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Orientation of quantum Cayley trees and applications

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Abstract. We introduce the quantum Cayley graphs associated to quantum discrete groups and study them in the case of trees. We focus in particular on the notion of quantum ascending orientation and describe the associated space of edges at infinity, which is an outcome of the non-involutivity of the edge-reversing operator and vanishes in the classical case. We end with applications to Property AO and K-theory.

0. Introduction

The original motivation of this paper is Cuntz' result on the K-amenability of free groups [7], and the geometric proof of this result given by the more general paper of Julg and Valette [8] on groups acting on trees with amenable stabilizers. Natural quantum analogues of the free groups are the free quantum groups defined by Wang and van Daele [16] and studied by Banica [5]. Moreover, equivariant KK-theory can be generalized to the case of coactions of Hopf C^*-algebras [2], and the notion of K-amenability carries over to this quantum framework without difficulty [17]. It is therefore natural to ask whether free quantum groups are K-amenable.

To apply the method of Julg and Valette in this framework, one needs a quantum geometric object to play the role of the tree acted upon by the quantum group under consideration. In the case of amalgamated free products of amenable discrete quantum groups, the construction of a quantum analogue of the Bass-Serre tree was achieved in [18] and could be used to prove the K-amenability of these amalgamated free products. In the case of the free quantum groups, the needed objects should be generalizations of the Cayley graphs of the free groups. The main goal of this paper is to define a notion of Cayley graph for discrete quantum groups, and to study its geometric properties in the case of the free quantum groups.

We will give in the last section applications of this study: a proof of the property of Akemann and Ostrand and the construction of a KK-theoretic element γ for free quantum groups. To prove that these quantum groups are K-amenable, it remains to prove that $\gamma = 1$. We refer the reader to the last section for more historical remarks and references about Property AO and K-amenability.

The paper is organized as follows:

1. In the first section, we recall some notation and formulae concerning discrete quantum groups and classical graphs.

2. The second section is a technical one about fusion morphisms of free quantum groups, and its results are only used in the proofs of Sections 6 and 7. The reader will probably like to skip over this section at first.

3. In the third section, we give the definition of the Cayley graphs of discrete quantum groups and state some basic results about them.

4. We then restrict ourselves to the case of Cayley trees. We introduce and characterize this notion in the fourth section, where we also study the natural ascending orientation of such a tree.

5. In the fifth section, we study more precisely the space of geometric edges of a quantum Cayley tree and we find that the projection of ascending edges onto geometric ones is not necessarily injective.

6. We show more precisely in the sixth section that the obstruction to this injectivity is the existence of a natural space of (geometric) edges at infinity, which vanishes in the classical case.

7. In the seventh section, we equip this space with a natural representation of the free quantum group under consideration, thus turning it into an interesting geometric object on its own.

8. Finally the last section deals with applications, as explained above.

Acknowledgments. Most of the results of Sections 2–5 and, in the special case of the orthogonal free quantum groups, of Sections 6, 7 and 8.2, were included in my PhD thesis [17] at the University Paris 7. I would like to thank my advisor, Prof. G. Skandalis, for having directed me to the beautiful paper [8] of Julg and Valette, and for his precious help in the redaction of my thesis.

The generalization and continuation of these results, as well as the redaction of the present article, were completed during a one-year stay at the University of Münster, where I could benefit from a postdoctoral position. It is a pleasure to thank Prof. J. Cuntz for his friendly hospitality and for the very stimulating atmosphere of his team.

1. Notation

The general framework of this paper will be the theory of compact quantum groups due to Woronowicz [22]. In fact we will use it, from the dual point of view, as a theory of discrete quantum groups. Let us fix the notation for the rest of the paper. The starting object is a unital Hopf C^*-algebra (S, δ) such that $\delta(S)(1 \otimes S)$ and $\delta(S)(S \otimes 1)$ are dense in $S \otimes S$. Such a Hopf C^*-algebra will be called a Woronowicz C^*-algebra. One of the key results of the theory is the existence of a unique Haar state h on (S,δ) [20]. We will put $\delta^2 = (\mathrm{id} \otimes \delta)\delta = (\delta \otimes \mathrm{id})\delta$ and similarly $\delta^3 = (\mathrm{id} \otimes \mathrm{id} \otimes \delta)\delta^2$.

We denote by $\mathscr{C}$ the category of corepresentations of (S,δ) on finite-dimensional Hilbert spaces, by $\mathrm{Irr}\,\mathscr{C}$ a set of representatives of irreducible corepresentations modulo equivalence, and by $1_{\mathscr{C}} = \mathrm{id}_{\mathbb{C}} \otimes 1_S$ the trivial corepresentation. We will denote by H_α and $v_\alpha \in B(H_\alpha) \otimes S$ the Hilbert space and the corepresentation associated to an object $\alpha \in \mathscr{C}$. The category $\mathscr{C}$ is equipped with direct sum, tensor product and conjugation operations: for the first and the second ones we refer to [20], and we give now some precisions about the third one which is slightly more involved.

Let (e_i) be an orthonormal basis of H_α. The conjugate object $\bar\alpha$ of $\alpha \in \mathscr{C}$ is characterized, up to isomorphism, by the existence of a conjugation map $j_\alpha : H_\alpha \to H_{\bar\alpha}$, $\zeta \mapsto \bar\zeta$ which is an anti-isomorphism such that $t_\alpha : 1 \mapsto \sum e_i \otimes \bar e_i$ and $t'_\alpha : \bar\zeta \otimes \xi \mapsto (\zeta|\xi)$ are resp. elements of $\mathrm{Mor}(1, \alpha \otimes \bar\alpha)$ and $\mathrm{Mor}(\bar\alpha \otimes \alpha, 1)$. We put $F_\alpha = j_\alpha^* j_\alpha$ and we say that j_α is normalized if $\mathrm{Tr}\, F_\alpha = \mathrm{Tr}\, F_\alpha^{-1}$. This positive number, which is also equal to $\|t_\alpha(1)\|^2$, does not depend on the normalized map j_α. It is called the quantum dimension of α and is denoted by M_α. When α is in $\mathrm{Irr}\,\mathscr{C}$, we can assume that $\bar\alpha$ is in $\mathrm{Irr}\,\mathscr{C}$, and the possible conjugation maps j_α only differ by a scalar. We have then $\bar{\bar\alpha} = \alpha$, and if $\alpha \neq \bar\alpha$ one can choose normalized conjugation maps $j_\alpha, j_{\bar\alpha}$ such that $j_{\bar\alpha} j_\alpha = 1$. If $\alpha = \bar\alpha$ one has $j_\alpha^2 = \pm 1$ for every normalized j_α.

The coefficients of the corepresentations v_α span a dense subspace $\mathscr{S} \subset S$ which turns out to be a Hopf $*$-algebra. We denote by m its multiplication, and by $\varepsilon : \mathscr{S} \to \mathbb{C}$ and $\kappa : \mathscr{S} \to \mathscr{S}$ its co-unit and its antipode. Notice that κ is not involutive in general. In this regard, an important role is played by a family $(f_z)_{z\in\mathbb{C}}$ of multiplicative linear forms on $\mathscr{S}$, which are also related to the non-triviality of the modular properties of h. We will need in this paper the following formulae in the Hopf *-algebra $\mathscr{S}$:

$$(1) \qquad \forall x \in \mathscr{S} \quad (\mathrm{id} \otimes \varepsilon) \circ \delta(x) = (\varepsilon \otimes \mathrm{id}) \circ \delta(x) = x,$$

$$(2) \qquad \forall x \in \mathscr{S} \quad m \circ (\mathrm{id} \otimes \kappa) \circ \delta(x) = m \circ (\kappa \otimes \mathrm{id}) \circ \delta(x) = \varepsilon(x)1.$$

Let $\Lambda_h : S \to H$ be the GNS construction of the Haar state h, denote by $\lambda : S \to B(H)$ the corresponding GNS representation and by S_{red} its image. The Kac system of the compact quantum group (S,δ) is given by the following formulae, where $f \star x := (\mathrm{id} \otimes f)\delta(x)$ is the convolution product of $f \in \mathscr{S}^*$ and $x \in \mathscr{S}$:

$$(3) \qquad V : (\Lambda_h \otimes \Lambda_h)(x \otimes y) \mapsto (\Lambda_h \otimes \Lambda_h)\big(\delta(x) 1 \otimes y\big),$$

$$(4) \qquad U : \Lambda_h(x) \mapsto \Lambda_h\big(f_1 \star \kappa(x)\big).$$

Let us recall the following notation and formulae from the general theory of multiplicative unitaries [3]. The unitary $V \in B(H \otimes H)$ is multiplicative, meaning that $V_{12}V_{13}V_{23} = V_{23}V_{12}$, and for any $\omega \in B(H)_*$ one puts $L(\omega) = (\omega \otimes \mathrm{id})(V)$ and $\rho(\omega) = (\mathrm{id} \otimes \omega)(V)$. On the other hand, U is an involutive unitary on H such that $\tilde V = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma$ and $\hat V = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ are again multiplicative unitaries. Moreover the irreducibility property holds: $\big(\Sigma(1 \otimes U)V\big)^3 = 1$ or, equivalently, $\hat V V \tilde V = (U \otimes 1)\Sigma$. Here Σ denotes the flip operator, and we use the leg numbering notation.

The reduced C^*-algebra S_{red} coincides with the closure of $L(B(H)_*)$ in $B(H)$, and we similarly denote by $\hat{S}$ the closure of $\rho(B(H)_*)$. Both can be made Hopf C^*-algebras by the following formulae:

$$\delta_{\mathrm{red}}(s) = V(s \otimes 1)V^* = \hat{V}^*(1 \otimes s)\hat{V}, \tag{5}$$

$$\hat{\delta}(\hat{s}) = V^*(1 \otimes \hat{s})V = \tilde{V}(\hat{s} \otimes 1)\tilde{V}^*. \tag{6}$$

Notice that the reduction homomorphism $\lambda : S \to S_{\mathrm{red}}$ induces then an isomorphism between the dense Hopf *-algebras of both Woronowicz C^*-algebras. Besides, the unitary V lies in $M(\hat{S} \otimes S_{\mathrm{red}})$ and we have the following commutation relations inside $B(H)$: $[S_{\mathrm{red}}, US_{\mathrm{red}}U] = [\hat{S}, U\hat{S}U] = 0$. There is also a full version of S [3] and we will say that S is a full Woronowicz C^*-algebra when it coincides with its full version.

Finally, the structure of the dual C^*-algebra $\hat{S}$ is very easy to describe: it is isomorphic to the direct sum over $\alpha \in \mathrm{Irr}\,\mathscr{C}$ of the matrix C^*-algebras $B(H_\alpha)$. We will denote by $p_\alpha \in B(H)$ the corresponding minimal central projections of $\hat{S}$, except the one associated to the trivial corepresentation $1_{\mathscr{C}}$ which will be denoted by p_0.

Let us recall some facts about free quantum groups. The definition was given in [19], [16]: let $n \geqq 2$ be an integer, and Q an invertible matrix in $M_n(\mathbb{C})$, the C^*-algebra $A_u(Q)$ is then the universal unital C^*-algebra generated by n^2 elements $u_{i,j}$ and the relations that make $U = (u_{i,j})$ and $Q\bar{U}Q^{-1} = Q(u_{i,j}^*)Q^{-1} \in M_n(A_u(Q))$ unitary. The C^*-algebra $A_o(Q)$ is defined similarly with the relations making U unitary and $Q\bar{U}Q^{-1}$ equal to U. We will write $S = A_o(Q)$ or $A_u(Q)$ when there is no need to distinguish the unitary and orthogonal versions. It is easy to see that S carries a unique Woronowicz C^*-algebra structure (S, δ) for which U is a corepresentation.

The corepresentation theory of $A_u(Q)$ was fully described in [5] in the following way. The set of representatives $\mathrm{Irr}\,\mathscr{C}$ can be identified with the free monoid on two generators u and $\bar{u}$ in such a way that the corepresentation associated to u is equivalent to U and the following recursive rules hold:

$$\alpha u \otimes \bar{u}\alpha' = \alpha u\bar{u}\alpha' \oplus \alpha \otimes \alpha', \quad \alpha\bar{u} \otimes u\alpha' = \alpha\bar{u}u\alpha' \oplus \alpha \otimes \alpha',$$

$$\alpha u \otimes u\alpha' = \alpha uu\alpha', \quad \alpha\bar{u} \otimes \bar{u}\alpha' = \alpha\bar{u}\bar{u}\alpha', \quad \overline{\alpha u} = \bar{u}\bar{\alpha}, \quad \overline{\alpha\bar{u}} = u\bar{\alpha}.$$

The corepresentation theory of $A_o(Q)$ is even simpler. We assume in this case that $Q\bar{Q}$ is a scalar matrix, otherwise the fundamental corepresentation U is not irreducible. The set $\mathrm{Irr}\,\mathscr{C}$ can then be identified with $\mathbb{N}$ in such a way that the corepresentation associated to α_1 is equivalent to U and the fusion and conjugation rules read as in the representation theory of $\mathrm{SU}(2)$:

$$\alpha_k \otimes \alpha_l = \alpha_{|k-l|} \oplus \alpha_{|k-l|+2} \oplus \cdots \oplus \alpha_{k+l-2} \oplus \alpha_{k+l}, \quad \overline{\alpha_k} = \alpha_k.$$

Let us finally fix some terminology concerning classical graphs. Following [14], a graph $\mathfrak{g}$ will be given by a set of vertices $\mathfrak{v}$, a set of edges $\mathfrak{e}$, an endpoints map $e : \mathfrak{e} \to \mathfrak{v} \times \mathfrak{v}$

and a reversing map $\theta : \mathfrak{e} \to \mathfrak{e}$ which should be an involution such that $e \circ \theta = \sigma \circ e$. In this paper we denote by σ the flip map for spaces and C^*-algebras. If e is injective, the graph $\mathfrak{g} = (\mathfrak{v}, \mathfrak{e}, e, \theta)$ is isomorphic to the graph $(\mathfrak{v}, e(\mathfrak{e}), i_{\mathrm{can}}, \sigma)$, which we will call the simplicial realization of $\mathfrak{g}$—although it only comes from a simplicial complex when it has no loops, i.e. when $e(\mathfrak{e})$ doesn't meet the diagonal.

