THE CONNES EMBEDDING PROPERTY
FOR QUANTUM GROUP VON NEUMANN ALGEBRAS

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ABSTRACT. For a compact quantum group $\mathbb{G}$ of Kac type, we study the existence of a Haar trace-preserving embedding of the von Neumann algebra $L^\infty(\mathbb{G})$ into an ultrapower of the hyperfinite II_1-factor (the Connes embedding property for $L^\infty(\mathbb{G})$). We establish a connection between the Connes embedding property for $L^\infty(\mathbb{G})$ and the structure of certain quantum subgroups of $\mathbb{G}$ and use this to prove that the II_1-factors $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$ associated to the free orthogonal and free unitary quantum groups have the Connes embedding property for all $N \geq 4$. As an application, we deduce that the free entropy dimension of the standard generators of $L^\infty(O_N^+)$ equals 1 for all $N \geq 4$. We also mention an application of our work to the problem of classifying the quantum subgroups of O_N^+.

1. INTRODUCTION

The Connes embedding problem asks whether any finite von Neumann algebra with separable predual embeds into an ultrapower of the hyperfinite II_1-factor in a trace-preserving way. This question was raised by Connes in [16]. See [12, 31, 32] for nice introductions to this topic. This central question in the theory of operator algebras is still open and has ramifications in many other areas of mathematics, such as e.g. non-commutative probability theory, quantum information, and non-commutative algebraic geometry. In probabilistic terms, this question amounts to knowing whether any finite family of elements of a bounded tracial non-commutative probability space admits an asymptotic matrix model. In the framework of Voiculescu's free entropy theory, this amounts to asking about the existence of matricial microstates; see [38, 39].

The aim of this paper is to provide a new class of examples of Connes embeddable von Neumann algebras, namely von Neumann algebras arising from non-coamenable compact quantum groups of Kac type. Within the operator algebraic framework, arguably the most studied examples of compact quantum groups of Kac type include the free orthogonal quantum groups O_N^+ and the free unitary quantum groups U_N^+. Over the last two decades, this class of quantum groups has been extensively studied, and remarkable connections have emerged between these quantum groups and free probability theory. These connections occur at the level of quantum symmetries and asymptotic freeness results [3, 6, 10, 18, 19] and also at the operator

algebra level [9, 20, 22–24, 36]. In particular, the von Neumann algebras $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$ share many of the same structural properties with the free group factors: they are full type II$_1$-factors; they are strongly solid, and in particular they are prime and have no Cartan subalgebra; they have the Haagerup property and are weakly amenable with Cowling-Haagerup constant 1 (CMAP). But unlike the case of the free group factors, the II$_1$-factors $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$, $N \geq 3$, were not known to be Connes embeddable (i.e., to admit matricial microstates).

Our main result in this paper is that as soon as $N \geq 4$, these von Neumann algebras have the Connes embedding property. This result is presented as a corollary of a more general and new stability result of the Connes embedding property under certain quantum group theoretical operations. More precisely, we consider a generalization within the category of compact quantum groups, of the notion of a compact group being topologically generated by a pair of closed subgroups (see Definition 4). Using this technology, we show in Theorem 3.6 that if a Kac type compact quantum group $\mathbb{G}$ is topologically generated by a pair of quantum subgroups $\mathbb{G}_1, \mathbb{G}_2$ which have the additional property that $L^\infty(\mathbb{G}_1), L^\infty(\mathbb{G}_2)$ are Connes embeddable, then $L^\infty(\mathbb{G})$ is also Connes embeddable. The utility of Theorem 3.6 lies in the fact that it can be used to reduce the problem of verifying the Connes embedding property for $L^\infty(\mathbb{G})$ to the (possibly easier) problem of verifying the same property for the "smaller" algebras $L^\infty(\mathbb{G}_1), L^\infty(\mathbb{G}_2)$. When dealing with the specific examples of $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$, this theorem in fact allows us to establish the Connes embedding property via an induction procedure over the dimension parameter N. Another interesting feature of the embedding of $L^\infty(\mathbb{G})$ into an ultrapower of the hyperfinite II$_1$-factor given by Theorem 3.6 is that it is obtained somewhat explicitly in terms of the embeddings associated to the given quantum subgroups $\mathbb{G}_1, \mathbb{G}_2$. See Remark 4 for details. It is our hope that this observation can lead to a more systematic understanding of how to construct explicit matricial microstates for certain quantum group von Neumann algebras.

The paper is organized as follows. Section 1 contains preliminaries about compact and free quantum groups. Section 3 recalls facts about the Connes embedding property and relates this property for quantum group von Neumann algebras to the structure of quantum subgroups. In Section 4 the Connes embedding property for $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$, $N \geq 4$, is derived through the study of specific quantum subgroups. Finally, in Section 5 we consider some applications of our results to free entropy dimension and to the problem of classifying the quantum subgroups of O_N^+ which contain the classical orthogonal group O_N as a quantum subgroup.

2. Preliminaries

2.1. **Compact quantum groups.** In this section we recall some basic facts on compact quantum groups. We follow [43] and [29] and refer to these papers for the facts stated below.

A compact quantum group is a pair $\mathbb{G} = (A, \Delta)$ where A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a unital $*$-homomorphism satisfying

$$(\iota \otimes \Delta)\Delta = (\Delta \otimes \iota)\Delta \quad \text{(coassociativity)},$$
$$[\Delta(A)(1 \otimes A)] = [\Delta(A)(A \otimes 1)] = A \otimes A \quad \text{(non-degeneracy)},$$

where $[S]$ denotes the norm-closed linear span of a subset $S \subset A \otimes A$. Here and in the rest of the paper, the symbol $\otimes$ will denote the minimal tensor product of

C*-algebras, $\overline{\otimes}$ will denote the spatial tensor product of von Neumann algebras, and $\odot$ will denote the algebraic tensor product of complex associative algebras. The homomorphism Δ is called a *coproduct*. The C*-algebra A together with the coproduct Δ is often called a *Woronowicz C*-algebra*.

For any compact quantum group $\mathbb{G} = (A, \Delta)$, there exists a unique *Haar state* $h : A \to \mathbb{C}$ which satisfies the following left and right invariance property, for all $a \in A$:

$$(1) \qquad (h \otimes \iota)\Delta(a) = (\iota \otimes h)\Delta(a) = h(a)1.$$

We say that a compact quantum group $\mathbb{G}$ is *of Kac type* if h is a tracial state. Note that in general h may not be faithful on A. In any case, we can construct a GNS representation $\pi_h : A \to \mathcal{B}(L^2(\mathbb{G}))$, where $L^2(\mathbb{G})$ is the Hilbert space obtained by separation and completion of A with respect to the sesquilinear form $\langle a|b \rangle = h(a^*b)$, and π_h is the natural extension to $L^2(\mathbb{G})$ of the left multiplication action of A on itself. The C*-algebra

$$C_r(\mathbb{G}) = \pi_h(A) \subset \mathcal{B}(L^2(\mathbb{G}))$$

is called the *reduced C*-algebra of functions on* $\mathbb{G}$. Due to the invariance properties of the Haar state h, the coproduct Δ extends to a unital $*$-homomorphism $\Delta_r : C_r(\mathbb{G}) \to C_r(\mathbb{G}) \otimes C_r(\mathbb{G})$, making the pair $(C_r(\mathbb{G}), \Delta_r)$ a compact quantum group (with faithful Haar state), called the *reduced version of* $\mathbb{G}$. The *von Neumann algebra of* $\mathbb{G}$ is given by

$$L^\infty(\mathbb{G}) = C_r(\mathbb{G})'' \subseteq \mathcal{B}(L^2(\mathbb{G})).$$

We note that Δ_r extends to an injective normal $*$-homomorphism $\Delta_r : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, and the Haar state on $C_r(\mathbb{G})$ extends to a faithful normal Δ_r-invariant state on $L^\infty(\mathbb{G})$.

Let H be a Hilbert space and $u \in M(\mathcal{K}(H) \otimes A)$ be an invertible (unitary) multiplier. The multiplier u is called a *(unitary) representation* of $\mathbb{G}$ if, following the leg numbering convention,

$$(2) \qquad (\iota \otimes \Delta)u = u_{12}u_{13}.$$

If $\dim H = n < \infty$, then (after fixing an orthonormal basis of H) we can identify u with an invertible matrix $u = [u_{ij}] \in M_n(A)$ and (2) means exactly that

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \qquad (1 \leq i, j \leq n).$$

Of course the unit $1 \in A$ is always a representation of $\mathbb{G}$, called the *trivial representation*.

Let $u \in M(\mathcal{K}(H_1)\otimes A)$ and $v \in M(\mathcal{K}(H_2)\otimes A)$ be two representations of $\mathbb{G}$. Then their *direct sum* is the representation $u \oplus v \in M(\mathcal{K}(H_1 \oplus H_2)\otimes A)$, and their *tensor product* is the representation $u \otimes v := u_{13}v_{23} \in M(\mathcal{K}(H_1 \otimes H_2)\otimes A)$. An *intertwiner* between u and v is a bounded linear map $T : H_1 \to H_2$ such that $(T \otimes \iota)u = v(T \otimes \iota)$. The Banach space of all such intertwiners is denoted by $\mathrm{Hom}_{\mathbb{G}}(u, v)$. If there exists an invertible (unitary) intertwiner between u and v, they are said to be *(unitarily) equivalent*. A representation is said to be irreducible if its only self-intertwiners are the scalar multiples of the identity map. It is known that each irreducible representation of $\mathbb{G}$ is finite dimensional and every finite dimensional representation is equivalent to a unitary representation. In addition, every unitary representation is unitarily equivalent to a direct sum of irreducible representations.

