varepsilon} \mathbb{C}$, that Φ_1 maps to 0.

(3) Let $\tilde{\Phi}_i : H \rightarrow K_g \oplus \mathbb{C}$ be the operator which coincides with Φ_i on $(\eta_i \otimes_{\varepsilon} 1)^\perp$, but maps $(\eta_i \otimes_{\varepsilon} 1)$ to $1_{\mathbb{C}}$. Let us denote by π the sum of the GNS and the trivial representations of S on $K_g \oplus \mathbb{C}$, and by ρ the GNS representation on H . It is clear that $\tilde{\Phi}_i$ is a compact perturbation of $\Phi_i + 0 : H \rightarrow K_g \oplus \mathbb{C}$, so that the class of $x = (\pi, \rho, \tilde{\Phi}_i)$ in $KK(S, \mathbb{C})$ equals $\lambda^*(\alpha) - [\varepsilon]$. We will now prove that x is homotopic to a degenerate triple.

It is clear from the proof of point 2 that $\tilde{\Phi}_i$ intertwines the restrictions of π and ρ to S_i . Moreover one has $\tilde{\Phi}_1 = \tilde{\Phi}_2 \circ u$, where $u \in L(H)$ equals the identity on the subspaces $F_k^\circ \otimes_{\varepsilon} \mathbb{C}$ and exchanges $\eta_1 \otimes_{\varepsilon} 1$ and $\eta_2 \otimes_{\varepsilon} 1$. In particular, $\tilde{\Phi}_1$ intertwines $\rho(s)$ and $\pi(s)$ if $s \in S_1$, but $u\rho(s)u$ and $\pi(s)$ if $s \in S_2$.

Now u is unitary and self-adjoint, so that the expression $u_t = \cos t + iu \sin t$ defines a family of unitaries (u_t) . Because u commutes to $\rho(T)$, one can define a family of representations of $S = S_1 *_T S_2$ on H by the formulas $\rho_t(s) = u_t^* \rho(s) u_t$ for $s \in S_2$ and $\rho_t(s) = \rho(s)$ for $s \in S_1$. Leaving π and $\tilde{\Phi}_1$ unchanged, this gives a homotopy between x and a degenerate triple: by construction, $\tilde{\Phi}_1$ intertwines exactly ρ_1 and π and is unitary. \square

Remark. Germain [10] defines a topological property called “relative K -nuclearity” for inclusions of unital C^* -algebras $T \subset S_1$, $T \subset S_2$. This property mostly serves as a sufficient condition for the full free product with amalgamation $S = S_1 *_T S_2$ to be “dominated in K -theory” by the reduced free product, which is equivalent to K -amenability in the case of C^* -algebras of discrete quantum groups. However, when T is infinite-dimensional very little is known about relative K -nuclearity, even in our case of inclusions of C^* -algebras of amenable discrete quantum groups. In this paper we chose a different, more geometrical way to prove the K -amenability of our amalgamated free products.

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