The set of geometric, or non-oriented, edges of $\mathfrak{g}$ is the quotient $\mathfrak{e}_g$ of $\mathfrak{e}$ by the relation $a \sim \theta(a)$. An orientation of the graph is a subset $\mathfrak{e}_+ \subset \mathfrak{e}$ such that $\mathfrak{e}$ is the disjoint union of $\mathfrak{e}_+$ and $\theta(\mathfrak{e}_+)$. The quotient map evidently induces a bijection between any orientation and the set of geometric edges. When $\mathfrak{g}$ is a tree endowed with an origin α_0, we denote by $|\cdot|$ the distance to α_0 and the ascending orientation of $\mathfrak{g}$ is the set of edges a such that $e(a) = (\alpha, \beta)$ with $|\beta| > |\alpha|$.

Let Δ be a finite subset of a discrete group Γ such that $1 \notin \Delta$ and $\Delta^{-1} = \Delta$. The directional picture of the Cayley graph associated to (Γ, Δ) is given by $\mathfrak{v} = \Gamma$, $\mathfrak{e} = \Gamma \times \Delta$, $e(\alpha, \gamma) = (\alpha, \alpha\gamma)$ and $\theta(\alpha, \gamma) = (\alpha\gamma, \gamma^{-1})$. Its simplicial realization will be called the simplicial picture of the Cayley graph.

2. Complements on fusion morphisms

In [4], [5] a full description of the involutive semi-ring structure of the corepresentation theory of $A_u(Q)$ and $A_o(Q)$ was given by means of the fusion and conjugation rules on the set of irreducible objects up to equivalence. In this section we choose concrete representatives for the irreducible objects and compute explicitly isometric morphisms realizing the "basic" fusion rules. This section is a technical one and its results are only used in Sections 6 and 7: we advise the reader interested in quantum Cayley graphs to skip to the next section.

In the case of $A_u(Q)$, with $Q \in \mathrm{GL}_n(\mathbb{C})$ and $n \geqq 2$, let us choose $\gamma = u$ or $\bar{u}$, and put $\gamma_{2l} = \bar{\gamma}$, $\gamma_{2l+1} = \gamma$. We will mainly be interested in the corepresentations $\alpha_k = \gamma\bar{\gamma}\gamma \cdots \gamma_k$ (k terms) and $\alpha_{k,k'} = \gamma \cdots \gamma_k \otimes \gamma_{k+1} \cdots \gamma_{k+k'}$. As a matter of fact, the fusion rules of $A_u(Q)$ reduce to the relations $\alpha_{k+1,k'+1} = \alpha_{k+k'+2} \oplus \alpha_{k,k'}$ and trivial tensor products. In the orthogonal case, we will also put $\gamma_k = \gamma = \alpha_1$ for every $k \in \mathbb{N}$ and $\alpha_{k,k'} = \alpha_k \otimes \alpha_{k'}$, to simplify the exposition.

Let us now choose concrete corepresentation spaces H_k and $\overline{H}_k$ for the classes $\alpha_k, \bar{\alpha}_k$. We first take $H_0 = \mathbb{C}$, equipped with the corepresentation $1_{\mathbb{C}} \otimes 1_S$, and $H_\gamma = H_{\bar{\gamma}} = \mathbb{C}^n$, equipped with the corepresentations U or $\overline{U}$. For any $k \in \mathbb{N}^*$ we denote by $H_\gamma^{\otimes k}$ the tensor product corepresentation $H_\gamma \otimes H_{\bar{\gamma}} \otimes \cdots \otimes H_{\gamma_k}$, and we define H_k to be its unique sub-corepresentation equivalent to α_k. We proceed in the same way inside $H_{\bar{\gamma}_k}^{\otimes k}$ and $H_\gamma^{\otimes k} \otimes H_{\bar{\gamma}_k}^{\otimes k'}$ to get corepresentation spaces $\overline{H}_k$ and $H_{k,k'}$ representing $\bar{\alpha}_k$ and $\alpha_{k,k'}$. We will denote by $t_k, \bar{t}_k$ and t_δ the morphisms associated to normalized conjugation maps of $\alpha_k, \bar{\alpha}_k$ and $\delta \in \{\gamma, \bar{\gamma}\}$ respectively—we can and will assume in this section that $j_{\bar{\gamma}} j_\gamma = \pm 1$, and we denote by ∓ 1 the opposite sign. We put $m_k = M_{\alpha_k} = M_{\bar{\alpha}_k}$ and we call $(m_k)_k$ the sequence of quantum dimensions of the quantum group. Let us gather simple facts about them in the following lemma:

Lemma 2.1. (1) *Denote by* $T_l : H_\gamma^{\otimes k} \to H_\gamma^{\otimes k+2}$ *the morphism* $\mathrm{id}^{\otimes l} \otimes t_{\gamma_{l+1}} \otimes \mathrm{id}^{\otimes k-l}$. *We have then* $H_{k+2} = \bigcap_{l=0}^{k} \mathrm{Ker}\, T_l^* \subset H_\gamma^{\otimes k+2}$.

(2) *We have* $(\mathrm{id} \otimes t_{\bar{\delta}}^*)(t_\delta \otimes \mathrm{id}) = \pm \mathrm{id}_{H_\delta}$ *and* $t_\delta^* t_\delta = m_1 \, \mathrm{id}_{\mathbb{C}}$ *for* $\delta \in \{\gamma, \bar{\gamma}\}$.

(3) *For any* $k \in \mathbb{N}$ *we have* $m_k \geqq \dim H_k$, *with equality iff* F_k *is the identity. Moreover the equality* $m_1 = 2$ *happens only in the three cases*

$$A_o\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_o\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_u\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

up to isomorphism.

(4) *Put* $m_{-1} = 0$. *The sequence of quantum dimensions satisfies the induction equations* $m_1 m_k = m_{k+1} + m_{k-1}$ *for* $k \in \mathbb{N}$. *Moreover* $m_0 = 1$ *and* m_1 *is the geometric mean of* $\mathrm{Tr}\, Q^*Q$ *and* $\mathrm{Tr}(Q^*Q)^{-1}$.

Proof. The equality of Point (1) is true when $k = 2$ because $H_\gamma \otimes H_{\bar{\gamma}}$ is the orthogonal direct sum of $t_\gamma(\mathbb{C})$ and a subspace equivalent to α_2, and the general result follows by induction because $H_{k+1} = H_{1,k} \cap H_{k,1}$. The proof of Point (2) is an easy calculation. For Point (3), denote by a (resp. h) the arithmetic (resp. harmonic) mean of the eigenvalues of F_k: the normalization condition of j_k shows that $a = h^{-1}$ so that

$$m_k = a \dim H_k = \sqrt{a/h} \dim H_k \geqq \dim H_k.$$

For the equality case $m_1 = 2$, see [5]. The induction equation of Point (4) relies on the fusion rule $\alpha_k \otimes \alpha_1 = \alpha_{k-1} \oplus \alpha_{k+1}$ which implies that $\Sigma(j_k \otimes j_1)$ and $j_{k-1} \oplus j_{k+1}$ are normalized conjugation maps for the same corepresentation [21]. Finally, the formula for m_1 holds because the matrix Q defines in the canonical base of $\mathbb{C}^n$ a (non-normalized) conjugation map for H_1, by definition of $A_u(Q)$ [5]. □

We want now to give the explicit expression of an isometric morphism from $H_{p,p'}$ to $H_{p+1,p'+1}$, for any $p, p' \in \mathbb{N}$. Note that there is an evident morphism $\mathscr{T} : H_{p,p'} \to H_{p+1,p'+1}$ given by the formula

$$\mathscr{T} = (\pi_{p+1} \otimes \pi_{p'+1}) \circ (\mathrm{id}_{H_p} \otimes t_{\gamma_{p+1}} \otimes \mathrm{id}_{H_p'}),$$

where π_k denotes the orthogonal projection of $H_\gamma^{\otimes k}$ onto H_k. However $\mathscr{T}$ is not isometric, and its definition does not allow to compute easily the image of a vector $x \in H_{p,p'}$. In Proposition 2.2 we give an explicit and simple expression of $\mathscr{T}$, which allows us to compute its polar decomposition in Proposition 2.3. From this we finally deduce Lemmas 2.4 and 2.5 which will be used in the proofs of Lemmas 6.3 and 7.1 respectively.

We use more precisely the following "basic morphisms" from $H_\gamma^{\otimes p+p'}$ to $H_\gamma^{\otimes p+p'+2}$, indexed by $l \in [\![0, p]\!]$ and $l' \in [\![0, p']\!]$:

$$T_{l,l'} = (\mathrm{id}^{\otimes p+1} \otimes t_{\gamma_{p+2}}^* \otimes \mathrm{id}^{\otimes p'+1}) \circ (\mathrm{id}^{\otimes p-l} \otimes t_{\gamma_{p-l+1}} \otimes \mathrm{id}^{\otimes l+l'} \otimes t_{\gamma_{p+l'+1}} \otimes \mathrm{id}^{\otimes p'-l'}).$$

If $A = (a_{l,l'})$ is a $(p+1) \times (p'+1)$ matrix, we will write $T_A = \sum a_{l,l'} T_{l,l'}$. Besides, we have by Lemma 2.1 a simpler expression of $T_{l,l'}$ when l or l' equals zero:

$$T_{l,0} = \pm(\mathrm{id}^{\otimes p-l} \otimes t_{\gamma_{p-l+1}} \otimes \mathrm{id}^{\otimes p'+l}),$$
$$T_{0,l'} = \pm(\mathrm{id}^{\otimes p+l'} \otimes t_{\gamma_{p+l'+1}} \otimes \mathrm{id}^{\otimes p'-l'}).$$

Proposition 2.2. (1) *There is at most one matrix A, up to a scalar factor, such that T_A restricts to a non-zero morphism from $H_{p,p'}$ to $H_{p+1,p'+1}$. If this is the case, one can assume that $a_{0,0} = 1$ and one has then $T_A = \mathscr{T}$.*

(2) *The following matrix A satisfies the conditions of Point* (1):

$$a_{l,l'} = (\mp 1)^{l+l'} \frac{m_{p-l} m_{p'-l'}}{m_p m_{p'}}.$$

Proof. (1) It is not hard to check that the family $(T_{l,l'})$ is free, even when restricted to $H_{p,p'}$. Hence it suffices to prove that an admissible T_A is necessarily a multiple of $\mathscr{T}$. First of all, Point (1) of Lemma 2.1 shows that we have $\big(y \mid T_{l,l'}(x)\big) = 0$ for any $y \in H_{p+1,p'+1}$ and $(l,l') \neq (0,0)$. Hence if $T_A(x) \in H_{p+1,p'+1}$ we obtain

$$\|T_A(x)\|^2 = a_{0,0}\big(T_A(x) \mid T_{0,0}(x)\big) = a_{0,0}\big(T_A(x) \mid \mathscr{T}(x)\big). \tag{7}$$

In particular $a_{0,0}$ must be non-zero, and therefore we can assume that it equals 1. To conclude we observe that the irreducible subspaces of $H_{p,p'}$ (resp. $H_{p+1,p'+1}$) are pairwise inequivalent, so that the morphisms T_A and $\mathscr{T}$ must be proportional on each irreducible subspace of $H_{p,p'}$, and (7) finally shows that the corresponding proportionality coefficients all equal 1.