Denote by $\mathrm{Irr}(\mathbb{G})$ the collection of equivalence classes of finite dimensional irreducible unitary representations of $\mathbb{G}$. For each $\alpha \in \mathrm{Irr}(\mathbb{G})$, select a representative unitary representation $u^\alpha = [u_{ij}^\alpha] \in M_{n_\alpha}(A)$. The linear subspace $\mathrm{Pol}(\mathbb{G}) \subseteq A$ spanned by $\{u_{ij}^\alpha : \alpha \in \mathrm{Irr}(\hat{\mathbb{G}}),\ 1 \leq i, j \leq n_\alpha\}$ is a dense $*$-subalgebra of A, called *the algebra of polynomial functions on* $\mathbb{G}$. $\mathrm{Pol}(\mathbb{G})$ is in fact a Hopf $*$-algebra with coproduct $\Delta_0 : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{G})$ given by restriction of the coproduct Δ. The *antipode* $S : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})^{\mathrm{op}}$ is the automorphism given by $(\iota \otimes S)(u^\alpha) = (u^\alpha)^*$, and the *counit* is the $*$-character $\epsilon : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ given by $(\iota \otimes \epsilon)(u^\alpha) = 1$. For any compact quantum group $\mathbb{G}$, the Haar state h is always faithful on $\mathrm{Pol}(\mathbb{G})$. Moreover $\mathbb{G}$ is Kac type if and only if $S^2 = \iota$.

The universal enveloping C*-algebra of $\mathrm{Pol}(\mathbb{G})$ is denoted by $C^u(\mathbb{G})$. By universality, the coproduct on $\mathrm{Pol}(\mathbb{G})$ extends continuously to a coproduct Δ_u on $C^u(\mathbb{G})$, making $(C^u(\mathbb{G}), \Delta_u)$ a compact quantum group (the *universal version of* $\mathbb{G}$). A compact quantum group $\mathbb{G}$ is called *coamenable* if the Haar state is faithful on $C^u(\mathbb{G})$. When $\mathbb{G}$ is of Kac type, this is equivalent to $L^\infty(\mathbb{G})$ being an injective finite von Neumann algebra [35].

Given a pair of compact quantum groups $\mathbb{G}, \mathbb{H}$, we call $\mathbb{H}$ a *quantum subgroup* of $\mathbb{G}$ if there exists a surjective $*$-homomorphism $\pi : C^u(\mathbb{G}) \to C^u(\mathbb{H})$ intertwining the respective coproducts: $\Delta_{u,\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \Delta_{u,\mathbb{G}}$. Given two compact quantum groups $\mathbb{G}_1, \mathbb{G}_2$ the *dual free product* $\mathbb{G}_1 \hat{*} \mathbb{G}_2$ of $\mathbb{G}_1$ and $\mathbb{G}_2$ is given by the reduced free product algebra $A = C_r(\mathbb{G}_1) *_r C_r(\mathbb{G}_2)$ with respect to the Haar states, endowed with the unique coproduct extending the ones of $C_r(\mathbb{G}_1)$ and $C_r(\mathbb{G}_2)$.

A compact quantum group $\mathbb{G}$ is called a *compact matrix quantum group* if there exists a finite dimensional unitary representation $u = [u_{ij}] \in M_n(A)$ whose matrix elements generate A as a C*-algebra. Such a representation u is called a *fundamental representation* of $\mathbb{G}$. In this case $\mathrm{Pol}(\mathbb{G})$ is simply the $*$-algebra generated by $(u_{ij})_{1 \leq i,j \leq n}$.

Remark 1. Associated to any compact quantum group $\mathbb{G}$ one can construct a unique *dual discrete quantum group* $\hat{\mathbb{G}}$. See [1, 33] for an introduction to the basic theory of discrete-compact quantum group duality. Although we do not use the technology of discrete quantum groups here, it is useful to note that through the discrete-compact quantum group duality, the algebras $\mathrm{Pol}(\mathbb{G})$, $C^u(\mathbb{G})$, $C_r(\mathbb{G})$, $L^\infty(\mathbb{G})$ introduced before play the role of the familiar algebras $\mathbb{C}[\Gamma]$, $C^*(\Gamma)$, $C_r^*(\Gamma)$, $\mathcal{L}(\Gamma) = \lambda(\Gamma)''$ (respectively) associated to a discrete group Γ. With this terminology, we see that the discrete quantum group associated with a dual free product $\mathbb{G}_1 \hat{*} \mathbb{G}_2$ can be interpreted as the free product quantum group $\hat{\mathbb{G}}_1 * \hat{\mathbb{G}}_2$, and this justifies our notation.

2.2. **Free orthogonal and free unitary quantum groups.** We now introduce the free orthogonal and free unitary quantum groups which form the central objects of study in this paper. These quantum groups were first introduced in the operator algebraic framework by Wang [42]. Purely algebraic versions of these objects were also introduced by Dubois-Violette and Launer in [21].

Let $N \geq 2$. The *free orthogonal quantum group* O_N^+ is the compact quantum group given (in universal form) by the pair $(C^u(O_N^+), \Delta_u)$, where

$$C^u(O_N^+) = C^*\big((u_{ij})_{1 \leq i,j \leq N} \mid u = [u_{ij}] \text{ is unitary in } M_N(C^u(O_N^+)) \ \& \ \bar{u} = u\big),$$

where $\bar{u} = [u_{ij}^*]$. The coproduct Δ_u is defined so that u becomes a unitary representation of O_N^+. That is, $\Delta_u(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}$ for each $1 \leq i, j \leq N$. Note

that the abelianization of $C_u(O_N^+)$ is naturally isomorphic to the C*-algebra of continuous functions on the compact Lie group O_N. In particular, O_N is a quantum subgroup of O_N^+.

The *free unitary quantum group* U_N^+ is defined in the same fashion as O_N^+, except that we no longer assume that the generators of $C^u(U_N^+)$ are self-adjoint. More precisely, we define

$$C^u(U_N^+) = C^*\big((u_{ij})_{1 \leq i,j \leq N} \mid u = [u_{ij}] \ \& \ \bar{u} \text{ are unitary}\big).$$

Similarly, U_N is a quantum subgroup of U_N^+.

From the above definitions, it follows that for $\mathbb{G} = O_N^+, U_N^+$, the antipode $S :$ $\text{Pol}(\mathbb{G}) \to \text{Pol}(\mathbb{G})$, $S(u_{ij}) = u_{ji}^*$ satisfies $S^2 = \iota$. In particular, the Haar states $h_{\mathbb{G}}$ are tracial.

2.3. Invariant theory for O_N^+.

Let $N \geq 2$ and u be the fundamental representation of O_N^+ acting on the Hilbert space $H = \mathbb{C}^N$. In this section we briefly recall the structure of the intertwiner spaces $\text{Hom}_{O_N^+}(u^{\otimes k}, u^{\otimes l})$, $k, l \in \mathbb{N}_0$, as described in [6] (see also [2]). We start with a couple of definitions.

Definition 1. Let $k, l \in \mathbb{N}_0$ be such that $k + l \in 2\mathbb{N}_0$. We denote by $NC_2(k,l)$ the set of *non-crossing pair partitions* of k upper points and l lower points, that is, partitions that can be represented by diagrams formed by an upper row of k points, a lower row of l points, and $(k+l)/2$ non-crossing strings joining pairs of points. The vector space of (k,l) *Temperley-Lieb diagrams* is the abstract complex vector space $\text{TL}(k,l)$ freely spanned by $NC_2(k,l)$.

Consider now the Hilbert space $H = \mathbb{C}^N$ and denote by $(e_i)_{i=1}^N$ its standard basis. Each diagram $p \in NC_2(k,l)$ acts as a linear map $T_p : H^{\otimes k} \to H^{\otimes l}$ given by

$$(3) \qquad T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1 \cdots j_l = 1}^N \begin{pmatrix} i_1 \cdots i_k \\ p \\ j_1 \cdots j_l \end{pmatrix} e_{j_1} \otimes \cdots \otimes e_{j_l},$$

where the middle symbol is 1 if all strings of p join pairs of equal indices and is 0 if not. We denote by $TL_N(k,l) \subseteq \mathcal{B}(H^{\otimes k}, H^{\otimes l})$ the subspace spanned by the maps T_p, $p \in NC_2(k,l)$. This subspace is related to O_N^+-intertwiners as follows.

Theorem 2.1 ([2,6]). *Let u be the fundamental representation of O_N^+. Then for all $N \geq 2$,*

$$\text{Hom}_{O_N^+}(u^{\otimes k}, u^{\otimes l}) = TL_N(k,l).$$

Moreover the family of linear maps $(T_p)_{p \in NC_2(k,l)}$ is linearly independent as soon as $N \geq 2$.

If the first index is zero we omit it and we denote $NC_2(k) = NC_2(0,k)$, $TL(k) = TL(0,k)$. When there is no risk of confusion we will denote by $\text{Fix}_k = \text{Fix}(u^{\otimes k}) = \text{Hom}_{O_N^+}(1, u^{\otimes k}) \subseteq H^{\otimes k}$ the subspace of fixed vectors for the representation $u^{\otimes k}$ of O_N^+, and according to Theorem 2.1 we have for $N \geq 2$:

$$\text{Fix}_k = \text{span}\,\{T_p \mid p \in NC_2(k)\} = TL_N(k).$$

We also recall that Theorem 2.1 has a classical counterpart dating back to Brauer [11]. We denote by $P_2(k,l)$ the set of all pair partitions of k upper points and l

lower points, and we observe that T_p can still be defined for any $p \in P_2(k,l)$. Then we have

$$\text{Hom}_{O_N}(v^{\otimes k}, v^{\otimes l}) = \text{span }\{T_p \mid p \in P_2(k,l)\},$$

where v is the fundamental representation of O_N on $H = \mathbb{C}^N$. Note however that the maps T_p, for $p \in P_2(k,l)$, are not linearly independent for any N as soon as $k + l$ is big enough.

Remark 2. In what follows, we will not need to discuss the details of the invariant theory of U_N^+, although it was thoroughly described in [3,6]. We will, however, use the fact that if $w = [w_{ij}]$ and $\bar{w} = [w_{ij}^*]$ denote the fundamental representation of U_N^+ and its conjugate, acting on $H = \mathbb{C}^N$ and $\bar{H}$ (respectively), then after canonically identifying H and $\bar{H}$ in the obvious way, we have the equality of intertwiner spaces

$$\text{Hom}_{U_N^+}(1, (w \otimes \bar{w})^{\otimes k}) = \text{Hom}_{O_N^+}(1, u^{\otimes 2k}) \qquad (k \geq 1).$$

3. The Connes embedding property

Before specializing to quantum groups, let us first recall a few basic things about the Connes embedding property in the context of unital $*$-algebras.