(2) We will express the condition that $T_A(x)$ should be in $H_{p+1,p'+1}$ for any $x \in H_{p,p'}$ using Point (1) of Lemma 2.1: for any $k \in [\![1,p]\!]$ and $k' \in [\![1,p']\!]$ we should have $T_{k,0}^* T_A(x) = T_{0,k'}^* T_A(x) = 0$. We therefore compute, for $l \in [\![0,p]\!]$ and $l' \in [\![0,p']\!]$:

$$\begin{aligned}
\pm T_{k,0}^* T_{l,l'}(x) &= (\mathrm{id}^{\otimes p-k} \otimes t_{\gamma_{p-k+1}}^* \otimes \mathrm{id}^{\otimes k-1} \otimes t_{\gamma_p}^* \otimes \mathrm{id}^{\otimes p'+1}) \\
&\quad \circ (\mathrm{id}^{\otimes p-l} \otimes t_{\gamma_{p-l+1}} \otimes \mathrm{id}^{\otimes l+l'} \otimes t_{\gamma_{p+l'+1}} \otimes \mathrm{id}^{\otimes p'-l'})(x) \\
&= \begin{cases} 0 & \text{if } k \leqq l-2 \text{ or } k \geqq l+2, \\ \pm T_{1,0}^* T_{0,l'}(x) & \text{if } k = l-1 \text{ or } l+1, \\ m_1 T_{1,0}^* T_{0,l'}(x) & \text{if } k = l \text{ (see Lemma 2.1).} \end{cases}
\end{aligned}$$

As a result, we get the following sufficient conditions on A:

$$\forall k \in [\![1,p]\!],\ l' \in [\![0,p']\!] \quad m_1 a_{k,l'} \pm (a_{k-1,l'} + a_{k+1,l'}) = 0,$$

if one agrees to put $a_{p+1,l'} = 0$. We recognize the induction equations satisfied by the sequence $(m_{p-i})_{0 \leqq i \leqq p+1}$, up to a sign change. Therefore these conditions mean that the columns of A should be proportional to $\big((\mp 1)^p m_p, \dots, \mp m_1, 1\big)$. Symmetrically the conditions $T_{0,k'}^* T_A(x) = 0$ are equivalent to the lines of A being proportional to $\big((\mp 1)^{p'} m_{p'}, \dots, \mp m_1, 1\big)$. The matrix of the statement satisfies these conditions, hence the associated morphism T_A maps $H_{p,p'}$ to $H_{p+1,p'+1}$. □

Proposition 2.3. *Let $q \in [\![0, \min(p, p')]\!]$ and denote by $G \subset H_{p,p'}$ the subspace equivalent to $\alpha_{p+p'-2q}$. One has then*

$$\|\mathcal{T}_{|G}\|^2 = \frac{m_{p+1}m_{p'} - m_{p-q}m_{p'-q-1}}{m_p m_{p'}}.$$

Proof. Let $z \in G$ be a unit vector. Because G is irreducible and $\mathcal{T}$ is a morphism, it is enough to compute the number $N^q_{p,p'} = \|\mathcal{T}(z)\|^2$. Of course we will use the expression $\mathcal{T} = T_A$ of Proposition 2.2. We start from the formula (7) and notice that $T_{l,l'}(z)$ is orthogonal to $T_{0,0}(z) \in H_p \otimes H_{\bar{\gamma}_p} \otimes H_{\gamma_p} \otimes H_{p'}$ whenever $l \geqq 1$ or $l' \geqq 1$. Hence

$$\|T_A(z)\|^2 = \big(T_A(z) \mid T_{0,0}(z)\big) = \sum_{l,l'=0,1} a_{l,l'}\big(T_{l,l'}(z) \mid T_{0,0}(z)\big).$$

When l or l' equals zero, we can use the formulae for $T^*_{l,l'}T_{0,0}(z)$ obtained in the proof of Proposition 2.2. The term $l = l' = 1$ will be a recursive one. Let us denote by $T'_{k,k'}$ and $\mathcal{T}' = (\pi_p \otimes \pi_{p'}) \circ T'_{0,0}$ the morphisms analogous to $T_{k,k'}$ and $\mathcal{T}$ for the inclusion $H_{p-1,p'-1} \to H_{p,p'}$. We remark that

$$T^*_{0,0}T_{1,1} = \pm T^*_{0,0}(\mathrm{id}^{\otimes p-1} \otimes t_{\gamma_p} \otimes t_{\gamma_p} \otimes \mathrm{id}^{\otimes p'-1})T'^*_{0,0} = T'_{0,0}T'^*_{0,0},$$

so that $\big(T_{1,1}(z) \mid T_{0,0}(z)\big) = \|T'^*_{0,0}(z)\|^2 = \|\mathcal{T}'^*(z)\|^2$. Putting all together, we get the relation

$$N^q_{p,p'} = m_1 - \frac{m_{p-1}}{m_p} - \frac{m_{p'-1}}{m_{p'}} + \frac{m_{p-1}m_{p'-1}}{m_p m_{p'}} N^{q-1}_{p-1,p'-1}$$

$$\Leftrightarrow m_{p'}(m_p N^q_{p,p'} - m_{p+1}) = m_{p'-1}(m_{p-1}N^{q-1}_{p-1,p'-1} - m_p).$$

Hence the left-hand side quantity is invariant under simultaneous shifts of the three indices p, p' and q. Note that the above relation is still valid when $q = 0$ if one puts $N^{-1}_{k,k'} = 0$ for any k and k': as a matter of fact, in this case z lies in $H_{p+p'}$ and in particular $\mathcal{T}'^*(z) = 0$. One can therefore shift $q+1$ times the indices and obtain the desired identity:

$$m_{p'}(m_p N^q_{p,p'} - m_{p+1}) = -m_{p'-q-1}m_{p-q}. \quad \square$$

Lemma 2.4. *Let $H_{k-1,1,k} \subset H_\gamma^{\otimes k-1} \otimes H_{\gamma_k} \otimes H_{\bar{\gamma}_k}^{\otimes k}$ be the tensor product of the respective subspaces equivalent to α_{k-1}, γ_k and $\bar{\alpha}_k$, with $k \in \mathbb{N}^*$. Let $t \in \mathrm{Mor}(H_{k-2}, H_{k-1,1})$ be an injection and denote by*

— *G_1 the subspace of $(t \otimes \mathrm{id})(H_{k-2,k}) \subset H_{k-1,1,k}$ equivalent to α_{2l},*

— *G_2 the subspace of $H_{k-1,k+1} \subset H_{k-1,1,k}$ equivalent to α_{2l}.*

Then the norm of the orthogonal projection from G_1 onto G_2 equals $\sqrt{1 - \dfrac{m_l m_{l-1}}{m_k m_{k-1}}}$.

Proof. Let $A = (a_{l,l'})$ and $\mathcal{T} = T_A$ be the matrix and the morphism of Proposition 2.2 in the case $(p, p') = (k-2, k)$. We denote by $A' = (a'_{l,l'})$ the matrix given by $a'_{l,0} = a_{l,0}$

and $a'_{l,l'} = 0$ if $l' \geqq 1$, and we remark that we have then $T_{A'} = (\mathscr{T}' \otimes \mathrm{id})$, where $\mathscr{T}'$ is the morphism of Proposition 2.2 for $(p, p') = (k-2, 0)$. Hence if $x \in H_\gamma^{\otimes 2k-2}$ is a vector in the subspace of $H_{k-2,k}$ equivalent to α_{2l}, we have $T_{A'}(x) \in G_1$ and $T_A(x) \in G_2$. The orthogonal projection of G_1 onto G_2 being a morphism, it is a multiple of an isometry, so that its norm equals

$$\frac{|(T_{A'}(x) \mid T_A(x))|}{\|T_{A'}(x)\| \, \|T_A(x)\|} = \frac{\|T_A(x)\|^2}{\|T_{A'}(x)\| \, \|T_A(x)\|} = \frac{\|\mathscr{T}(x)\|}{\|(\mathscr{T}' \otimes \mathrm{id})(x)\|},$$

because the terms $T_{l,l'}(x)$ with $l' \geqq 1$ are orthogonal to $T_A(x)$. We finally compute the value of the last quotient thanks to Proposition 2.3, with $(p, p', q) = (k-2, k, k-1-l)$ and $(p, p', q) = (k-2, 0, 0)$:

$$\frac{\|\mathscr{T}(x)\|^2}{\|(\mathscr{T}' \otimes \mathrm{id})(x)\|^2} = \frac{m_{k-1}m_k - m_{l-1}m_l}{m_{k-2}m_k} \frac{m_{k-2}}{m_{k-1}} = 1 - \frac{m_l m_{l-1}}{m_k m_{k-1}}. \quad \square$$

Lemma 2.5. *Let $H_{1,k,k'} \subset H_\gamma \otimes H_{\bar\gamma}^{\otimes k} \otimes H_{\gamma_k}^{\otimes k'}$ be the tensor product of the respective subspaces equivalent to $\gamma, \bar\gamma\gamma \cdots \bar\gamma_k$ (k terms) and $\gamma_k\gamma_{k+1} \cdots \gamma_{k+k'-1}$ (k' terms), with $k, k' \in \mathbb{N}^*$. Let $t \in \mathrm{Mor}(H_{k-1}, H_1 \otimes H_k)$ be an injection and denote by*

— *G_1 the subspace of $(t \otimes \mathrm{id})(H_{k-1,k'}) \subset H_{1,k,k'}$ equivalent to $\alpha_{k+k'-1}$,*

— *G_2 the subspace of $H_{1,k+k'} \subset H_{1,k,k'}$ equivalent to $\alpha_{k+k'-1}$.*

Then the norm of the orthogonal projection from G_1 to G_2 equals $\sqrt{1 - \frac{m_{k'-1}}{m_{k+k'-1}m_k}}$.

Proof. Like in the previous proof we will use the morphisms $\mathscr{T} : H_{0,k+k'-1} \to H_{1,k+k'}$ and $\mathscr{T}' : H_{0,k-1} \to H_{1,k}$ studied in Proposition 2.2. We notice that G_1 (resp. G_2) is the image of $H_{k+k'-1}$ by $(\mathscr{T}' \otimes \mathrm{id})$ (resp. $\mathscr{T}$), so that the norm of the projection we are interested in is given by

$$\frac{|(\mathscr{T}(x) \mid (\mathscr{T}' \otimes \mathrm{id})(x))|}{\|\mathscr{T}(x)\| \, \|(\mathscr{T}' \otimes \mathrm{id})(x)\|} = \frac{\|\mathscr{T}(x)\|}{\|(\mathscr{T}' \otimes \mathrm{id})(x)\|},$$

for the same reason as above. Proposition 2.3 with $(p, p', q) = (0, k+k'-1, 0)$ and $(0, k-1, 0)$ gives then

$$\frac{\|\mathscr{T}(x)\|^2}{\|(\mathscr{T}' \otimes \mathrm{id})(x)\|^2} = \frac{(m_1 m_{k+k'-1} - m_{k+k'-2})m_{k-1}}{m_{k+k'-1}(m_1 m_{k-1} - m_{k-2})} = \frac{m_{k+k'}m_{k-1}}{m_{k+k'-1}m_k}.$$

The result follows then from the identity $m_{k+k'-1}m_k = m_{k+k'}m_{k-1} + m_{k'-1}$, which is easy to prove by induction, or by noticing that the irreducible subobjects of $H_{k+k'-1,k}$ are the same as for $H_{k+k',k-1}$, up to the one equivalent to $H_{k'-1}$. $\square$

3. Quantum Cayley graphs

In this section we introduce the notion of Cayley graph for discrete quantum groups. In fact the classical notion can be generalized into two different directions, coming from the two different pictures introduced in Section 1. The quantum generalization of the simplicial picture is still a classical graph. On the contrary, the ℓ^2-spaces of the directional picture give rise in the quantum case to a quantum object, in the spirit of non-commutative geometry.

In the following definition, we use freely the notation of Section 1. In particular, S and $\hat{S}$ are the dual Hopf C^*-algebras of a compact quantum group—S being unital—, H is the GNS space of the Haar state of S, p_α is the minimal central projection of $\hat{S}$ corresponding to an irreducible corepresentation $\alpha \in \operatorname{Irr}\mathscr{C}$ and $p_0 = p_{1_\mathscr{C}}$.

Definition 3.1. Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$.

(1) The classical Cayley graph $\mathfrak{g}$ associated with (S, p_1) is given in simplicial form by $\mathfrak{v} = \operatorname{Irr}\mathscr{C}$ and $\mathfrak{e} = \{(\alpha, \alpha') \in \mathfrak{v}^2 \mid \hat{\delta}(p_{\alpha'})(p_\alpha \otimes p_1) \neq 0\}$.

(2) The hilbertian quantum Cayley graph associated with (S, p_1) is the 4-uplet (H, K, E, Θ) where $K = H \otimes p_1H$, $E = V_{|K} \in B(K, H \otimes H)$ and $\Theta = \tilde{V}(1 \otimes U)_{|K} \in B(K)$.

Let us introduce some more objects associated with this quantum graph. We denote by ϵ the linear form on p_1H defined by $\epsilon\big(\Lambda_h(x)\big) = \Lambda_h\big(\varepsilon(x)\big)$.

(3) The source and target operators of the hilbertian quantum Cayley graph are $E_1 = (\mathrm{id} \otimes \epsilon)$ and $E_2 = E_1 \circ \Theta \in B(K, H)$.

(4) The quantum ℓ^2-space of geometric edges is $K_g = \operatorname{Ker}(\Theta + \mathrm{id})$.

Remarks 3.2. (1) The central projections p_1 that match the hypotheses of Definition 3.1 are sums of projections p_α over finite subsets $\mathscr{D} \subset \operatorname{Irr}\mathscr{C}$ such that $\overline{\mathscr{D}} = \mathscr{D}$ and $1_\mathscr{C} \notin \mathscr{D}$. The elements of $\mathfrak{e}$ are then the ordered pairs of vertices (α, α') for which there exist $\gamma \in \mathscr{D}$ such that $\alpha' \subset \alpha \otimes \gamma$. Note that this set of edges is symmetric, thanks to the equivalence $\alpha' \subset \alpha \otimes \gamma \Leftrightarrow \alpha \subset \alpha' \otimes \bar{\gamma}$ (Jacobi duality). In this paper, the classical Cayley graph will mainly be used as a tool for the study of the quantum one.

(2) The hilbertian quantum Cayley graph will be more useful for our purposes because he naturally carries representations of the discrete quantum group under consideration: the C^*-algebra S acts on H via the GNS representation, and we let it act trivially on p_1H. Moreover the operators Θ, E_1 and E_2 commute to these representations, and in particular K_g is also endowed with a natural representation of S. The commutation properties to the action of $\hat{S}$ will be examined in Proposition 3.7.

(3) The identity $\hat{V}V\tilde{V} = (U \otimes 1)\Sigma$ provides us with another expression for the reversing operator: $\Theta = (\Sigma\hat{V}V)^*$. Moreover the identity $\tilde{V}^* = (\hat{J} \otimes J)\tilde{V}(\hat{J} \otimes J)$, where $J, \hat{J}$ are the modular conjugations of S and $\hat{S}$ [10], shows that $(\hat{J} \otimes \hat{J})\Theta(\hat{J} \otimes \hat{J}) = \Theta^* = \Theta^{-1}$. But the main fact about the reversing operator is its non-involutivity in the quantum case, see Proposition 3.4—in fact in this proposition it is enough to consider the restriction of $\tilde{V}(1 \otimes U)$ to $H \otimes p_1H$, as soon as $\mathscr{D}$ generates $\mathscr{C}$. □

Example 3.3 (classical case). Suppose $S = C^*(\Gamma)$ for some discrete group Γ, with the co-commutative coproduct given by $\delta(\gamma) = \gamma \otimes \gamma$ for all $\gamma \in \Gamma \subset C^*(\Gamma)$. Then $\operatorname{Irr}\mathscr{C}$

identifies with Γ in such a way that $v_\gamma \simeq \mathrm{id}_{\mathbb{C}} \otimes \gamma$ for every $\gamma \in \Gamma$, and the tensor product (resp. the conjugation) of corepresentations then coincides with the product (resp. the inverse) of Γ. In particular, the inclusion $\alpha' \subset \alpha \otimes \gamma$ reduces in this case to an equality $\alpha' = \alpha\gamma$, so that the classical graph of Definition 3.1 is nothing but the simplicial picture of the Cayley graph associated to (Γ, Δ), with $\Delta = \mathscr{D}$.

Besides, one has $H = \ell^2(\Gamma)$, $S_{\mathrm{red}} = C^*_{\mathrm{red}}(\Gamma)$, $\hat{S} = c_0(\Gamma)$ and the projections p_α correspond to the characteristic functions $\mathbb{1}_\alpha \in \hat{S}$ of the points of Γ. Moreover one has the following expressions for the Kac system of (S, δ): $V(\mathbb{1}_\alpha \otimes \mathbb{1}_\beta) = \mathbb{1}_\alpha \otimes \mathbb{1}_{\alpha\beta}$ and $U(\mathbb{1}_\alpha) = \mathbb{1}_{\alpha^{-1}}$. From this it is easy to see that the hilbertian quantum graph of Definition 3.1 is nothing but the ℓ^2-object associated to the directional picture $(\mathfrak{v}, \mathfrak{e}, e, \theta)$ of the Cayley graph of (Γ, Δ). The only non-trivial check concerns the reversing operator: according to Proposition 3.4, one has $\Sigma \hat{V} V = V^* \Sigma V = E^* \Sigma E$ so that $\Theta^* = \Theta$ is the classical reversing operator. □

Proposition 3.4. *Let (H, V, U) be an irreducible Kac system* [3]. *Then the multiplicative unitary V is co-commutative iff $\hat{V} = \Sigma V^* \Sigma$ iff $\tilde{V}(1 \otimes U)$ is involutive.*

Proof. The direct implications are easy to check in the underlying locally compact groups. Conversely, assume that $\hat{V} = \Sigma V^* \Sigma$. Then for any $x = (\mathrm{id} \otimes \omega)(V) \in \hat{S}_{\mathrm{red}}$, one also has $x = U(\mathrm{id} \otimes \omega)(V^*)U \in U\hat{S}_{\mathrm{red}}U \subset \hat{S}'_{\mathrm{red}}$, hence $\hat{S}_{\mathrm{red}}$ is commutative. Replacing V by $\tilde{V}$ one gets the dual version of this result: if $\tilde{V} = \Sigma V^* \Sigma$, then V is commutative. Now, $\tilde{V}(1 \otimes U)$ is involutive iff $\tilde{V}(1 \otimes U) = (1 \otimes U)\tilde{V}^*$ iff $\Sigma(1 \otimes U)\tilde{V}(1 \otimes U)\Sigma = \Sigma \tilde{V}^* \Sigma$, which implies by the previous "dual" statement that $\tilde{V}$ is commutative, hence V is co-commutative. □

Let us give now alternative expressions for the reversing, source and target operators in terms of the Hopf $*$-algebra structure of $\mathscr{S}$, and study the intertwining properties of these operators relatively to the representations of the dual Hopf C^*-algebra $\hat{S}$.

Lemma 3.5. *Let $x, y \in \mathscr{S} \subset S_{\mathrm{red}}$, we have*

$$\tilde{V}(1 \otimes U) \circ (\Lambda_h \otimes \Lambda_h)(x \otimes y) = (\Lambda_h \otimes \Lambda_h) \circ (\mathrm{id} \otimes \kappa)\big((x \otimes 1)\delta(y)\big).$$

Proof. In this proof we will write Θ in place of $\tilde{V}(1 \otimes U)$, although we do not necessarily restrict ourselves to K. We have

$$\Theta \circ (\Lambda_h \otimes \Lambda_h)(x \otimes y) = (x \otimes \mathrm{id}) \circ \Theta \circ (\Lambda_h \otimes \Lambda_h)(1 \otimes y),$$

so that it suffices to consider the case when $x = 1$. Let us use the expressions (3) and (4) of U and V:

$$\begin{aligned}\Theta \circ (\Lambda_h \otimes \Lambda_h)(1 \otimes y) &= \Sigma(1 \otimes U)V(1 \otimes U) \circ (\Lambda_h \otimes \Lambda_h)\big(f_1 \star \kappa(y) \otimes 1\big) \\ &= \Sigma(1 \otimes U) \circ (\Lambda_h \otimes \Lambda_h) \circ \delta\big(f_1 \star \kappa(y)\big).\end{aligned}$$

It is easy to check that $\delta(f_z \star a) = \big(\mathrm{id} \otimes (f_z \star)\big)\big(\delta(a)\big)$, hence

$$\begin{aligned}\Theta \circ (\Lambda_h \otimes \Lambda_h)(1 \otimes y) &= \Sigma(1 \otimes U) \circ (\Lambda_h \otimes \Lambda_h)\big(\mathrm{id} \otimes (f_1 \star)\big)\big(\delta(\kappa(y))\big) \\ &= \Sigma(1 \otimes U) \circ (\Lambda_h \otimes \Lambda_h)\big(\kappa \otimes (f_1 \star)\kappa\big)\sigma\delta(y).\end{aligned}$$

One recognizes then $(1 \otimes U)^2 = 1 \otimes 1$:

$$\Theta \circ (\Lambda_h \otimes \Lambda_h)(x \otimes y) = \Sigma \circ (\Lambda_h \otimes \Lambda_h) \circ (\kappa \otimes \mathrm{id})\sigma\delta(y)$$
$$= (\Lambda_h \otimes \Lambda_h) \circ (\mathrm{id} \otimes \kappa)\delta(y). \quad \square$$

Proposition 3.6. *Let ϵ be the linear form on $\Lambda_h(\mathscr{S})$ defined by $\epsilon \circ \Lambda_h = \Lambda_h \circ \varepsilon$. We have the following identities*:

(1) *$E_1 = (\mathrm{id} \otimes \epsilon) \circ V$ and $E_2 = (\epsilon \otimes \mathrm{id}) \circ V$ on $K \cap (\Lambda_h \otimes \Lambda_h)(\mathscr{S} \otimes \mathscr{S})$.*

(2) *$E_2 \circ (\Lambda_h \otimes \Lambda_h)(x \otimes y) = \Lambda_h(xy)$ for $x \otimes y \in \mathscr{S} \otimes \mathscr{S}$, and $E_2 \circ \Theta = E_1$.*

Proof. The co-unit ε being multiplicative, we have for all x and y in $\mathscr{S}$

$$(\mathrm{id} \otimes \varepsilon)\big(\delta(x)(1 \otimes y)\big) = \varepsilon(y)(\mathrm{id} \otimes \varepsilon)\delta(x) = \varepsilon(y)x = (\mathrm{id} \otimes \varepsilon)(x \otimes y),$$

hence $E_1 = (\mathrm{id} \otimes \epsilon) \circ V$. In the same way one can write, using the identity $\varepsilon \circ \kappa = \varepsilon$ and Equation (1):

$$(\mathrm{id} \otimes \varepsilon)(\mathrm{id} \otimes \kappa)\big((x \otimes 1)\delta(y)\big) = (\mathrm{id} \otimes \varepsilon)\big((x \otimes 1)\delta(y)\big)$$
$$= xy = (\varepsilon \otimes \mathrm{id})\big(\delta(x)(1 \otimes y)\big),$$

which yields $E_2 \circ (\Lambda_h \otimes \Lambda_h)(x \otimes y) = \Lambda_h(xy)$ and $E_2 = (\epsilon \otimes \mathrm{id}) \circ V$, thanks to the definition of E_2 and the expression of Θ given by Lemma 3.5. Now, using these results and Equation (2), we can proceed to the last computation, where $m : \mathscr{S} \otimes \mathscr{S} \to \mathscr{S}$ denotes the multiplication of $\mathscr{S}$:

$$E_2 \circ \Theta \circ (\Lambda_h \otimes \Lambda_h)(x \otimes y) = \Lambda_h\big(x\big(m(\mathrm{id} \otimes \kappa)\delta(y)\big)\big)$$
$$= \varepsilon(y)\Lambda_h(x) = E_1 \circ (\Lambda_h \otimes \Lambda_h)(x \otimes y). \quad \square$$

Proposition 3.7. *Let us define $\hat{\pi}_2 : \hat{S}^{\otimes 2} \to L(H)$ by the formula $\hat{\pi}_2(x \otimes x') = x(Ux'U)$. Similarly, let us denote by $\hat{\pi}_4 : \hat{S}^{\otimes 4} \to L(K)$ the homomorphism such that $\hat{\pi}_4(x \otimes y \otimes y' \otimes x') = (x \otimes y)(Ux'U \otimes Uy'U)$, and let us put $\hat{\delta}' = \hat{\pi}_4 \circ (1 \otimes 1 \otimes \hat{\delta})$, so that $\hat{\delta}'(x) = (U \otimes U)\Sigma\hat{\delta}(x)\Sigma(U \otimes U)$. One has then, for any $x \in \hat{S}$:*

(1) *$\Theta \circ (x \otimes 1) = \hat{\delta}(x) \circ \Theta$,*

(2) *$\Theta \circ (1 \otimes x) = (1 \otimes UxU) \circ \Theta$,*

(3) *$\Theta \circ \hat{\delta}'(x) = (UxU \otimes 1) \circ \Theta$,*

(4) *$E_2 \circ \hat{\delta}(x) = x \circ E_2$ and $E_2 \circ \hat{\delta}'(x) = UxU \circ E_2$.*

Hence Θ intertwines the representations $\hat{\pi}_4 \circ (\mathrm{id} \otimes \mathrm{id} \otimes \hat{\delta})$ and $\hat{\pi}_4 \circ (\hat{\delta} \otimes \mathrm{id} \otimes \mathrm{id})$ of $\hat{S} \otimes \hat{S} \otimes \hat{S}$ on K. In particular Θ commutes to $\hat{\pi}_4 \circ \hat{\delta}^3$. Similarly, E_2 intertwines the representations $\hat{\pi}_2$ and $\hat{\pi}_4 \circ (\hat{\delta} \otimes \hat{\delta})$ of $\hat{S} \otimes \hat{S}$.