3.1. Connes embeddable tracial $*$-algebras.
Let A be a unital $*$-algebra and $\tau : A \to \mathbb{C}$ a (not necessarily faithful) tracial state. A can be endowed with a sesquilinear form $\langle x, y \rangle := \tau(x^*y)$. Let $N_\tau := \{x \in A : \langle x, x \rangle = 0\}$. By the Cauchy-Schwarz inequality, $N_\tau := \{x \in A : \langle x, y \rangle = 0 \ \forall y \in A\}$, and therefore N_τ is a linear subspace of A and $\langle \cdot, \cdot \rangle$ can be defined naturally on the quotient space A/N_τ, where it is a non-degenerate sesquilinear form. The resulting completion of A/N_τ is denoted by $L^2(A, \tau)$. We denote by $\Lambda_\tau : A \to A/N_\tau \subset L^2(A, \tau)$ the quotient map. If A is generated as a $*$-algebra by elements $(x_i)_{i \in I}$ such that $\tau((x_i^*x_i)^n)^{1/n}$ is bounded for each i, then there exists a unital $*$-homomorphism $\pi_\tau : A \to B(L^2(A, \tau))$ satisfying

$$\pi_\tau(x)\Lambda_\tau(y) = \Lambda_\tau(xy) \qquad (x, y \in A).$$

The representation π_τ is usually called the GNS representation of A with respect to the tracial state τ. Taking double commutants, we obtain from A a von Neumann algebra $\pi_\tau(A)'' \subseteq \mathcal{B}(L^2(A, \tau))$, and the original state τ extends by continuity to a faithful normal tracial state on $\pi_\tau(A)''$ still denoted by τ. Throughout this paper, we will always assume that our tracial $*$-algebras (A, τ) are such that π_τ exists and that the von Neumann algebra $\pi_\tau(A)''$ has a separable predual.

Let us briefly recall the ultrapower construction for the hyperfinite II_1-factor. Let R denote the hyperfinite II_1-factor and τ_R its unique faithful normal tracial state. Let ω be a free ultrafilter on $\mathbb{N}$ and let $I_\omega \subseteq \ell^\infty(\mathbb{N}, R)$ be the ideal consisting of those sequences $(x_n)_{n=1}^\infty$ such that $\lim_{n \to \omega} \tau_R((x_n)^*x_n) = 0$. Then the *ultrapower of the hyperfinite II_1-factor (along the ultrafilter ω)* is the quotient $R^\omega := \ell^\infty(\mathbb{N}, R)/I_\omega$, which turns out to be a II_1-factor with faithful normal tracial state $\tau_{R^\omega}((x_n)_n + I_\omega) = \lim_{n \to \omega} \tau_R(x_n)$. We now come to the fundamental concept of this paper.

Definition 2. Let A be a unital $*$-algebra equipped with a tracial state τ. The state τ is said to have the *Connes embedding property* if the finite von Neumann algebra $(\pi_\tau(A)'', \tau)$ can be embedded into an ultrapower R^ω of the hyperfinite II_1-factor R in a trace-preserving way. We write $\mathrm{CEP}(A)$ for the set of such tracial states $\tau : A \to \mathbb{C}$.

Since our point of view and motivation are those of matricial microstates, let us also recall the following definition.

Definition 3. Let A be a unital $*$-algebra equipped with a faithful tracial state τ. If $X = (x_1, \ldots, x_n)$ is a finite subset of $A_{sa} := \{x \in A : x^* = x\}$, we say that X *has matricial microstates (relative to τ)* if for every $m \in \mathbb{N}$ and every $\epsilon > 0$, there is a $k \in \mathbb{N}$ and self-adjoint matrices $a_1, \ldots, a_n \in M_k(\mathbb{C})$ such that whenever $1 \le p \le m$ and $i_1, \ldots, i_p \in \{1, \ldots, n\}$, we have

$$(4) \qquad |\mathrm{tr}_k(a_{i_1} a_{i_2} \cdots a_{i_p}) - \tau(x_{i_1} x_{i_2} \cdots x_{i_p})| < \epsilon,$$

where tr_k is the normalized trace on $M_k(\mathbb{C})$ satisfying $\mathrm{tr}_k(1) = 1$.

The following von Neumann algebraic result connecting the existence of matricial microstates to the Connes embedding property is well known; see for example [14, Prop. 3.3]:

Proposition 3.1. *Let M be a von Neumann algebra with separable predual equipped with a faithful normal tracial state τ. Then the following are equivalent:*

(1) $\tau \in \mathrm{CEP}(M)$ *(i.e., M has the Connes embedding property).*
(2) *Every finite subset $X \subset M_{sa}$ has matricial microstates relative to τ.*
(3) *If $Y \subset M_{sa}$ is a generating set for M, then every finite subset $X \subset Y$ has matricial microstates.*

In particular, if $Y \subset M_{sa}$ is a finite generating set of M, then the above conditions are equivalent to Y having matricial microstates.

The following lemma gives some important stability properties of $\mathrm{CEP}(A)$ that will be essential in the sequel.

Lemma 3.2. *Let (A, τ) and $(A_i, \tau_i)_{i=1,2}$ be unital $*$-algebras equipped with tracial states τ and $(\tau_i)_{i=1,2}$ respectively. The following assertions are true.*

(1) *If $B \subseteq A$ is a unital $*$-subalgebra and $\tau \in \mathrm{CEP}(A)$, then $\tau|_B \in \mathrm{CEP}(B)$.*
(2) *If $\pi : A_1 \to A_2$ is a unital $*$-homomorphism such that $\tau_2 \circ \pi = \tau_1$ and $\tau_1 \in \mathrm{CEP}(A_1)$, then $\tau_2|_{\pi(A_1)} \in \mathrm{CEP}(\pi(A_1))$.*
(3) *If $\tau_1 \in \mathrm{CEP}(A_1)$ and $\tau_2 \in \mathrm{CEP}(A_2)$, then $\tau_1 \otimes \tau_2 \in \mathrm{CEP}(A_1 \odot A_2)$ and $\tau_1 * \tau_2 \in \mathrm{CEP}(A_1 * A_2)$, where $*$ denotes the reduced free product of tracial unital $*$-algebras [30].*
(4) *If $(\tau_n)_{n \in \mathbb{N}} \subset \mathrm{CEP}(A)$ is a sequence such that the pointwise limit $\tau := \lim_{n \to \infty} \tau_n$ exists, then $\tau \in \mathrm{CEP}(A)$.*

Proof. (1) and (2) follow from the fact that the Connes embedding property is stable under (trace-preserving) inclusions of von Neumann algebras. (3) follows from the fact that the Connes embedding property is stable under tensor products and free products of von Neumann algebras with respect to tracial states [34, 40]. (4) is a direct ultra product construction. Alternately, this readily follows from the definition of matricial microstates. □

3.2. **Hyperlinear discrete quantum groups.** Now let $\mathbb{G}$ be a compact quantum group (of Kac type) and consider the unital Hopf $*$-algebra $A = \mathrm{Pol}(\mathbb{G})$, with coproduct Δ (from now on we drop the notation Δ_0 and simply write Δ). Given a tracial state $\tau : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$, we will write $\tau \in \mathrm{CEP}(\mathbb{G})$ if $\tau \in \mathrm{CEP}(\mathrm{Pol}(\mathbb{G}))$. Our main interest is in determining when the Haar state $h_{\mathbb{G}}$ belongs to $\mathrm{CEP}(\mathbb{G})$, i.e., when $(L^{\infty}(\mathbb{G}), h_{\mathbb{G}})$ is a Connes embeddable von Neumann algebra. In this case (following the analogies with discrete groups given in Remark 1), we will say that the dual discrete quantum group $\hat{\mathbb{G}}$ is *hyperlinear*.

We start with a crucial but elementary lemma. Given two states τ_1, τ_2 on $\mathrm{Pol}(\mathbb{G})$, recall that their *convolution product*

$$\tau_1 \star \tau_2 := (\tau_1 \otimes \tau_2) \circ \Delta$$

is again a state on $\mathrm{Pol}(\mathbb{G})$ and $\tau_1 \star \tau_2$ is tracial if both τ_1, τ_2 are.

Lemma 3.3. *If $\tau_1, \tau_2 \in \mathrm{CEP}(\mathbb{G})$, then $\tau_1 \star \tau_2 \in \mathrm{CEP}(\mathbb{G})$.*

Proof. Let $\sigma = \tau_1 \otimes \tau_2|_{\Delta(\mathrm{Pol}(\mathbb{G}))}$. Then it follows from Lemma 3.2 (1)-(2) that $\sigma \in \mathrm{CEP}(\Delta(\mathrm{Pol}(\mathbb{G})))$. Moreover, since $\Delta^{-1} : \Delta(\mathrm{Pol}(\mathbb{G})) \to \mathrm{Pol}(\mathbb{G})$ is a $*$-isomorphism such that $(\tau_1 \star \tau_2) \circ \Delta^{-1} = \sigma$, another application of Lemma 3.2 (2) gives $\tau_1 \star \tau_2 \in \mathrm{CEP}(\mathbb{G})$. $\qquad\square$

As one might expect, duals of coamenable compact quantum groups of Kac type are always hyperlinear.

Lemma 3.4. *Let $\mathbb{G}$ be a coamenable compact quantum group of Kac type. Then $\hat{\mathbb{G}}$ is hyperlinear.*

Proof. If $\mathbb{G}$ is of Kac type and coamenable, it follows from Ruan [35, Proposition 2.3] that $(L^{\infty}(\mathbb{G}), h_{\mathbb{G}})$ is a hyperfinite tracial von Neumann algebra. In particular, this implies that there is a Haar state-preserving embedding of $L^{\infty}(\mathbb{G})$ into the hyperfinite II_1-factor R. Since R embeds trivially in R^{ω} in a trace-preserving way, we are done. $\qquad\square$

3.3. **Quantum subgroups and a stability result for hyperlinearity.** In this section we present a new stability result for hyperlinear discrete quantum groups (Theorem 3.6). The main conceptual tool here is a quantization of the notion of a compact group being topologically generated by a pair of closed subgroups. The results of this section will be applied to specific examples in the next section.

Let $\mathbb{G}$ be a compact quantum group and $\mathbb{G}_1 \leq \mathbb{G}$ a quantum subgroup (given by a surjective $*$-homomorphism of Woronowicz C*-algebras $\pi : C^u(\mathbb{G}) \to C^u(\mathbb{G}_1)$). Recall that any representation u of $\mathbb{G}$ induces a representation $u^{\mathbb{G}_1} := (\iota \otimes \pi)u$ of the quantum subgroup $\mathbb{G}_1$. When considering spaces of intertwiners, note that we always have the inclusions

$$(5) \qquad\qquad \mathrm{Hom}_{\mathbb{G}}(u, v) \subseteq \mathrm{Hom}_{\mathbb{G}_1}(u^{\mathbb{G}_1}, v^{\mathbb{G}_1})$$

for any pair of representations u, v of $\mathbb{G}$. If, moreover, we have equality in (5) for every u, v, then it follows that $\mathbb{G}$ and $\mathbb{G}_1$ are isomorphic compact quantum groups. This leads us to the following quantum analogue of a compact group being topologically generated by a pair of closed subgroups.

Definition 4. Let $\mathbb{G}$ be a compact quantum group and $\mathbb{G}_1, \mathbb{G}_2 \leq \mathbb{G}$ a pair of quantum subgroups. We say that $\mathbb{G}$ is *topologically generated by $\mathbb{G}_1$ and $\mathbb{G}_2$* (and write $\mathbb{G} = \langle \mathbb{G}_1, \mathbb{G}_2 \rangle$) if

$$\mathrm{Hom}_{\mathbb{G}}(u, v) = \mathrm{Hom}_{\mathbb{G}_1}(u^{\mathbb{G}_1}, v^{\mathbb{G}_1}) \cap \mathrm{Hom}_{\mathbb{G}_2}(u^{\mathbb{G}_2}, v^{\mathbb{G}_2})$$

for every pair of finite dimensional unitary representations u, v of $\mathbb{G}$.

The following proposition shows that the condition $\mathbb{G} = \langle \mathbb{G}_1, \mathbb{G}_2 \rangle$ can be characterized purely in terms of a relation between the Haar states on $\mathbb{G}_1, \mathbb{G}_2$, and $\mathbb{G}$. If v is a representation of $\mathbb{G}$ we denote by $\mathrm{Fix}(v) = \mathrm{Hom}_{\mathbb{G}}(1, v)$ the space of fixed vectors of v.

Proposition 3.5. *Let C be a class of representations of $\mathbb{G}$ such that any irreducible representation of $\mathbb{G}$ is equivalent to a subrepresentation of some element of C. Let $\mathbb{G}_1, \mathbb{G}_2 \leq \mathbb{G}$. Denote by $h = h_{\mathbb{G}}$ the Haar state of $\mathbb{G}$ and by $h_i = h_{\mathbb{G}_i} \circ \pi_i$ the state on $C^u(\mathbb{G})$ induced by the Haar state of $\mathbb{G}_i$. Then the following conditions are equivalent.*

    (1) $\mathbb{G} = \langle \mathbb{G}_1, \mathbb{G}_2 \rangle$.
    (2) $\mathrm{Fix}(v^{\mathbb{G}_1}) \cap \mathrm{Fix}(v^{\mathbb{G}_2}) = \mathrm{Fix}(v)$ *for all $v \in C$.*
    (3) *On $\mathrm{Pol}(\mathbb{G})$, we have $h = \lim_{k \to \infty} (h_1 \star h_2)^{\star k}$ (pointwise).*

Proof. (1) $\implies$ (2) is clear.

(2) $\implies$ (1). We have $\mathrm{Fix}(v \oplus w) = \mathrm{Fix}(v) \oplus \mathrm{Fix}(w)$ and, if w is the restriction of v on a subspace $K \subset H$, $\mathrm{Fix}(w) = \mathrm{Fix}(v) \cap K$. Hence the property $\mathrm{Fix}(v^{\mathbb{G}_1}) \cap \mathrm{Fix}(v^{\mathbb{G}_2}) = \mathrm{Fix}(v)$ holds for any finite dimensional subrepresentation of v. (1) now follows by Frobenius reciprocity (see [43, Proposition 3.4]): we have indeed $\mathrm{Hom}_{\mathbb{G}}(u, v) \simeq \mathrm{Fix}(v \otimes \bar{u})$ and similarly for the restrictions to $\mathbb{G}_1, \mathbb{G}_2$.

(2) $\implies$ (3). Let $u \in \mathcal{B}(H) \otimes C^u(\mathbb{G})$ be a finite dimensional unitary representation of $\mathbb{G}$ and consider the operators $P = (\iota \otimes h)(u)$ and $P_i = (\iota \otimes h_i)(u)$ $(i = 1, 2)$ in $\mathcal{B}(H)$. Then P, P_1, P_2 are orthogonal projections with range equal to $\mathrm{Fix}\, u, \mathrm{Fix}\, u^{\mathbb{G}_1}, \mathrm{Fix}\, u^{\mathbb{G}_2}$, respectively. In particular, $\lim_{k \to \infty} (P_1 P_2)^k$ exists in $\mathcal{B}(H)$ and is the orthogonal projection with range equal to $\mathrm{Fix}(u^{\mathbb{G}_1}) \cap \mathrm{Fix}(u^{\mathbb{G}_2})$. As a result $P = \lim_{k \to \infty} (P_1 P_2)^k$. But since

$$(6) \qquad\qquad (P_1 P_2)^k = (\iota \otimes (h_1 \star h_2)^{\star k})(u),$$

we conclude that $h = \lim_{k \to \infty} (h_1 \star h_2)^{\star k}$ on every matrix element of every finite dimensional unitary representation of $\mathbb{G}$. This proves the assertion.

(3) $\implies$ (2). Similarly, (6) shows that $P = \lim_{k \to \infty} (P_1 P_2)^k$, hence $\mathrm{Fix}(u^{\mathbb{G}_1}) \cap \mathrm{Fix}(u^{\mathbb{G}_2}) = Fix(u)$. $\qquad\qquad\square$

Remark 3. At this point it is worthwhile pointing out the connection between our notion of topological generation by subgroups and the concept of an inner faithful representation of a Woronowicz C*-algebra. Let $\mathbb{G}$ be a compact quantum group and B a unital C*-algebra. Recall that a *-homomorphism $\alpha : C^u(\mathbb{G}) \to B$ is *inner faithful* if $\mathrm{Ker}\,\alpha$ does not contain any non-zero Hopf *-ideal. Equivalently, for any factorization $\alpha = \tilde{\alpha} \circ \pi$ with $\pi : C^u(\mathbb{G}) \to C^u(\mathbb{H})$, a surjective morphism of Woronowicz C*-algebras, we have in fact that π is an isomorphism. More generally, the *Hopf image* of α is the "biggest" quantum subgroup $(\mathbb{H}, \pi)$ of $\mathbb{G}$ such that α factors through $\pi : C^u(\mathbb{G}) \to C^u(\mathbb{H})$ (cf. [4]) and α is inner faithful iff its Hopf image is $(\mathbb{G}, \iota)$.

With this terminology, it follows that $\mathbb{G}$ is topologically generated by $(\mathbb{H}_1, \pi_1)$, $(\mathbb{H}_2, \pi_2)$ iff $\alpha := (\pi_1 \otimes \pi_2) \circ \Delta : C^u(\mathbb{G}) \to C^u(\mathbb{H}_1) \otimes C^u(\mathbb{H}_2)$ is inner faithful. Indeed, by [4, Theorem 8.6] α is inner faithful iff $\mathrm{Fix}_{\mathbb{G}}(v) = \mathrm{Fix}(v^\alpha)$ for all representations v of $\mathbb{G}$, where $v^\alpha = v_{12}^{\mathbb{H}_1} v_{13}^{\mathbb{H}_2}$. Then we have

$$\xi \in \mathrm{Fix}(v^\alpha) \;\Leftrightarrow\; v_{12}^{\mathbb{H}_1 *}(\xi \otimes 1 \otimes 1) = v_{13}^{\mathbb{H}_2}(\xi \otimes 1 \otimes 1)$$

$$\Leftrightarrow\; v^{\mathbb{H}_1}(\xi \otimes 1) = \xi \otimes 1 = v^{\mathbb{H}_2}(\xi \otimes 1)$$

so that $\mathrm{Fix}(v^\alpha) = \mathrm{Fix}(v^{\mathbb{H}_1}) \cap \mathrm{Fix}(v^{\mathbb{H}_2})$.

More generally, we say that a quantum subgroup $(\mathbb{H}, \pi)$ is topologically generated by $(\mathbb{H}_1, \pi_1)$ and $(\mathbb{H}_2, \pi_2)$ if it is the Hopf image of $\alpha = (\pi_1 \otimes \pi_2) \circ \Delta$. If $(\mathbb{H}, \pi)$, $(\mathbb{H}_1, \pi_1)$ are two quantum subgroups of $\mathbb{G}$, we say that $\mathbb{H}$ contains $\mathbb{H}_1$ if π_1 factors through π. From the definitions, we have that if $\mathbb{H} = (\mathbb{H}, \pi)$ contains $\mathbb{H}_1$ and $\mathbb{H}_2$, then it also contains the subgroup generated by $\mathbb{H}_1$ and $\mathbb{H}_2$. Indeed, writing $\pi_1 = \rho_1 \circ \pi$ and $\pi_2 = \rho_2 \circ \pi$ we have $\alpha = (\rho_1 \otimes \rho_2) \circ (\pi \otimes \pi) \circ \Delta_{\mathbb{G}} = (\rho_1 \otimes \rho_2) \circ \Delta_{\mathbb{H}} \circ \pi$ and π is a morphism of Woronowicz C*-algebras; hence $\mathbb{H}$ contains the Hopf image of α.

We now turn to an interesting corollary of Proposition 3.5, which shows that it is possible to deduce the Connes embeddability of $L^\infty(\mathbb{G})$ from the Connes embeddability of the von Neumann algebras associated to its quantum subgroups.