Proof. Point (1) results from the identity $\hat{\delta}(x) = \tilde{V}(x \otimes 1)\tilde{V}^*$. Writing $E_2 = (\mathrm{id} \otimes \epsilon) \circ \Theta^{-1}$, it implies the first relation of Point (4). For Point (3), one uses the formula $\Theta = (\Sigma \hat{V} V)^*$ and the fact that V commutes to $U\hat{S}U \otimes 1$:

$$\begin{aligned}(UxU \otimes 1)\Theta &= (UxU \otimes 1)V^*\hat{V}^*\Sigma = V^*(UxU \otimes 1)\hat{V}^*\Sigma \\ &= V^*\hat{V}^*(U \otimes U)\tilde{V}(x \otimes 1)\tilde{V}^*(U \otimes U)\Sigma \\ &= V^*\hat{V}^*(U \otimes U)\hat{\delta}(x)(U \otimes U)\Sigma = V^*\hat{V}^*\Sigma\hat{\delta}'(x) = \Theta\hat{\delta}'(x).\end{aligned}$$

Composing on the left by $(\mathrm{id} \otimes \epsilon)$, one obtains the second relation of Point (4). For Point (2), simply notice that $\tilde{V}$ is in $M(USU \otimes \hat{S})$, and hence commutes to $1 \otimes U\hat{S}U$. □

4. Ascending orientation

In the case of a classical tree, the ascending orientation associated to a chosen origin defines a subspace K_+ of the ℓ^2-space of edges K. The aim of this section is to introduce and study such a subspace in the case of quantum Cayley graphs. The next definition relies on the links between the quantum and classical Cayley graphs, the latter one being endowed with the origin $1_{\mathscr{C}}$.

Definition 4.1. Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0 p_1 = 0$. Assume that the classical Cayley graph associated with (S, p_1) is a tree, and denote by $|\cdot|$ the distance to the origin $1_{\mathscr{C}}$ in this tree.

(1) For any $n \in \mathbb{N} \backslash \{0, 1\}$ we put $p_n = \sum \{p_\alpha \,|\, |\alpha| = n\} \in Z(\hat{S})$.

(2) We call $p_{\star+} = \sum (p_n \otimes p_1)\hat{\delta}(p_{n+1})$ and $p_{+\star} = \sum (p_n \otimes p_1)\hat{\delta}'(p_{n+1})$ the left and right ascending projections. Put $p_{\star-} = 1 - p_{\star+}$, $p_{-\star} = 1 - p_{+\star}$.

(3) We call $p_{++} = p_{+\star}p_{\star+}$ the ascending projection of the quantum Cayley tree, and we denote by K_{++} its image. We define similarly

$$p_{+-} = p_{+\star}p_{\star-}, \quad p_{-+} = p_{-\star}p_{\star+} \quad \text{and} \quad p_{--} = p_{-\star}p_{\star-},$$

$$K_{+-} = p_{+-}K, \quad K_{-+} = p_{-+}K \quad \text{and} \quad K_{--} = p_{--}K.$$

Remarks 4.2. (1) We have $|\alpha| = 1 \Leftrightarrow \alpha \in \mathscr{D}$ and $|\alpha| = 0 \Leftrightarrow \alpha = 1_{\mathscr{C}}$. In particular the first point of Definition 4.1 is consistent with the notation p_0 and p_1 used in Definition 3.1.

(2) Take $\alpha \in \mathrm{Irr}\,\mathscr{C}$ with $|\alpha| = n + 1$. We have $\hat{\delta}(p_\alpha) = \sum \hat{\delta}(p_\alpha)(p_\beta \otimes p_{\beta'})$, where the sum goes over the ordered pairs (β, β') such that $\alpha \subset \beta \otimes \beta'$. Hence

$$\hat{\delta}(p_\alpha)(p_n \otimes p_1) = \hat{\delta}(p_\alpha)(p_{\alpha'} \otimes p_1),$$

where α' is the vertex preceding α in the classical Cayley graph $\mathfrak{g}$. Hence we get the following expression of $p_{\star+}$, in terms of the classical ascending orientation $\mathfrak{e}_+$ of $\mathfrak{g}$:

$$p_{\star+} = \sum_{(\alpha', \alpha) \in \mathfrak{e}_+} V^*(p_{\alpha'} \otimes p_\alpha)V(\mathrm{id} \otimes p_1).$$

Recall that V plays the role of the endpoints operator, which implements in the co-commutative case the equivalence between the simplicial and directional pictures of the Cayley graph.

(3) Let J (resp. $\hat{J}$) be the modular conjugation on H induced by the involution of S (resp. $\hat{S}$). We know from [10] that $U = \hat{J}J$, $[J, p_n] = 0$ and $(J \otimes J)\hat{\delta}(p_n)(J \otimes J) = \Sigma\hat{\delta}(p_n)\Sigma$. From this we can deduce the following relation between $p_{+\star}$ and $p_{\star+}$:

$$p_{+\star} = (\hat{J} \otimes \hat{J})p_{\star+}(\hat{J} \otimes \hat{J}).$$

Hence $p_{+\star}$ and $p_{\star+}$ come from the same projection of $M(\hat{S} \otimes \hat{S})$ acting respectively on the left and on the right of K. In particular they commute and are equal in the co-commutative case. □

In the next proposition we examine the links between the ascending projections and the reversing and target operators. The first point of the proposition shows that the reversing operator switches the left and right versions of the quantum ascending projections: this is the reason why it is necessary to use both $p_{\star+}$ and $p_{+\star}$ in the general case.

Proposition 4.3. *We use the hypothesis and notation of Definition* 4.1.

(1) *We have* $p_{\star-} = \Theta p_{+\star}\Theta^*$ *and* $p_{-\star} = \Theta^* p_{\star+}\Theta$. *More precisely*:

$$\Theta p_{+\star}(p_n \otimes \mathrm{id}) = (p_{n+1} \otimes \mathrm{id})p_{\star-}\Theta,$$
$$\Theta p_{-\star}(p_n \otimes \mathrm{id}) = (p_{n-1} \otimes \mathrm{id})p_{\star+}\Theta.$$

(2) *We have* $E_2 p_{+-} = E_2 p_{-+} = 0$ *and*

$$p_n E_2 = E_2(p_{n-1} \otimes \mathrm{id})p_{++} + E_2(p_{n+1} \otimes \mathrm{id})p_{--}.$$

Proof. (1) We put $u = \mathrm{Ad}(U)$. Using the formulae $\tilde{V}(p_\alpha \otimes 1)\tilde{V}^* = \hat{\delta}(p_\alpha)$ and $V^*(1 \otimes p_\alpha)V = \hat{\delta}(p_\alpha)$ for the dual coproduct, and the fact that p_n commutes to U, we see that

$$\tilde{V}^*(p_n \otimes 1)\tilde{V} = (U \otimes 1)\Sigma V^*\Sigma(Up_nU \otimes 1)\Sigma V\Sigma(U \otimes 1)$$
$$= (u \otimes \mathrm{id})\sigma\big(V^*(1 \otimes p_n)V\big) = (u \otimes \mathrm{id})\sigma\hat{\delta}(p_n).$$

We use this expression in conjunction with the definition of Θ:

$$\Theta p_{+\star}(p_n \otimes 1)\Theta^* = \tilde{V}(p_n \otimes p_1)(u \otimes \mathrm{id})\sigma\hat{\delta}(p_{n+1})\tilde{V}^*$$
$$= (1 \otimes p_1)\hat{\delta}(p_n)\tilde{V}(u \otimes \mathrm{id})\sigma\hat{\delta}(p_{n+1})\tilde{V}^* = \hat{\delta}(p_n)(p_{n+1} \otimes p_1).$$

To prove that the last expression is equal to $p_{\star-}(p_{n+1} \otimes p_1)$, it is enough to check that we obtain $p_{\star-}$ by summing it over n. But $(p_n \otimes p_1)\hat{\delta}(p_{n'})$ vanishes as soon as $n' \neq n \pm 1$, so that

$$\sum (p_{n+1} \otimes p_1)\hat{\delta}(p_n) + p_{\star+} = \sum \big((p_{n+1} \otimes p_1)\hat{\delta}(p_n) + (p_n \otimes p_1)\hat{\delta}(p_{n+1}) \big)$$
$$= \big(\sum (p_n \otimes p_1) \big) \big(\sum \hat{\delta}(p_{n'}) \big) = \mathrm{id}_K.$$

(2) The last point of Proposition 3.7 shows that $p_k E_2(p_l \otimes p_1)$ equals $E_2\hat{\delta}(p_k)(p_l \otimes p_1)$. In particular $E_2(p_n \otimes p_1)p_{\star+} = p_{n+1}E_2 p_{\star+}$ and similarly

$$E_2(p_n \otimes p_1)p_{\star-} = p_{n-1}E_2 p_{\star-}.$$

On the other hand one has, using Propositions 3.6 and 4.3:

$$E_2(p_n \otimes p_1)p_{-\star} = E_1\Theta(p_n \otimes p_1)p_{-\star} = E_1(p_{n-1} \otimes p_1)p_{\star+}\Theta$$
$$= p_{n-1}E_1 p_{\star+}\Theta = p_{n-1}E_2 p_{-\star},$$

and similarly

$$E_2(p_n \otimes p_1)p_{+\star} = p_{n+1}E_2 p_{+\star}.$$

As a result $E_2(p_n \otimes p_1)p_{-+}$ equals both $p_{n+1}E_2 p_{-+}$ and $p_{n-1}E_2 p_{-+}$, so that it must vanish, and in the same way $E_2(p_n \otimes p_1)p_{+-} = 0$. In particular $p_n E_2 = p_n E_2(p_{++} + p_{--})$, and the last statement of the proposition results then from the identities

$$p_n E_2 p_{\star+} = E_2(p_{n-1} \otimes p_1)p_{\star+} \quad \text{and} \quad p_n E_2 p_{\star-} = E_2(p_{n+1} \otimes p_1)p_{\star-}$$

that we proved above. □

Until now we have used a very minimal notion of "tree" for our quantum Cayley graphs, namely the fact that the corresponding classical Cayley graph should be a classical tree. However this notion is too weak for our purposes, because it doesn't take into account multiplicity issues that appear in the quantum case. More precisely, let us define the "full" classical Cayley graph $\mathfrak{G}$ associated to (S, p_1) in the following way:

$$\mathfrak{v} = \operatorname{Irr}\mathscr{C}, \quad \mathfrak{e} = \{(\alpha, \alpha', \gamma, i) \mid \gamma \in \mathscr{D}, \alpha' \subset \alpha \otimes \gamma \text{ with multiplicity order } i\},$$
$$e(\alpha, \alpha', \gamma, i) = (\alpha, \alpha'), \quad \theta(\alpha, \alpha', \gamma, i) = (\alpha', \alpha, \bar{\gamma}, i).$$

Here $\mathscr{D}$ stands for the set of corepresentations associated with p_1, like in Remark 3.2.1. The image of $\mathfrak{G}$ by e is the classical Cayley graph $\mathfrak{g}$ of Definition 3.1, but the map e needs not to be injective in general. The component γ of an edge $(\alpha, \alpha', \gamma, i)$ is called the direction of the edge. In the rest of this paper, we will assume that the full classical Cayley graph $\mathfrak{G}$ with origin $1_\mathscr{C}$ is a "directional tree", meaning that it is a tree and that the ascending edges starting from a given vertex have pairwise different directions.

In Lemma 4.4 we state some basic results about classical Cayley graphs and give a corepresentation-theoretic formulation of the extra assumptions introduced above. Proposition 4.7 shows that our framework is the right one for the study of free quantum groups, i.e. free products of orthogonal and unitary free quantum groups [16], [5]. Finally we prove that the quantum ascending orientation $K_{++} \subset K$ behaves nicely in this framework: the target operator induces a bijection between ascending edges and vertices orthogonal to the origin, exactly like in the classical case.

Lemma 4.4. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{g}$ is a tree and denote by $(\alpha \otimes \gamma)_+$ (resp. $(\alpha \otimes \gamma)_-$) the sum of the subobjects of $(\alpha \otimes \gamma)$ which are further from (resp. closer to) $1_{\mathscr{C}}$ than α.*

(1) *For every $\alpha \in \operatorname{Irr}\mathscr{C}$ one has $|\alpha| = |\bar{\alpha}|$ in $\mathfrak{g}$.*

(2) *The full classical Cayley graph $\mathfrak{G}$ is a directional tree iff*

— *for all $\alpha \in \operatorname{Irr}\mathscr{C}$ and $\gamma \in \mathscr{D}$, $(\alpha \otimes \gamma)_+$ is irreducible or zero and*

— *for all $\alpha \in \operatorname{Irr}\mathscr{C}$ and $\gamma \neq \gamma' \in \mathscr{D}$, $(\alpha \otimes \gamma)_+$ and $(\alpha \otimes \gamma')_+$ are inequivalent or zero.*

(3) *We assume that $\mathfrak{G}$ is a directional tree. For any $(\alpha, \gamma) \in \operatorname{Irr}\mathscr{C} \times \mathscr{D}$, one has $(\alpha \otimes \gamma)_+ = 0$ iff $\dim \gamma = 1$ and α is the target of an ascending edge with direction $\bar{\gamma}$.*

(4) *If $\mathfrak{G}$ is a directional tree and (α, β) is an ascending edge then $\dim \beta \geqq \dim \alpha$, with equality iff the corresponding direction $\gamma \in \mathscr{D}$ has dimension 1.*

Proof. (1) For this first point $\mathfrak{g}$ does not need to be a tree. Because $\bar{\mathscr{D}} = \mathscr{D}$, it is enough to prove the following property: $|\alpha| \leqq n$ iff there exist elements $\gamma_1, \dots, \gamma_n \in \mathscr{D}$ such that $\alpha \subset \gamma_1 \otimes \cdots \otimes \gamma_n$. We proceed by induction over n: for $n = 0$ the property is satisfied because $\alpha \subset 1_{\mathscr{C}} \Leftrightarrow \alpha = 1_{\mathscr{C}}$. Assume now that the property is satisfied for a given $n \geqq 0$ and consider an $\alpha \in \operatorname{Irr}\mathscr{C}$ such that $|\alpha| = n + 1$. By definition of $\mathfrak{g}$ there exist $\beta \in \mathfrak{v}$ and $\gamma \in \mathscr{D}$ such that $|\beta| = n$ and $\alpha \subset \beta \otimes \gamma$, and the induction hypothesis for β gives the desired inclusion $\alpha \subset \gamma_1 \otimes \cdots \otimes \gamma_n \otimes \gamma$. Assume conversely that $\alpha \subset \gamma_1 \otimes \cdots \otimes \gamma_{n+1}$ and let (β_k) be a maximal orthogonal family of irreducible subobjects of $\gamma_1 \otimes \cdots \otimes \gamma_n$. Because α is irreducible the inclusion $\alpha \subset \bigoplus(\beta_k \otimes \gamma_{n+1})$ implies that $\alpha \subset \beta_k \otimes \gamma_{n+1}$ for some k. By induction hypothesis one has $|\beta_k| \leqq n$, hence $|\alpha| \leqq n + 1$.