Theorem 3.6. *Let $\mathbb{G}$ be a compact quantum group of Kac type and assume $\mathbb{G} = \langle \mathbb{G}_1, \mathbb{G}_2 \rangle$ for some pair of quantum subgroups $\mathbb{G}_1, \mathbb{G}_2 \leq \mathbb{G}$. If $\hat{\mathbb{G}}_1$ and $\hat{\mathbb{G}}_2$ are hyperlinear, then so is $\hat{\mathbb{G}}$.*

Proof. Consider the Haar states $h_{\mathbb{G}_1}$ and $h_{\mathbb{G}_2}$ and the associated states h_1, h_2 on $\mathrm{Pol}(\mathbb{G})$. Since the GNS construction for h_i yields $(L^\infty(\mathbb{G}_i), h_{\mathbb{G}_i})$, which is Connes embeddable by assumption, we conclude that $h_1, h_2 \in \mathrm{CEP}(\mathbb{G})$. By Lemma 3.3 $(h_1 \star h_2)^{\star k} \in \mathrm{CEP}(\mathbb{G})$ for all $k \in \mathbb{N}$. Since $\mathbb{G} = \langle \mathbb{G}_1, \mathbb{G}_2 \rangle$, an application of Proposition 3.5 and Lemma 3.2(4) shows that $h_{\mathbb{G}} = \lim_{k \to \infty}(h_1 \star h_2)^{\star k}$ belongs to $\mathrm{CEP}(\mathbb{G})$. $\qquad\square$

Remark 4. An examination of Proposition 3.5, Theorem 3.6 and their proofs shows that under the above assumptions, one can build matricial microstates for generators of $L^\infty(\mathbb{G})$ using matricial microstates for elements of tensor products of $L^\infty(\mathbb{G}_1)$ and $L^\infty(\mathbb{G}_2)$ (which, by assumption, are known to exist!). To see this, note that by a standard ultraproduct argument along the lines of [14, Prop. 3.3], it suffices to exhibit a Haar-state-preserving embedding of $L^\infty(\mathbb{G})$ into a tracial ultraproduct of tensor products of the von Neumann algebras $(L^\infty(\mathbb{G}_1), h_{\mathbb{G}_1})$ and $(L^\infty(\mathbb{G}_2), h_{\mathbb{G}_2})$.

To this end, let M_k be the finite von Neumann algebra $(L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_2))^{\overline{\otimes}k}$ equipped with the faithful normal trace-state $\tau_k := (h_{\mathbb{G}_1} \otimes h_{\mathbb{G}_2})^{\otimes k}$, for each $k \in \mathbb{N}$. Let ω be a fixed free ultrafilter on $\mathbb{N}$ and consider the tracial ultraproduct

$$M = \prod_{k \in \mathbb{N}}^{\omega}(M_k, \tau_k) := \Big(\prod_{k \in \mathbb{N}}^{\ell^\infty}(M_k, \tau_k)\Big)/I_\omega,$$

where $I_\omega = \{(x_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}}^{\ell^\infty}(M_k, \tau_k) : \lim_{k \to \omega} \tau_k(x_k^* x_k) = 0\}$. Note that M is a finite von Neumann algebra with faithful normal tracial state $\tau_\omega(x) = \lim_{k \to \omega} \tau_k(x_k)$, where $x = (x_k)_{k \in \mathbb{N}}^{\cdot} \in M$ denotes the equivalence class of $(x_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}}^{\ell^\infty}(M_k, \tau_k)$.

Denote by $\sigma_k : \mathrm{Pol}(\mathbb{G}) \to M_k$ the unital $*$-homomorphism

$$\sigma_k(x) = \left((\pi_{h_{\mathbb{G}_1}} \circ \pi_1) \otimes (\pi_{h_{\mathbb{G}_2}} \circ \pi_2) \right)^{\otimes k} \circ \Delta^{2k-1}(x) \qquad (x \in \mathrm{Pol}(\mathbb{G})),$$

where $\Delta^r := (\iota \otimes \Delta) \circ \Delta^{r-1} : \mathrm{Pol}(\mathbb{G}) \to Pol(\mathbb{G})^{\otimes(r+1)}$ is the r-fold iterated coproduct, $\pi_i : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}_i)$ is the surjective $*$-homomorphism identifying $\mathbb{G}_i$ as a quantum subgroup of $\mathbb{G}$, and $\pi_{h_{\mathbb{G}_i}} : \mathrm{Pol}(\mathbb{G}_i) \to L^\infty(\mathbb{G}_i)$ is the GNS representation associated to $h_{\mathbb{G}_i}$. Since $\mathbb{G} = \langle \mathbb{G}_1, \mathbb{G}_2 \rangle$ by assumption, we have $h_{\mathbb{G}} = \lim_{k \to \infty} \tau_k \circ \sigma_k$, and therefore the $*$-homomorphism

$$\sigma : (\mathrm{Pol}(\mathbb{G}), h_{\mathbb{G}}) \to (M, \tau_\omega); \qquad \sigma(x) = (\sigma_k(x))^{\cdot}_{k \in \mathbb{N}} \qquad (x \in \mathrm{Pol}(\mathbb{G}))$$

is trace-preserving and extends uniquely to a trace-preserving normal injective $*$-homomorphism $\sigma : (L^\infty(\mathbb{G}), h_{\mathbb{G}}) \to (M, \tau_\omega)$.

4. THE CONNES EMBEDDING PROPERTY FOR $L^\infty(O_N^+)$ AND $L^\infty(U_N^+)$

In this section we apply the general theory of the previous sections to study the hyperlinearity of the discrete quantum groups dual to O_N^+ and U_N^+, $N \geq 2$. In particular, we prove that O_N^+ is topologically generated by certain canonical pairs of quantum subgroups of lower rank. The results of this section may be of independent interest, particularly with respect to the problem of classifying *all quantum subgroups $O_N \leq \mathbb{G} \leq O_N^+$* (see Section 5 for more on this).

Below we will consider the following list of quantum subgroups of O_N^+. Recall that we denote by $u \in B(H) \otimes C^u(O_N^+)$ the fundamental representation of O_N^+, with $H = \mathbb{C}^N$, and let us also put $S_1 = \{\xi \in \mathbb{R}^N \mid \|\xi\| = 1\} \subset H$.

(1) The classical orthogonal group $O_N \leq O_N^+$, given by the Woronowicz C*-morphism $\pi_{O_N} : C^u(O_N^+) \to C(O_N)$ whose kernel is generated by commutators.

(2) The classical permutation group $\mathfrak{S}_N \leq O_N^+$, given by the Woronowicz C*-morphism $\pi_{\mathfrak{S}_N} : C^u(O_N^+) \to C(\mathfrak{S}_N)$ whose kernel is generated by the commutators together with the elements $(u_{ij} - u_{ij}^2)_{1 \leq i,j \leq N}$.

(3) The free product quantum subgroups $O_a^+ \hat{*} O_b^+ \leq O_N^+$ for $a + b = N$, given by the Woronowicz C*-morphism $\pi_{a,b} : C^u(O_N^+) \to C^u(O_a^+ \hat{*} O_b^+)$ which sends the $a \times a$ upper left (resp. $b \times b$ lower right) corner of the fundamental representation of O_N^+ to the fundamental representation of O_a^+ (resp. O_b^+), and all other entries to 0.

(4) The quantum stabilizer subgroups $O_{N-1}^{+,\xi} \leq O_N^+$ for $\xi \in S_1$, given by the Woronowicz C*-morphisms $\pi_\xi : C^u(O_N^+) \to C^u(O_{N-1}^{+,\xi})$ obtained by completing ξ into an orthonormal basis and sending the corresponding generator u_{11} to 1. Note that $O_{N-1}^{+,\xi} \simeq O_{N-1}^+$ for all ξ.

The main theorems of this section are as follows.

Theorem 4.1. *Let $N \geq 4$. Then the following assertions are true.*

(1) $O_N^+ = \langle O_N, O_{N-1}^{+,\xi} \rangle$ *for each $\xi \in S_1$.*

(2) $O_N^+ = \langle O_{N-1}^{+,\xi_1}, O_{N-1}^{+,\xi_2} \rangle$ *for any pair of linearly independent vectors $\xi_1, \xi_2 \in S_1$.*

Theorem 4.2. *Let $n \geq 2$ be a non-negative integer. Then $O_{2n}^+ = \langle \mathfrak{S}_{2n}, O_n^+ \hat{*} O_n^+ \rangle$.*

Before proving Theorems 4.1 and 4.2 we state their applications to hyperlinearity.

Corollary 4.3. *Let $N = 2$ or $N \geq 4$. Then $\widehat{O_N^+}$ is hyperlinear.*

Proof. The hyperlinearity of $\widehat{O_2^+}$ follows from Lemma 3.4. For the case $N = 4$, note that $O_4^+ = \langle \mathfrak{S}_4, O_2^+ \hat{*} O_2^+ \rangle$ by Theorem 4.2. Since $L^\infty(\mathfrak{S}_4)$ and $L^\infty(O_2^+ \hat{*} O_2^+) = (L^\infty(O_2^+), h_{O_2^+}) * (L^\infty(O_2^+), h_{O_2^+})$ are both Connes embeddable, we conclude that $\widehat{O_4^+}$ is hyperlinear by Theorem 3.6. Finally, the cases $N \geq 5$ follow by induction using Theorem 4.1 and Theorem 3.6. $\square$

Using a structure result of Banica [3, Théorème 1], we can easily deduce the hyperlinearity of $\widehat{U_N^+}$ from the corresponding result for $\widehat{O_N^+}$.

Theorem 4.4. *Let $N = 2$ or $N \geq 4$. Then $\widehat{U_N^+}$ is hyperlinear.*

Proof. From [3, Théorème 1], there exists a trace-preserving embedding

$$(L^\infty(U_N^+), h_{U_N^+}) \hookrightarrow (L^\infty(\mathbb{T}) * L^\infty(O_N^+), \tau * h_{O_N^+}),$$

where τ denotes integration with respect to the Haar probability measure on $\mathbb{T}$. The Connes embeddability of $L^\infty(U_N^+)$ now follows from the Connes embeddability of $L^\infty(\mathbb{T}), L^\infty(O_N^+)$ and Lemma 3.2 (3). $\square$

Remark 5. We expect that Theorem 4.1 holds when $N = 3$ (and therefore that $\widehat{O_3^+}, \widehat{U_3^+}$ are hyperlinear). However, the following proof method seems to break down in this case. See also Remark 6.

The remainder of this section is devoted to proving the above quantum subgroup generation results for O_N^+.

4.1. **Proofs of Theorems 4.1 and 4.2.** We begin by developing some tools for the proof of Theorem 4.1.

Recall that we denote $\mathrm{Fix}_k = \mathrm{Hom}_{O_N^+}(1, u^{\otimes k})$ where u is the fundamental representation of O_N^+, and let us denote similarly

$$\mathrm{Fix}_k^\xi = \mathrm{Hom}_{O_N^{+,\xi}}(1, u^{\otimes k}) = \mathrm{Hom}_{O_{N-1}^+}(1, (\iota \otimes \pi_\xi)(u)^{\otimes k}) \subseteq H^{\otimes k},$$

$$\mathrm{Fix}_k^{O_N} = \mathrm{Hom}_{O_N}(1, (\iota \otimes \pi_{O_N})(u)^{\otimes k}) \subset H^{\otimes k}.$$

According to Proposition 3.5, Theorem 4.1 is equivalent to the equalities $\mathrm{Fix}_k^\xi \cap \mathrm{Fix}_k^{O_N} = \mathrm{Fix}_k^{\xi_1} \cap \mathrm{Fix}_k^{\xi_2} = \mathrm{Fix}_k$ for all k. Hence we start by describing the subspaces Fix_k^ξ.

Let $NC_{2,1}(k)$ be the set of non-crossing partitions of $\{1, \ldots, k\}$ consisting of blocks with cardinality at most 2. In what follows, a block of $p \in NC_{2,1}(k)$ with cardinality equal to 1 will be called a *singleton*, and a block with cardinality equal to 2 will be called a *pair*. We also denote by $NC_{2,1}^s(k) \subset NC_{2,1}(k)$ the subset of non-crossing partitions containing exactly s singletons, so that $NC_2(k) = NC_{2,1}^0(k)$. For $p \in NC_{2,1}(k)$ and i a k-tuple we put $\delta_i^p = 1$ if $i_l = i_m$ for all pairs $\{l, m\} \in p$,

and $\delta_i^p = 0$ else. Then we associate to $p \in NC_{2,1}^s(k)$ a linear map $T_p : H^{\otimes s} \to H^{\otimes k}$ as follows:

$$T_p(\xi_1 \otimes \cdots \otimes \xi_s) = \sum_{i_j=1}^N \delta_i^p(e_{i_1} \otimes \cdots \otimes \xi_1 \otimes \cdots \otimes \xi_s \otimes \cdots \otimes e_{i_k}),$$

where we put a term ξ_i at position l if $\{l\}$ is the ith singleton in p, and a term e_{i_l} else. In other words, T_p is the usual map associated to the pair partition (possibly with crossings) $p' \in P_2(s, k)$ obtained from p by attaching a vertical segment to each singleton. We will also denote $T_p = T_p^1$ and consider the variant T_p^2 (resp. T_p^3) where the indices i_j range from 2 to N (resp. 3 to N). Finally we denote by $S : H^{\otimes l} \to H^{\otimes l}$ the "symmetrizing" operators

$$S : \xi_1 \otimes \cdots \otimes \xi_l \mapsto \sum_{\sigma \in \mathfrak{S}_l} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(l)}.$$

Lemma 4.5. *Denote by $v = (\iota \otimes \pi_{O_N})(u)$ the fundamental representation of O_N and fix $\xi \in S_1$.*

(1) *We have, for any $N \geq 2$, $k \in \mathbb{N}$:*

$$\mathrm{Fix}_k^\xi = \mathrm{span} \ \{T_p(\xi^{\otimes s}) \mid p \in NC_{2,1}^s(k), 0 \leq s \leq k\}.$$

(2) *The vectors $T_p(\xi^{\otimes s})$ for $p \in NC_{2,1}(k)$ are linearly independent if $N \geq 3$.*

(3) *We have $T_p \in \mathrm{Hom}_{O_N}(v^{\otimes s}, v^{\otimes k})$ for all $p \in NC_{2,1}^s(k)$.*

Proof. Consider $e_1 = \xi$ as the first vector of an ONB $(e_1, \ldots, e_n)$. By definition we have $(\iota \otimes \pi_\xi)(u) = 1 \oplus w$ in the decomposition $H = \mathbb{C}e_1 \oplus e_1^\perp$, where w is equivalent to the fundamental representation of O_{N-1}^+. As a result $(\iota \otimes \pi_\xi)(u)^{\otimes k}$ decomposes into pairwise orthogonal subrepresentations equivalent to $w^{\otimes k-s}$, $0 \leq s \leq k$. We know that the subspace of fixed vectors of $w^{\otimes k-s}$ is spanned by the elements $T_q^2(1)$, $q \in NC_2(k-s)$. Now identifying $w^{\otimes k-s}$ with a subrepresentation of $(\iota \otimes \pi_\xi)(u)^{\otimes k}$ corresponds to inserting vectors ξ at s fixed legs of the tensor product $H^{\otimes k}$, and this maps $T_q^2(1)$ to $T_r^2(\xi^{\otimes s})$, where r is obtained from q by inserting s singletons at fixed places. In this way we obtain all partitions of $NC_{2,1}^s(k)$.

We know by Theorem 2.1 that the vectors $T_q^2(1)$, $q \in NC_2(k-s)$, are linearly independent for $N - 1 \geq 2$. Since we have decomposed $(\iota \otimes \pi_\xi)(u)^{\otimes k}$ into orthogonal subrepresentations, this implies that the family of vectors $T_r^2(\xi^{\otimes s})$, $r \in NC_{2,1}(k)$, is linearly independent. Now we observe that the vectors $T_p(\xi^{\otimes s}) = T_p^1(\xi^{\otimes s})$, $p \in NC_{2,1}(k)$, can be decomposed as linear combinations of the vectors $T_r^2(\xi^{\otimes s})$ by writing

$$\sum_{i=1}^N e_i \otimes e_i = \xi \otimes \xi + \sum_{i=2}^N e_i \otimes e_i$$

at each pair of legs of $H^{\otimes k}$ determined by the pairs in p. Note that the partitions $r \neq p$ used to decompose in this way a vector $T_p(\xi^{\otimes s})$ have strictly more singletons than p, so that the decomposition matrix is block triangular (with respect to the value of s) with identity blocks on the diagonal. This implies that the family $T_p(\xi^{\otimes s})$, $p \in NC_{2,1}(k)$, is linearly independent and spans the same subspace as the vectors $T_r^2(\xi^{\otimes s})$. This proves the first two assertions.

Finally, for any $p \in NC_{2,1}^s(k)$ we know that $T_p = T_{p'}$ for a suitable partition $p' \in P_2(s,k)$ (see above) and that the maps $T_{p'}$ are O_N-intertwiners (see the end of Section 2.3). □

We now reduce Theorem 4.1 to a linear independence problem:

Proposition 4.6. *For $N \geq 3$ and $k \in \mathbb{N}$, the following are equivalent:*

(1) *We have $\mathrm{Fix}_k^{\xi_1} \cap \mathrm{Fix}_k^{\xi_2} = \mathrm{Fix}_k$ for some (or any) pair of linearly independent vectors $\xi_1, \xi_2 \in S_1$.*

(2) *We have $\mathrm{Fix}_k^{\xi} \cap \mathrm{Fix}_k^{O_N} = \mathrm{Fix}_k$ for some (or any) $\xi \in S_1$.*

(3) *The vectors $T_p(S(e_1 \otimes \cdots \otimes e_1 \otimes e_2))$, $p \in NC_{2,1}(k) \setminus NC_2(k)$, are linearly independent.*

Proof. We first recall that two different stabilizer subgroups $O_{N-1} < O_N$ generate O_N. Indeed, for $1 \leq i < j \leq N$, calling $R_{i,j,\theta}$ the rotation of angle θ, between the canonical basis vectors e_i and e_j it is known that $R_{i,j,\theta}$ generates SO_N if one takes all $1 \leq i < j \leq N, \theta \in [0, 2\pi)$.

Without loss of generality – at the possible cost of involving conjugation by rotations – we can assume that the first copy of O_{N-1} fixes e_N and the second copy fixes e_1. One can check that $R_{1,N,\theta}$ can be obtained as a conjugation of $R_{1,N-1,\theta}$ by $R_{N-1,N,\pi}$. This implies that any two copies of $SO_{N-1} < SO_N$ generate SO_N. The fact that two different copies of $O_{N-1} < O_N$ generate O_N follows from the fact that we can find in O_{N-1} an isometry that takes SO_{N-1} to $O_{N-1} \backslash SO_{N-1}$ by left multiplication.

Now let $x \in \mathrm{Fix}_k^{\xi_1} \cap \mathrm{Fix}_k^{\xi_2}$ be given. Then x is fixed by the two copies of O_{N-1} inside the quantum subgroups O_{N-1}^{+,ξ_1} and O_{N-1}^{+,ξ_2}; hence it is fixed by O_N by the previous paragraph. On the other hand, for $g \in O_N$ we have $\mathrm{Fix}_k^{g\xi} = g \cdot \mathrm{Fix}_k^{\xi} := v(g)^{\otimes k} \mathrm{Fix}_k^{\xi}$, where $v = u^{O_N}$ is the fundamental representation of O_N. As a result, if $x \in \mathrm{Fix}_k^{O_N} \cap \mathrm{Fix}_k^{\xi}$, then x lies in Fix_k^{ζ} for any $\zeta \in S_1$. This shows that $\mathrm{Fix}_k^{\xi_1} \cap \mathrm{Fix}_k^{\xi_2}$ and $\mathrm{Fix}_k^{O_N} \cap \mathrm{Fix}_k^{\xi}$ are equal and independent of the choice of ξ and the linearly independent pair ξ_1, ξ_2 in S_1.