(2) Recall that the endpoints map e induces a morphism from $\mathfrak{G}$ onto $\mathfrak{g}$, the latter one being a tree. Therefore $\mathfrak{G}$ is a tree iff e is injective, and it is enough to check it on the ascending orientation $\mathfrak{e}_+ \subset \mathfrak{e}$: this leads to the condition that the subobjects $(\alpha \otimes \gamma)_+$, for a given α, should have pairwise different subobjects without multiplicity. The tree $\mathfrak{G}$ is then directional with respect to the origin $1_{\mathscr{C}}$ iff the corepresentations $(\alpha \otimes \gamma)_+$ have at most one subobject.

(3) and (4) We proceed again by induction on the distance to the origin: let (α, β) be an ascending edge with direction γ and assume that $\dim \beta \geqq \dim \alpha$, with equality iff $\dim \gamma > 1$. Take $\gamma' \in \mathscr{D}$, the assumption on $\mathfrak{G}$ shows that $(\beta \otimes \gamma') = (\beta \otimes \gamma')_+$ or $(\beta \otimes \gamma') = \alpha \oplus (\beta \otimes \gamma')_+$. In the first case, which can only happen when $(\beta \otimes \gamma')_+ \neq 0$, one has clearly $\dim(\beta \otimes \gamma')_+ \geqq \dim \beta$ with equality iff $\dim \gamma' = 1$. On the other hand, we are in

the second case iff $\gamma' = \bar{\gamma}$, because of the equivalence $\alpha \subset \beta \otimes \gamma' \Leftrightarrow \beta \subset \alpha \otimes \bar{\gamma}'$. Moreover one has then $(\beta \otimes \gamma')_+ = 0$ iff $\dim \beta \dim \gamma' = \dim \alpha$, which is equivalent to $\dim \gamma' = 1$ by induction hypothesis. If on the contrary $\dim \gamma' = \dim \gamma > 1$, the strict case of the induction hypothesis gives

$$\dim(\beta \otimes \gamma')_+ = \dim \beta \dim \gamma' - \dim \alpha \geqq 2 \dim \beta - \dim \alpha > \dim \beta. \quad \square$$

Proposition 4.5. *Let S be a full Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0 p_1 = 0$. If the full classical Cayley graph $\mathfrak{G}$ is a directional tree, then*

— S is a free product of a finite number of free Woronowicz C^*-algebras $A_o(Q_i)$ and $A_u(Q'_j)$, with $Q_i \bar{Q}_i \in \mathbb{C}\,\mathrm{id}$ and Q'_j invertible,

— p_1 is the sum of the central supports of the respective fundamental corepresentations of these Woronowicz C^*-algebras.

Conversely the full classical Cayley graph $\mathfrak{G}$ of any such pair (S, p_1) is a directional tree.

Proof. The classical graph $\mathfrak{g}$ being a tree, the set $\mathrm{Irr}\,\mathscr{C}$ of its vertices lies in one-to-one correspondence with the set of paths without half-turns starting from the origin $1_{\mathscr{C}}$. Because $\mathfrak{g}$ is isomorphic to the full graph $\mathfrak{G}$, these paths are characterized by the finite sequences of the directions they follow. Finally, Lemma 4.4 shows that the finite sequences (γ_i) of elements of $\mathscr{D}$ that arise in such a way are exactly the ones that fulfill the condition $\gamma_{i+1} \neq \bar{\gamma}_i$ or $\dim \gamma_{i+1} > 1$ for each i.

For every pair $\{\gamma, \bar{\gamma}\} \subset \mathscr{D}$ with $\gamma = \bar{\gamma}$ (resp. $\gamma \neq \bar{\gamma}$), the universal property of free quantum groups gives a Hopf homomorphism from some $A_o(Q)$ (resp. $A_u(Q)$) onto S, where Q is a matrix such that $Q\bar{Q} \in \mathbb{C}\,\mathrm{id}$ (resp. is invertible). By universality of free products, one obtains then a surjective Hopf homomorphism $\Phi : F \to S$, where F is some finite free product of free quantum groups. By definition, for each factor $A_o(Q), A_u(Q) \subset F$ the fundamental corepresentation U and its conjugate are mapped by $\mathrm{id} \otimes \Phi$ onto the corresponding pair $\{\gamma, \bar{\gamma}\} \subset \mathscr{D}$.

On the other hand, the starting remarks on the structure of $\mathfrak{g}$ show that $\mathrm{Irr}\,\mathscr{C}$ is the monoid generated by $\mathscr{D}$ and the relations $\{\gamma\bar{\gamma} = \bar{\gamma}\gamma = 1 \mid \gamma \in \mathscr{D}, \dim \gamma = 1\}$. Hence Φ induces a bijection between $\mathrm{Irr}\,\mathscr{C}$ and the set $\mathrm{Irr}\,\mathscr{F}$ of irreducible corepresentations of F (up to equivalence)—see [19], [4], [5] for the description of $\mathrm{Irr}\,\mathscr{F}$ and notice that $A_o(Q)$ and $A_u(Q)$ are respectively isomorphic to $C^*(\mathbb{Z}/2\mathbb{Z})$ and $C^*(\mathbb{Z})$ when $\dim Q = 1$. This proves, using [21] and the fact that we are dealing with full Woronowicz C^*-algebras, that Φ is injective. The statement that $\mathfrak{G}$ is a directional tree for any free product of free quantum groups follows easily from the above mentioned description of $\mathrm{Irr}\,\mathscr{F}$. $\square$

Example 4.6 (free quantum groups). Let us picture the simplest cases of Proposition 4.5. When $\dim Q > 1$ and $Q\bar{Q} \in \mathbb{C}\,\mathrm{id}$, the classical Cayley graph of $A_o(Q)$ endowed with its fundamental corepresentation is the half line with vertices at the integers. When $\dim Q > 1$ and Q is invertible, the classical Cayley graph of $A_u(Q)$ is drawn in Figure 1. $\square$

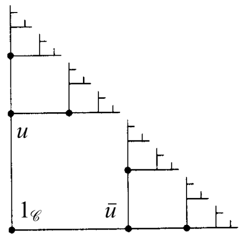

Figure 1. Classical Cayley graph of the unitary free quantum group.

Proposition 4.7. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree. Then the restriction of E_2 to K_{++} is injective and its image is $(1 - p_0)H$.*

Proof. Let $\alpha \in \operatorname{Irr}\mathscr{C}$ and $\gamma \in \mathscr{D}$ be such that $|\alpha| = n$ and $(\alpha \otimes \gamma)_+ \neq 0$. The subspace $(p_\alpha \otimes p_\gamma)K$ is irreducible with respect to the representation $\hat{\pi}_4$ of $\hat{S}^{\otimes 4}$, and equivalent to $(\alpha \otimes \gamma) \otimes \overline{(\alpha \otimes \gamma)}$. By definition, $p_{\star+}(p_\alpha \otimes p_\gamma) = \hat{\delta}(p_{n+1})(p_\alpha \otimes p_\gamma)$, so that $p_{\star+}(p_\alpha \otimes p_\gamma)K$ is equivalent to $(\alpha \otimes \gamma)_+ \otimes \overline{(\alpha \otimes \gamma)}$ with respect to the representation $\hat{\pi}_4 \circ (\hat{\delta} \otimes \mathrm{id} \otimes \mathrm{id})$ of $\hat{S}^{\otimes 3}$. Similarly, and thanks to the first point of Lemma 4.4, the subspace $p_{+\star}(p_\alpha \otimes p_\gamma)K$ is equivalent to $(\alpha \otimes \gamma) \otimes \overline{(\alpha \otimes \gamma)_+}$ with respect to the representation $\hat{\pi}_4 \circ (\mathrm{id} \otimes \mathrm{id} \otimes \hat{\delta})$. Finally, $p_{++}(p_\alpha \otimes p_\gamma)K$ is equivalent to $(\alpha \otimes \gamma)_+ \otimes \overline{(\alpha \otimes \gamma)_+}$ for the representation $\hat{\pi}_4 \circ (\hat{\delta} \otimes \hat{\delta})$ of $\hat{S} \otimes \hat{S}$, and therefore irreducible by hypothesis.

Recall now from Proposition 3.7 that E_2 intertwines $\hat{\pi}_4 \circ (\hat{\delta} \otimes \hat{\delta})$ and $\hat{\pi}_2$. Hence the restriction of E_2 to $(p_\alpha \otimes p_\gamma)K_{++}$ is a multiple of an isometry, and one can compute the corresponding norm by considering the image of particular vectors, for instance characters. One has by Proposition 3.6

$$\begin{aligned} E_2 p_{++}(\chi_\alpha \otimes \chi_\gamma) &= E_2 p_{\star+}(\chi_\alpha \otimes \chi_\gamma) = E_2 \hat{\delta}(p_{n+1})(\chi_\alpha \otimes \chi_\gamma) \\ &= p_{n+1} E_2(\chi_\alpha \otimes \chi_\gamma) = p_{n+1}(\chi_{\alpha \otimes \gamma}) \\ &= p_{n+1}(\chi_{(\alpha \otimes \gamma)_-} + \chi_{(\alpha \otimes \gamma)_+}) = \chi_{(\alpha \otimes \gamma)_+}. \end{aligned}$$

The norm in H of the character of an irreducible corepresentation equals 1 (cf. [20], th. 5.8), so that one eventually gets the following lower bound for the norm of $E_2 p_{++}(p_\alpha \otimes p_\gamma)$:

$$(8) \qquad \|E_2 p_{++}(p_\alpha \otimes p_\gamma)\| = \frac{\|\chi_{(\alpha \otimes \gamma)_+}\|}{\|p_{++}(\chi_\alpha \otimes \chi_\gamma)\|} \geqq \frac{\|\chi_{(\alpha \otimes \gamma)_+}\|}{\|\chi_\alpha \otimes \chi_\gamma\|} = 1.$$

To conclude, let us remark that $E_2 p_{++}$ maps the respective orthogonal subspaces $p_{++}(p_\alpha \otimes p_\gamma)K$ onto the subspaces $p_{(\alpha \otimes \gamma)_+}H$, which are pairwise different by hypothesis, hence orthogonal, and whose sum equals $(1 - p_0)H$. The operator E_2 is therefore injective and has dense image in $(1 - p_0)H$, but this image is closed by (8). □

Remark 4.8. We will need in Section 8.1 to have a slightly more general and more precise result than (8). Let $\mathscr{H}$ be the algebraic direct sum of the subspaces $p_k H$, and let $\mathscr{E}_2$ be the operator defined on $\mathscr{H} \otimes \mathscr{H}$ in the same way as E_2, that is, by Proposition 3.6, coming from the multiplication of S. Let us also extend $p_{\star +}$ to $\mathscr{H} \otimes \mathscr{H}$ by putting $\mathscr{P}_{\star +} = \sum \hat{\delta}(p_{n+k})(p_n \otimes p_k)$: we have then $\mathscr{E}_2 \mathscr{P}_{\star +}(\mathrm{id} \otimes p_1) = E_2 p_{\star +} = E_2 p_{++}$. The first arguments of the preceding proof are still valid if one replaces γ with any $\beta \in \operatorname{Irr} \mathscr{C}$: as a matter of fact the hypothesis implies that $\alpha \otimes \beta$ contains at most one subobject δ with $|\delta| = |\alpha| + |\beta|$. If such a δ exists one gets the following generalization of (8):

$$\|\mathscr{E}_2 \mathscr{P}_{\star +}(p_\alpha \otimes p_\beta)\| = \|\hat{\delta}(p_\delta)(\chi_\alpha \otimes \chi_\beta)\|^{-1}.$$

Let us remark that χ_α (resp. $t_\alpha(1)$) generates the invariant line of $p_\alpha H$ (resp. $H_\alpha \otimes H_{\bar{\alpha}}$) with respect to the action of $\hat{S}$. Moreover one has $\|\chi_\alpha\| = 1$ and $\|t_\alpha(1)\| = \sqrt{M_\alpha}$, so that χ_α in fact corresponds to $t_\alpha(1)/\sqrt{M_\alpha}$ in the isomorphism $p_\alpha H \simeq H_\alpha \otimes H_{\bar{\alpha}}$, up to a phase factor. Consequently, in the isomorphism $p_\alpha H \otimes p_\beta H \simeq H_\alpha \otimes H_\beta \otimes H_{\bar{\beta}} \otimes H_{\bar{\alpha}}$ the vector $\hat{\delta}(p_\delta)(\chi_\alpha \otimes \chi_\beta)$ corresponds to

$$\big(\hat{\delta}(p_\delta) \otimes \mathrm{id} \otimes \mathrm{id}\big)\big(t_{\alpha \otimes \beta}(1)\big)/\sqrt{M_\alpha M_\beta} = t_\delta(1)/\sqrt{M_\alpha M_\beta},$$

if one isometrically identifies H_δ with the equivalent subspace of $H_\alpha \otimes H_\beta$. We therefore get the following exact formula, from which (8) can be recovered by noticing that $M_\delta \leqq M_\alpha \otimes M_\beta$:

$$\|\mathscr{E}_2 \mathscr{P}_{\star +}(p_\alpha \otimes p_\beta)\| = \sqrt{\frac{M_\alpha M_\beta}{M_\delta}}. \quad \square$$

5. Geometric edges

In this section we will study the Hilbert space $K_g = \operatorname{Ker}(\Theta + \mathrm{id})$ when the classical Cayley graph $\mathfrak{G}$ is a directional tree. We consider Proposition 4.7 as an evidence that K_{++} provides a good notion of "quantum ascending edges", and we would similarly like to know whether K_g provides a good notion of "quantum geometric edges". By this we mean that there should be exactly one geometric edge for each ascending edge, which can be more rigorously expressed in the hilbertian framework by the fact that the restriction $p_{++} : K_g \to K_{++}$ should be invertible.

Of course the study of $K_g = \operatorname{Ker}(\Theta + \mathrm{id})$ is closely related to the problem of the non-involutivity of the reversing operator Θ. The next proposition provides a "weak involutivity" property which we will use for the proof of Theorem 5.3, as well as a technical corollary obtained in Lemma 5.2. Notice that $(p_{++} + p_{--})K$ behaves as a subspace of "quasi-classical" quantum edges in this regard.

Proposition 5.1. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $U p_1 U = p_1$ and $p_0 p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree. Then we have, for all $n \in \mathbb{N}$:*

$$(p_{++} + p_{--})\Theta^n(p_{++} + p_{--}) = (p_{++} + p_{--})\Theta^{-n}(p_{++} + p_{--}).$$

Proof. Inserting $\mathrm{id} = p_{++} + p_{+-} + p_{-+} + p_{--}$ between the occurrences of $\Theta^{\pm 1}$ in $\Theta^{\pm n}$ and developing, the statement of the theorem becomes an equality between two sums of terms looking like $p_{\epsilon'_0,\epsilon'_0}\Theta^{\pm 1}p_{\epsilon_1,\epsilon'_1}\Theta^{\pm 1}\cdots\Theta^{\pm 1}p_{\epsilon_n,\epsilon_n}$. We will in fact prove that these terms are pairwise equal: for $\epsilon_i, \epsilon'_i \in \{+,-\}$ with $i \in [\![0,n]\!]$, one has

$$p_{\epsilon'_0,\epsilon'_0}\Theta p_{\epsilon_1,\epsilon'_1}\Theta\cdots\Theta p_{\epsilon_{n-1},\epsilon'_{n-1}}\Theta p_{\epsilon_n,\epsilon_n} = p_{\epsilon'_0,\epsilon'_0}\Theta^{-1}p_{\epsilon'_1,\epsilon_1}\Theta^{-1}\cdots\Theta^{-1}p_{\epsilon'_{n-1},\epsilon_{n-1}}\Theta^{-1}p_{\epsilon_n,\epsilon_n}. \tag{9}$$

Let us proceed by induction over $n \in \mathbb{N}$, calling "rank 0" the trivial equality $p_{\epsilon',\epsilon'}p_{\epsilon,\epsilon} = p_{\epsilon',\epsilon'}p_{\epsilon,\epsilon}$. Choose $n \geqq 1$. As a first step, assume that there exists $k \in [\![1,n-1]\!]$ such that $\epsilon_k = \epsilon'_k$. Then the conclusion results straightforwardly from two applications of the induction hypothesis at ranks k and $n-k$, with $(\epsilon_i,\epsilon'_i)_{0\leqq i\leqq k}$ and $(\epsilon_i,\epsilon'_i)_{k\leqq i\leqq n}$ respectively.

We assume now that $\epsilon_i = -\epsilon'_i$ for each i. If one side of (9) is non-zero, we necessarily have $(\epsilon_i,\epsilon'_i) = (-\epsilon'_0,-\epsilon_n)$ for all indices i: as a matter of fact, Proposition 4.3 shows that the equalities $\epsilon_{i+1} = -\epsilon'_i$ are required for the products in (9) not to vanish. In particular, we have then $\epsilon'_0 = -\epsilon_n$. This proves that the equalities from the first step are sufficient to get the identities $p_{--}\Theta^n p_{--} = p_{--}\Theta^{-n}p_{--}$ and $p_{++}\Theta^n p_{++} = p_{++}\Theta^{-n}p_{++}$. Moreover, taking the adjoint allows to switch from $\epsilon'_0 = -1$ to $\epsilon'_0 = 1$, so that it only remains to prove the equality

$$p_{++}\Theta p_{-+}\Theta\cdots\Theta p_{-+}\Theta p_{--} = p_{++}\Theta^{-1}p_{+-}\Theta^{-1}\cdots\Theta^{-1}p_{+-}\Theta^{-1}p_{--}.$$

By adding terms from the first step we rather focus on the following equivalent equality:

$$p_{++}\Theta^n p_{--} + p_{--}\Theta^n p_{--} \stackrel{?}{=} p_{++}\Theta^{-n}p_{--} + p_{--}\Theta^{-n}p_{--}.$$

Using the fact from Proposition 4.7 that the target operator E_2 is injective on K_{++}, we can compose on the left by E_2 and use Proposition 4.3 to get another equivalent equality: $E_2\Theta^n p_{--} = E_2\Theta^{-n}p_{--}$. But this is true since we have, from Proposition 3.6 and the definition of E_2: $E_2\Theta^2 = E_1\Theta = E_2$, hence $E_2\Theta^{2k} = E_2\Theta^{-2k} = E_2$ and $E_2\Theta = E_2\Theta^{-1}$. □

Lemma 5.2. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S,p_1) is a directional tree. Then there exists a unique unitary operator $W : K_{+-} \to K_{-+}$ such that*

$$\forall k \in \mathbb{N} \quad W(p_{+-}\Theta)^k p_{++} = (p_{-+}\Theta^{-1})^k p_{++}.$$

Moreover we have $Wp_{+-}\Theta = p_{-+}\Theta^{-1}W$ and $p_{--}\Theta = p_{--}\Theta^{-1}W$ on K_{+-}.

Proof. Let X (resp. X') be the operator from $K_{++}\otimes\ell_2(\mathbb{N})$ to K_{+-} (resp. K_{-+}) defined by $X(\xi\otimes e_k) = 2^{-k}(p_{+-}\Theta)^k\xi$ (resp. $X'(\xi\otimes e_k) = 2^{-k}(p_{-+}\Theta^{-1})^k\xi$). Thanks to the coefficients 2^{-k}, the operators X and X' are bounded, and it is easy to see that their adjoints are resp. given by

$$X^* = \sum 2^{-k}T_k p_{++}(\Theta^{-1}p_{+-})^k \quad \text{and} \quad X'^* = \sum 2^{-k}T_k p_{++}(\Theta p_{-+})^k,$$

where we put $T_k(\xi) = \xi\otimes e_k$ for any $\xi \in K_{++}$. Let ζ be an element of $\operatorname{Ker} X^*$, for every k

and n we have $p_{++}(\Theta^{-1}p_{+-})^k(p_n \otimes \mathrm{id})\zeta = 0$. In particular $(\Theta^{-1}p_{+-})^n(p_n \otimes p_1)\zeta$ vanishes: by Proposition 4.3 it is an element of $(p_0 \otimes \mathrm{id})K$, which is contained in K_{++}. By a finite descending induction on $k \in [\![0, n]\!]$, we deduce that

$$(\Theta^{-1}p_{+-})^k(p_n \otimes p_1)\zeta = p_{++}(\Theta^{-1}p_{+-})^k(p_n \otimes p_1)\zeta + \Theta(\Theta^{-1}p_{+-})^{k+1}(p_n \otimes p_1)\zeta$$

vanishes, and in particular $(p_n \otimes \mathrm{id})\zeta = 0$ for any n. Hence X^* is injective and X has dense image. In the same way, X' has dense image.

To prove the existence and the uniqueness of W, which is characterized by the identity $WX = X'$, it is therefore enough to show that $\|X\eta\| = \|X'\eta\|$ for any $\eta \in K_{++} \otimes \ell_2(\mathbb{N})$, or as well, that $X^*X = X'^*X'$. We will work on each subspace $K_{++} \otimes e_i$ separately, and we are thus led to prove for every k and l the equality

$$p_{++}(\Theta^{-1}p_{+-})^l(p_{+-}\Theta)^k p_{++} = p_{++}(\Theta p_{-+})^l(p_{-+}\Theta^{-1})^k p_{++}, \tag{10}$$

which can also be written

$$\begin{aligned} &p_{++}\Theta^{-1}p_{+-}\cdots p_{+-}\Theta^{-1}(1 - p_{--})\Theta p_{+-}\cdots p_{+-}\Theta p_{++} \\ &\quad = p_{++}\Theta p_{-+}\cdots p_{-+}\Theta(1 - p_{--})\Theta^{-1}p_{-+}\cdots p_{-+}\Theta^{-1}p_{++}. \end{aligned}$$

We proceed by induction on $\min(k, l)$ and distribute $(1 - p_{--})$ on both sides: the terms coming from p_{--} are equal thanks to Equation (9) of Proposition 5.1, and the terms coming from 1 are equal by induction hypothesis. When $kl = 0$ but $(k, l) \neq (0, 0)$, both sides of (10) vanish, and when $k = l = 0$, (10) is trivial.

Because X has dense image, it suffices to check the equalities $Wp_{+-}\Theta = p_{-+}\Theta^{-1}W$ and $p_{--}\Theta = p_{--}\Theta^{-1}W$ on the image of $(p_{+-}\Theta)^k p_{++}$, for every k. The first one follows immediately from the definition of W:

$$(Wp_{+-}\Theta)(p_{+-}\Theta)^k p_{++} = W(p_{+-}\Theta)^{k+1}p_{++} = (p_{-+}\Theta^{-1})^{k+1}p_{++}$$

and

$$(p_{-+}\Theta^{-1}W)(p_{+-}\Theta)^k p_{++} = p_{-+}\Theta^{-1}(p_{-+}\Theta^{-1})^k p_{++} = (p_{-+}\Theta^{-1})^{k+1}p_{++}.$$

For the second one, we use furthermore Equation (9) from Proposition 5.1:

$$\begin{aligned} (p_{--}\Theta^{-1}W)(p_{+-}\Theta)^k p_{++} &= p_{--}\Theta^{-1}(p_{-+}\Theta^{-1})^k p_{++} \\ &= (p_{--}\Theta)(p_{+-}\Theta)^k p_{++}. \quad \square \end{aligned}$$

Theorem 5.3. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree. Then the orthogonal projection from K_g to K_{++} is injective and its image is given by*

$$p_{++}K_g = \{\zeta \in K_{++} \mid \exists \eta \in K_{+-}\ (\mathrm{id} + p_{+-}\Theta)(\eta) = p_{+-}\Theta\zeta\}.$$

Proof. By definition, a vector $\xi \in K$ lies in K_g iff $\Theta(\xi) = -\xi$, which we split in two equations: $p_{\star -}\xi = -\Theta p_{+\star}\xi$ and $p_{-\star}\xi = -\Theta^{-1}p_{\star +}\xi$. Let us first analyze these conditions with respect to the decomposition $K = \bigoplus (p_k \otimes \mathrm{id})K$, using Proposition 4.3:

$$\forall n \in \mathbb{N} \quad p_{\star -}(p_n \otimes \mathrm{id})\xi = -\Theta p_{+\star}(p_{n-1} \otimes \mathrm{id})\xi,$$

$$\forall n \in \mathbb{N} \quad p_{-\star}(p_n \otimes \mathrm{id})\xi = -\Theta^{-1}p_{\star +}(p_{n-1} \otimes \mathrm{id})\xi.$$

If $p_{++}\xi = 0$, this gives a linear induction equation for $\big((p_n \otimes \mathrm{id})\xi\big)_n$, and since $(p_0 \otimes \mathrm{id})\xi = p_{++}(p_0 \otimes \mathrm{id})\xi = 0$ the whole sequence vanishes. Hence the restriction of p_{++} to K_g is injective.