According to the previous lemma, any $x \in \mathrm{Fix}_k^{\xi}$ can be written $x = \sum \lambda_p T_p(\xi^{\otimes s})$ in a unique way, and since Fix_k is spanned by the vectors $T_p(1)$, $p \in NC_2(k)$, we have $x \in \mathrm{Fix}_k$ **iff** $\lambda_p = 0$ for all $p \in NC_{2,1}^s(k)$, $s > 0$. Besides, if x is O_N-invariant we have $x = g \cdot x = \sum \lambda_p T_p((g\xi)^{\otimes s})$ for all $g \in O_N$, so that the map $\mathcal{T}_\lambda : \mathbb{R}^N \to (\mathbb{R}^N)^{\otimes k}$, $\xi \mapsto \sum \lambda_p T_p(\xi^{\otimes s})$ is constant on S_1. Hence the second assertion in the statement is equivalent to the implication (I) "$\mathcal{T}_\lambda$ constant on $S_1 \Rightarrow \lambda_p = 0$ for $p \in NC_{2,1}^s(k)$, $s > 0$" for all $\lambda : NC_{2,1}(k) \to \mathbb{C}$.

Now we differentiate: $\mathcal{T}_\lambda$ is constant on S_1 **iff** $d_\xi \mathcal{T}_\lambda(\eta) = 0$ for all $\xi \in S_1$, $\eta \perp \xi$. Moreover by O_N-covariance of $\mathcal{T}_\lambda$ we have $g \cdot d_\xi \mathcal{T}_\lambda(\eta) = d_{g\xi} \mathcal{T}_\lambda(g\eta)$, and since O_N acts transitively on pairs of normed orthogonal vectors, $\mathcal{T}_\lambda$ is constant on S_1 **iff** $d_{e_1} \mathcal{T}_\lambda(e_2) = 0$. Then we compute $d_\xi(\xi^{\otimes s})(\eta) = S(\xi \otimes \cdots \otimes \xi \otimes \eta)/(s-1)!$, hence

$$d_\xi \mathcal{T}_\lambda(\eta) = \sum_{s>0, \ p \in NC_{2,1}^s(k)} \frac{\lambda_p}{(s-1)!} \, T_p(S(\xi^{\otimes s-1} \otimes \eta)).$$

This shows the equivalence of the last assertion in the statement with the condition (I) above. □

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Proof of Theorem 4.1. We will verify the linear independence condition given in part (3) of Proposition 4.6. Consider the vectors $y_{p,i} = T_p^3(e_{i_1} \otimes \cdots \otimes e_{i_s}) \in H^{\otimes k}$ with $i_l = 1, 2$ and $p \in NC_{2,1}(k)$. They form a linearly independent family, which we shall denote by $\mathcal{C}$. Indeed if $\langle y_{p,i} | y_{q,j} \rangle \neq 0$, then p, q must have the same singletons and $i = j$. Moreover when this is the case, then $\langle y_{p,i} | y_{q,j} \rangle$ coincides with the scalar product $\langle T'_{p'}(1) | T'_{q'}(1) \rangle$ associated with the partitions p', $q' \in NC_2(k-s)$ obtained from p and q by removing singletons, where $T'_{p'}$ is the map analogous to $T_{p'}$, but in dimension $N-2$. Since $N-2 \geq 2$, the vectors $T'_{p'}(1)$ are linearly independent, and we can deduce that the Gram matrix of the family $\mathcal{C}$ is invertible (cf. Theorem 2.1).

Now consider the vectors $x_p = T_p(S(e_1 \otimes \cdots \otimes e_1 \otimes e_2))/(s-1)!$ from Proposition 4.6. Note that each x_p can be written as a (unique) linear combination of elements in $\mathcal{C}$. This follows from the definition of S and by writing $\sum_{i=1}^N e_i \otimes e_i = e_1 \otimes e_1 + e_2 \otimes e_2 + \sum_{i=3}^N e_i \otimes e_i$ as in the proof of Lemma 4.5. More precisely, if $p \in NC_{1,2}^s(k)$, then x_p decomposes into the sum of the s vectors $y_{p,i}$ with i taking the value 2 only once, and a linear combination of vectors $y_{q,j}$ with q having strictly more singletons that p. As a result, if we partially order the families $\mathcal{B} = (x_p)$ and $\mathcal{C} = (y_{p,i})$ according to the number of singletons s in p, the corresponding decomposition matrix for $\mathcal{B}$ in terms of the basis $\mathcal{C}$ will be block lower-triangular (with rectangular blocks), and each diagonal sub-block (one for each integer s) is itself block diagonal, with diagonal blocks which are non-zero columns (one for each partition $p \in NC_{2,1}^s(k)$). In particular, this decomposition matrix has maximal rank, and therefore $\mathcal{B}$ is linearly independent. $\qquad\square$

Remark 6. Although the proof above only applies for $N \geq 4$, it seems very likely that the linear independence condition introduced in Proposition 4.6, and hence Theorem 4.1, also holds at $N = 3$. This would imply the hyperlinearity of $\hat{O}_N^+$ and $\hat{U}_N^+$ for all $N \geq 2$, without relying on Theorem 4.2. In fact, we have strong numerical evidence that the family of vectors $T_p(e_1)$ associated to "one singleton" partitions $p \in NC_{2,1}^1(k)$ is linearly independent for all $N \geq 2$, and using similar techniques as above this would imply Theorem 4.1 for all $N \geq 3$.

We now provide a proof of Theorem 4.2.

Proof of Theorem 4.2. Let $u \in \mathcal{B}(\mathbb{C}^{2n}) \otimes C^u(O_{2n}^+)$ be the fundamental representation of O_{2n}^+, let $l \in \mathbb{N}$, and put $u_l = u^{\otimes l}$. Similarly, let w be the fundamental representation of U_{2n}^+ and put $w_l = w \otimes \bar{w} \otimes w \otimes \cdots$ (l-terms). In what follows, we will regard w_l and u_l as both acting on the same Hilbert space (cf. Remark 2). In particular, when l is even, we have $\mathrm{Fix}(u_l) = \mathrm{Fix}(w_l)$ under this identification. Moreover, from the description of the spaces of intertwiners for free products of compact quantum groups given in [27, Proposition 2.15], it also follows that $\mathrm{Fix}(u_l^{O_n^+ \hat{*} O_n^+}) = \mathrm{Fix}(w_l^{U_n^+ \hat{*} U_n^+})$ when l is even.

Now choose $x \in \mathrm{Fix}(u_l^{\mathfrak{S}_{2n}}) \cap \mathrm{Fix}(u_l^{O_n^+ \hat{*} O_n^+}) \subset (\mathbb{C}^{2n})^{\otimes l}$. Our goal, according to Proposition 3.5 (2), is to show that $x \in \mathrm{Fix}(u_l)$. Since $\mathrm{Fix}(u_l^{O_n^+ \hat{*} O_n^+}) = \mathrm{Fix}(u_l) = \{0\}$ when l is odd, we will assume $l = 2k$ ($k \in \mathbb{N}$) for the remainder of the proof. According to the discussion in the previous paragraph, we have $x \in \mathrm{Fix}(w_{2k}^{U_n^+ \hat{*} U_n^+}) \subset \mathrm{Fix}(w_{2k}^{U_n \times U_n})$, where the classical group $U_n \times U_n \leq U_n^+ \hat{*} U_n^+$ acts block-diagonally on $\mathbb{C}^{2n}$ with respect to a fixed orthonormal basis $(e_i)_{i=1}^{2n}$. Now we recall the following

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elementary group theoretic fact:

(7) U_{2n} is generated by the subgroups $\mathfrak{S}_{2n}$ and $U_n \times U_n$.

One can actually even show more, namely that for any $d \geq 2$, U_d is (algebraically) generated by the subgroups $\mathfrak{S}_d$ and U_2, where U_2 is viewed as sitting on the upper left corner of $d \times d$ matrices. This is trivial for $d = 2$. For general d, we proceed by induction over d: given an element of U_d, it is possible to multiply it on the left by $d - 1$ elements of type $\sigma U \sigma^{-1}$ where $\sigma \in \mathfrak{S}_d$ and $U \in U_2$ and ensure that the bottom element of the last column of the new element of U_d is 1. Indeed, the action by left multiplication leaves the columns invariant, and successive operations of the groups $U_2, (13)U_2(13), \ldots, (1d)U_2(1d)$ can be performed to ensure that the element of respective indices $(1, d), (2, d), \ldots, (d - 1, d)$ is sent to zero and, in turn, that the entry of index (d, d) is sent to 1. By orthogonality relations, the new matrix obtained also has zeros on all entries of the last row apart from the last one; therefore it sits in U_{d-1} viewed as the upper left corner of U_d. The general result follows by induction.

Applying fact (7), we finally conclude that $x \in \mathrm{Fix}(w_{2k}^{U_n^+ \hat{*} U_n^+}) \cap \mathrm{Fix}(w_{2k}^{U_{2n}})$. To finish the proof, we appeal to the following lemma, which is a special case of a very recent result of Chirvasitu [13]. We include a detailed proof for the convenience of the reader.   □

Lemma 4.7 ([13], Lemma 3.11). *With the notation and conventions as above, we have the equalities*

$$\mathrm{Fix}(w_{2k}^{U_n^+ \hat{*} U_n^+}) \cap \mathrm{Fix}(w_{2k}^{U_{2n}}) = \mathrm{Fix}(w_{2k}) = \mathrm{Fix}(u_{2k}) \qquad (k \in \mathbb{N}).$$

Proof. Let $H = \mathbb{C}^{2n}$, fix an orthonormal basis $(e_i)_{i=1}^{2n}$ for H and write $H = H_1 \oplus H_2$, where $H_1 = \mathrm{span}(e_1, \ldots, e_n)$ and $H_2 = \mathrm{span}(e_{n+1}, \ldots, e_{2n})$. Denote by $P_i \in \mathcal{B}(H)$ the orthogonal projection whose range is H_i. In the following, we will fix once and for all the linear isomorphism

(8) $\Phi : (H \otimes \bar{H})^{\otimes k} \to \mathcal{B}(H^{\otimes k})$

given by identifying an elementary tensor $e_{i_1} \otimes \overline{e_{i_2}} \otimes \cdots \otimes e_{i_{2k-1}} \otimes \overline{e_{i_{2k}}} \in (H \otimes \bar{H})^{\otimes k}$ with the rank-one operator

$$H^{\otimes k} \ni \xi \mapsto \Big(\bigotimes_{r=1}^{k} e_{i_{2r-1}} \Big) \langle \bigotimes_{r=1}^{k} e_{i_{2r}} \mid \xi \rangle \in H^{\otimes k}.$$

Recall that $\mathrm{Fix}(w_{2k}^{U_{2n}})$ is spanned by the vectors

$$T_p = \sum_i \delta_i^p (e_{i_1} \otimes \overline{e_{i_2}} \otimes \cdots \otimes e_{i_{2k-1}} \otimes \overline{e_{i_{2k}}}),$$

where $p \in P_2(2k)$ is a pair partition respecting the additional requirement that even points are connected to odd points.

When Φ is restricted to the subspace $\mathrm{Fix}(w_{2k}^{U_{2n}})$, we obtain an isomorphism

(9) $\mathrm{Fix}(w_{2k}^{U_{2n}}) \cong^{\Phi} \mathrm{Hom}((w^{U_{2n}})^{\otimes k}, (w^{U_{2n}})^{\otimes k}),$

which maps the vector T_p to a map T_q, where $q \in P_2(k, k)$ is obtained by connecting the even (respectively, odd) points of p to the input (respectively, output) points of q. We denote by $q \in S(k, k)$ the pair partitions obtained in this way: these are exactly the ones where input points are connected to output points via

the permutation specified by q. Note that one recovers the Schur-Weyl duality for unitary groups describing $\mathrm{Hom}((w^{U_{2n}})^{\otimes k}, (w^{U_{2n}})^{\otimes k})$ as the linear span of the operators $(T_q)_{q \in S(k,k)}$ which permute the tensor factors of $H^{\otimes k}$.

On the other hand, the image by Φ of the subspace $\mathrm{Fix}(w_{2k})$ is spanned by the maps T_q where q belongs to a subset $S'(k,k) \subset S(k,k) \subset P_2(k,k)$: namely the one corresponding to $p \in NC_2(2k) \subset P_2(2k)$. The only thing we will need to know about $S'(k,k)$ is that the family T_q, $q \in S'(k,k)$, is linearly independent as soon as $n \geq 2$. This is indeed the case since $(T_p)_{p \in NC_2(2k)}$ is linearly independent and Φ is an isomorphism.

Finally, for each $q \in S'(k,k)$ and each k-tuple $i = (i_1, \ldots, i_k) \in \{1,2\}^k$, we define a linear map $T_{q,i} := T_q P_i$, where $P_i := \bigotimes_{r=1}^{k} P_{i_r}$ is the orthogonal projection whose range is $H_i := \bigotimes_{r=1}^{k} H_{i_r}$. From the description of the intertwiner spaces of U_N^+ ($N \geq 2$) and their free products given in [6, Section 9] and [27, Proposition 2.15], respectively, it follows that the family $T_{q,i}$, $q \in S'(k,k)$, $i \in \{1,2\}^k$, forms a basis of the intertwiner space

$$\mathrm{Hom}((w^{U_n^+ \hat{*} U_n^+})^{\otimes k}, (w^{U_n^+ \hat{*} U_n^+})^{\otimes k}) \cong^{\Phi} \mathrm{Fix}(w_{2k}^{U_n^+ \hat{*} U_n^+}).$$

In view of the above isomorphisms, it remains to demonstrate that any linear map $T \in \mathrm{Hom}\,((w^{U_n^+ \hat{*} U_n^+})^{\otimes k}, (w^{U_n^+ \hat{*} U_n^+})^{\otimes k}) \cap \mathrm{Hom}((w^{U_{2n}})^{\otimes k}, (w^{U_{2n}})^{\otimes k})$ lies in fact in $\mathrm{Hom}(w^{\otimes k}, w^{\otimes k})$. By linear independence, T can be uniquely expressed as the sum

$$T = \sum_{i \in \{1,2\}^k} \sum_{q \in S'(k,k)} \lambda_{q,i} T_{q,i} \qquad (\lambda_{q,i} \in \mathbb{C}).$$

Denote by $1^k = (1,1,\ldots,1)$ the constant k-tuple. Since we can decompose each T_q as $T_q = \sum_{i \in \{1,2\}^k} T_{q,i}$, we may subtract $T_1 := \sum_{q \in S'(k,k)} \lambda_{q,1^k} T_q \in \mathrm{Hom}(w^{\otimes k}, w^{\otimes k})$ from T and consequently assume for the remainder that $TP_{1^k} = 0$.

Now consider an arbitrary k-tuple i. We claim that $T|_{H_i} = 0$. To see this, consider the linear map $g : H \to H_1$, $e_r \mapsto e_r$, $e_{r+n} \mapsto e_r$ for all $r = 1, \ldots, n$. Observe that the restriction $g^{\otimes k} : H_i \to H_{1^k}$ is an isomorphism and that $g^{\otimes k} T_{q,i} = T_{q,1^k} g^{\otimes k} P_i$. Moreover, since T is a U_{2n}-intertwiner it is a linear combination of maps T_r, $r \in S(k,k)$, and each of these maps verifies the relation $h^{\otimes k} T = T h^{\otimes k}$ for any $h \in \mathcal{B}(H)$. We then have

$$0 = T g^{\otimes k} P_i = g^{\otimes k} T P_i = g^{\otimes k} \sum_q \lambda_{q,i} T_{q,i} = \sum_q \lambda_{q,i} T_{q,1^k} g^{\otimes k} P_i.$$

Since the family $T_{q,1^k}$, $q \in S'(k,k)$, is linearly independent and $g^{\otimes k} P_i : H_i \to H_{1^k}$ is an isomorphism, we conclude that $\lambda_{q,i} = 0$ for each q. Finally, since $H^{\otimes k} = \bigoplus_i H_i$ and i was arbitrary, this implies $T = 0$, i.e., $T = T_1 \in \mathrm{Hom}(w^{\otimes k}, w^{\otimes k})$. $\qquad \square$

5. Applications

5.1. **Free entropy dimension.** In this section we present an application of our hyperlinearity results to the computation of the free entropy dimension of the canonical generators of $L^\infty(O_N^+)$. We refer the reader to the survey [41] for details on the various notions of free entropy dimension and related concepts.

Let Γ be a finitely generated discrete group with a finite symmetric system of generators $(g_i)_{i=1}^n$, and put $x_i = \Re\lambda(g_i), y_i = \Im\lambda(g_i) \in \mathcal{L}(\Gamma)$. In [17, Corollary 4.9], Connes and Shlyakhtenko showed that the (non-microstates) free entropy dimension $\delta^*(x_i, y_i)$ verifies the inequality

$$(10) \qquad \delta^*(x_i, y_i) \leq \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1,$$

where $\beta_k^{(2)}(\Gamma)$ is the kth ℓ^2-Betti number of Γ. On the other hand, by [8], the (modified) microstates free entropy dimension is known to satisfy the inequality

$$(11) \qquad \delta_0(x_i, y_i) \leq \delta^*(x_i, y_i).$$

Finally, if $\mathcal{L}(\Gamma)$ is diffuse and has the Connes embedding property, it was shown in [25, Corollary 4.7] that

$$(12) \qquad 1 \leq \delta_0(x_i, y_i).$$

All of the above inequalities apply to the case of quantum groups. More precisely, let $\mathbb{G}$ be a compact matrix quantum group of Kac type with diffuse Connes embeddable von Neumann algebra $L^\infty(\mathbb{G})$, let $(u^i)_{i=1}^n$ be a self-conjugate family of inequivalent irreducible unitary representations of $\mathbb{G}$ whose matrix elements generate $\mathrm{Pol}(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$, and let $(x_{kl}^i), (y_{kl}^i) \subset L^\infty(\mathbb{G})$ be the real and imaginary parts of the matrix elements of these representations (respectively). Then we have the chain of inequalities

$$(13) \qquad 1 \leq \delta_0(x_{kl}^i, y_{kl}^i) \leq \delta^*(x_{kl}^i, y_{kl}^i) \leq \beta_1^{(2)}(\mathbb{G}) - \beta_0^{(2)}(\mathbb{G}) + 1,$$

where $\beta_k^{(2)}(\mathbb{G})$ is the *kth ℓ^2-Betti number* of the compact quantum group $\mathbb{G}$ introduced by Kyed. See [26] and [28]. Putting all of this together, we obtain the following result.

Theorem 5.1. *The microstates (and non-microstates) free entropy dimension of $L^\infty(O_N^+)$ associated to the canonical generators $(u_{ij})_{1 \leq i,j \leq N}$ is 1 for all $N \geq 4$.*

Proof. It was proved in [15, 37] that the right hand side of inequality (13) is exactly 1. Together with the above discussion, the proof is complete. $\qquad\square$

5.2. **A remark on the classification of intermediate quantum subgroups between O_N and O_N^+.** Let $N \geq 3$ and consider the quantum group O_N^+. It is currently an open problem to determine all intermediate quantum subgroups $O_N \leq \mathbb{G} \leq O_N^+$. At this time, only one such intermediate quantum subgroup is known, namely the *half-liberated orthogonal quantum group* $O_N \leq O_N^* \leq O_N^+$ [7]. Moreover, it is known that the inclusion $O_N \leq O_N^*$ is maximal, meaning that there is no intermediate quantum group $O_N \leq \mathbb{G} \leq O_N^*$. See [5] for details. It is also conjectured in [5] that the inclusion $O_N^* \leq O_N^+$ is maximal. In this short section we state an easy consequence of Theorem 4.1, which can be regarded as shedding some light on this conjecture.

Let $k \geq 1$ and let $E \subset \mathbb{C}^N$ be a k-dimensional subspace. Generalizing the case $k = N - 1$ discussed in Section 4, we can canonically associate to E the quantum subgroup $O_{k,E}^+ \leq O_N^+$ (isomorphic to O_k^+ acting on E via its fundamental representation and acting trivially on $E^\perp$), as well as the corresponding subgroups $O_{k,E}^*, O_{k,E}$.

Theorem 5.2. *If $\mathbb{G} \leq O_N^+$ is a quantum subgroup containing O_N^* and $O_{3,E}^+$, then $\mathbb{G} = O_N^+$.*

Proof. From Theorem 4.1 it follows by a simple induction on $k = \dim E \geq 3$ that O_N^+ is generated by O_N and any subgroup $O_{k,E}^+$. In particular it is generated by O_N^* and $O_{3,E}^+$. $\qquad\square$

Acknowledgments

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