Now we use the decomposition $\mathrm{id} = p_{++} + p_{+-} + p_{-+} + p_{--}$ to get a new system equivalent to the conditions $p_{\star -}\xi = -\Theta p_{+\star}\xi$ and $p_{-\star}\xi = -\Theta^{-1}p_{\star +}$, which characterize vectors in K_g:

$$p_{+-}\xi = -p_{+-}\Theta p_{++}\xi - p_{+-}\Theta p_{+-}\xi, \tag{11}$$

$$p_{--}\xi = -p_{--}\Theta p_{++}\xi - p_{--}\Theta p_{+-}\xi, \tag{12}$$

$$p_{--}\xi = -p_{--}\Theta^{-1}p_{++}\xi - p_{--}\Theta^{-1}p_{-+}\xi, \tag{13}$$

$$p_{-+}\xi = -p_{-+}\Theta^{-1}p_{++}\xi - p_{-+}\Theta^{-1}p_{-+}\xi. \tag{14}$$

Let $\zeta \in K_{++}$ be as in the statement of the theorem: there exists $\eta \in K_{+-}$ such that $(\mathrm{id} + p_{+-}\Theta)(\eta) = p_{+-}\Theta\zeta$. Put $\xi = \zeta - \eta - W\eta + p_{--}\Theta(\eta - \zeta)$. In this case, the above system can be written in the following way:

$$-\eta = -p_{+-}\Theta\zeta + p_{+-}\Theta\eta, \tag{11'}$$

$$p_{--}\Theta(\eta - \zeta) = -p_{--}\Theta\zeta + p_{--}\Theta\eta, \tag{12'}$$

$$p_{--}\Theta(\eta - \zeta) = -p_{--}\Theta^{-1}\zeta + p_{--}\Theta^{-1}W\eta, \tag{13'}$$

$$-W\eta = -p_{-+}\Theta^{-1}\zeta + p_{-+}\Theta^{-1}W\eta. \tag{14'}$$

We can notice that (11′) amounts to the hypothesis on ζ and η, whereas (12′) is trivial. Proposition 5.1 and Lemma 5.2 show that (13′) is always satisfied. Finally the hypothesis on ζ and η yields $W\eta = Wp_{+-}\Theta\zeta - Wp_{+-}\Theta\eta$, and (14′) follows then from Lemma 5.2. Hence ξ lies in K_g and $\zeta = p_{++}\xi$ is in $p_{++}K_g$. The reverse inclusion can easily be obtained from (11): if ζ equals $p_{++}\xi$ with $\xi \in K_g$, we put $\eta = p_{+-}\xi$ and the above mentioned equation reads then $(\mathrm{id} + p_{+-}\Theta)(\eta) = p_{+-}\Theta\zeta$, as already noticed. □

6. Edges at infinity: the set

The expression for $p_{++}K_g$ obtained in Theorem 5.3 is trivial in the classical case because the projection p_{+-} vanishes then, but it has to be analyzed in greater detail in the quantum case. More precisely, we need to understand the interaction between p_{+-} and Θ, and we will see that it can be described by a purely quantum object: the space of "edges at infinity" K_∞, that we introduce in Definition 6.1.

This definition bases on the simple remark that the operator $p_{+-}\Theta p_{+-}$ maps $(p_k \otimes \mathrm{id})K_{+-}$ to $(p_{k+1} \otimes \mathrm{id})K_{+-}$ by Proposition 4.3, and acts therefore as a right shift in the decomposition of K_{+-} given by the distance to the origin in the classical Cayley graph. It is then very natural to introduce the associated inductive limit K_∞. Proposition 6.2 serves as a more precise motivation for this definition and shows that the existence of K_∞ is an obstruction to the surjectivity of $p_{++} : K_g \to K_{++}$. Notice that in the classical case, the subspaces $(p_k \otimes \mathrm{id})K_{+-}$ vanish, so that K_∞ equals zero.

Definition 6.1. Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph associated with (S, p_1) is a tree.

(1) Put $r = -p_{+-}\Theta p_{+-}$, $s = p_{+-}\Theta p_{++}$ and define the inductive limit Hilbert space $K_\infty = \varinjlim\big((p_k \otimes \mathrm{id})K_{+-}, r\big)$.

(2) Let R_k be the natural morphism from $(p_k \otimes \mathrm{id})K_{+-}$ to K_∞, and denote by R the linear map $\sum_{k \geqq 0} R_k$ defined on $\bigoplus_{\mathrm{alg}} (p_k \otimes \mathrm{id})K_{+-}$.

Proposition 6.2. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree.*

(1) *The map Rs extends to a co-isometry from K_{++} to K_∞.*

(2) *The subspace $p_{++}K_g$ is contained in* $\mathrm{Ker}\, Rs$. *Moreover if the R_k are injective one has, denoting by $p_{\geqq k}$ the sum $\sum_{i \geqq k} p_i \otimes \mathrm{id}$:*

$$p_{++}K_g = \{\zeta \in \mathrm{Ker}\, Rs \mid (\|R_k^{-1} Rs p_{\geqq k}\zeta\|)_k \in \ell^2(\mathbb{N})\}.$$

Proof. (1) We start with a simple computation, using Proposition 4.3:

$$\begin{aligned} rr^* + ss^* &= p_{+-}\Theta p_{+-}\Theta^* p_{+-} + p_{+-}\Theta p_{++}\Theta^* p_{+-} \\ &= p_{+-}\Theta p_{+\star}\Theta^* p_{+-} = p_{+-}\Theta\Theta^* p_{+-} = \mathrm{id}_{K_{+-}}. \end{aligned}$$

Notice that $R_0 = 0$ because $p_{+-}(p_0 \otimes \mathrm{id}) = 0$, and that $R_{k+1}r = R_k$ for any $k \in \mathbb{N}$, by definition. We have then, denoting by $p_{\leqq k}$ the sum $\sum_{i \leqq k} p_i \otimes \mathrm{id}$:

$$\begin{aligned} (Rsp_{\leqq k})(Rsp_{\leqq k})^* &= \sum_{i=0}^{k-1} R_{i+1} ss^* R_{i+1}^* = \sum_{i=0}^{k-1} R_{i+1}(1 - rr^*)R_{i+1}^* \\ &= \sum_{i=0}^{k-1} (R_{i+1}R_{i+1}^* - R_iR_i^*) = R_kR_k^*. \end{aligned}$$

The maps R_k being contractive, it follows that $Rsp_{\leqq k}$ and Rs itself are contractions. Because $p_{\leqq k}$ converges to the identity in the $*$-strong topology, $(Rsp_{\leqq k})(Rsp_{\leqq k})^*$ converges strongly to $(Rs)(Rs)^*$, and so it remains to show that $R_k R_k^*$ converges to the identity of K_∞. This is actually a general fact for contractive inductive limits: for any $l \geqq k \geqq 0$ and any $y \in (p_k \otimes \mathrm{id})K_{+-}$, we have

$$\|R_l R_l^*(R_k y) - R_k y\|^2 \leqq \|R_l^* R_k y - r^{l-k} y\|^2$$

and

$$\begin{aligned}\|R_l^* R_k y - r^{l-k} y\|^2 &= \|R_l^* R_k y\|^2 - 2\Re(R_l^* R_k y \,|\, r^{l-k} y) + \|r^{l-k} y\|^2 \\ &= \|R_l^* R_k y\|^2 - 2\|R_k y\|^2 + \|r^{l-k} y\|^2 \\ &\leqq \|r^{l-k} y\|^2 - \|R_k y\|^2.\end{aligned}$$

This upper bound tends to zero as l goes to infinity, by definition of the norm of K_∞. The union $\bigcup \operatorname{Im} R_k$ being dense in K_∞, this proves that $R_l R_l^* \to_s \mathrm{id}$.

(2) Let $\zeta \in p_{++}K_g$: by Theorem 5.3, there exists $\eta \in K_{+-}$ such that $(1-r)\eta = s\zeta$. This can also be written

$$\begin{aligned} &\forall k \in \mathbb{N}^* \quad (p_k \otimes \mathrm{id})\eta = s(p_{k-1} \otimes \mathrm{id})\zeta + r(p_{k-1} \otimes \mathrm{id})\eta \\ \Leftrightarrow\ &\forall k \in \mathbb{N}^* \quad (p_k \otimes \mathrm{id})\eta = \sum_{i=0}^{k-1} r^{k-i-1} s(p_i \otimes \mathrm{id})\zeta \\ \Rightarrow\ &\forall k \in \mathbb{N}^* \quad R_k(p_k \otimes \mathrm{id})\eta = Rsp_{\leqq k-1}\zeta. \end{aligned} \tag{15}$$

The right-hand side of this equality converges to $Rs\zeta$ when k goes to infinity, whereas the left-hand side tends to zero. Hence $p_{++}K_g \subset \operatorname{Ker} Rs$. Now, if the R_k are injective, the implication leading to (15) is an equivalence, so that a vector $\zeta \in K_{++}$ is in $p_{++}K_g$ iff (15) defines a vector $\eta \in K_{+-}$ iff the orthogonal sequence $(R_k^{-1} Rsp_{\leqq k-1}\zeta)_k$ is summable in K_{+-}. Finally, we have clearly $Rsp_{\leqq k-1}\zeta = -Rsp_{\geqq k}\zeta$ when ζ lies in $\operatorname{Ker} Rs$. □

The rest of this section will be devoted to a more detailed study of K_∞. We first want to compute exactly the weights of the "shift" $p_{+-}\Theta p_{+-}$: this is accomplished in Lemma 6.3 and relies on the technical results of Section 2. It is then easy to show that the maps R_k are indeed injective, and therefore that K_∞ is infinite-dimensional in the quantum case. Using the explicit result of Lemma 6.3 we are also able in Theorem 6.5 to make more precise the second statement of Proposition 6.2: it appears that K_∞ is the only obstruction the non-surjectivity of $p_{++} : K_g \to K_{++}$, except when the free product (S, δ) under consideration contains one of the "exceptional cases"

$$A_o\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_o\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_u\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $(\gamma_1, \ldots, \gamma_k)$ be a finite sequence of directions $\gamma_i \in \mathscr{D}$. There is at most one vertex $\alpha \in \operatorname{Irr}\mathscr{C}$ such that the geodesic from $1_\mathscr{C}$ to α follows successively these directions: we will then put $\alpha = \gamma_1 \cdots \gamma_k$. Now choose $\gamma \in \mathscr{D}$ and put $\gamma_{2l} = \bar{\gamma}$, $\gamma_{2l+1} = \gamma$. We denote by α_k the vertex $\gamma\bar{\gamma}\cdots\gamma_k$, when it exists. Lemma 4.4 shows that the set of values of k is $\{0, 1\}$ when

$\dim\gamma = 1$ and $\mathbb{N}$ otherwise. We define in both cases the associated projection $P_\gamma = \sum p_{\alpha_k} \otimes p_{\bar\gamma_k}$. It is a central element of $M(\hat S \otimes \hat S)$, hence it commutes to the projections $p_{\star+}$ and $p_{+\star}$. Moreover one has by Proposition 3.7:

$$\begin{aligned}\Theta P_\gamma &= \sum \Theta(p_{\alpha_k} \otimes p_{\bar\gamma_k}) = \sum \hat\delta(p_{\alpha_k})(1 \otimes p_{\gamma_k})\Theta \\ &= \sum \big(p_{\star+}(p_{\alpha_{k-1}} \otimes p_{\gamma_k}) + p_{\star-}(p_{\alpha_{k+1}} \otimes p_{\gamma_k})\big)\Theta = P_\gamma\Theta.\end{aligned}$$

On the other hand, the projections $p_{+\star}$ and $p_{\star+}$ commute respectively to the representations $\hat\pi_4 \circ (\hat\delta \otimes \mathrm{id} \otimes \mathrm{id})$ and $\hat\pi_4 \circ (\mathrm{id} \otimes \mathrm{id} \otimes \hat\delta)$ of $\hat S \otimes \hat S \otimes \hat S$ on K, by definition. In particular they both commute to the representation $\hat\pi_4 \circ \hat\delta^3$ of $\hat S$, as well as Θ: see Proposition 3.7. Hence $p_{+\star}, p_{\star+}$ and Θ all commute to the projections $q_l = \hat\pi_4\hat\delta^3(p_{2l})$. Let us recall in the case $\dim\gamma > 1$ that $(p_{\alpha_k} \otimes p_{\bar\gamma_k})K_{+-}$ is equivalent to $\alpha_{k-1} \otimes \bar\alpha_{k+1}$ with respect to $\hat\pi_4 \circ \hat\delta^3$, so that $q_l(p_k \otimes \mathrm{id})P_\gamma K_{+-}$ is non-zero iff $l \in [\![1,k]\!]$, and is then irreducible and equivalent to α_{2l}. In the rest of this section we will study the inductive system $\big((p_k \otimes \mathrm{id})K_{+-}, r\big)$ in the decomposition given by the projection $q_l P_\gamma$.

Lemma 6.3. *Let Θ be the reversing operator of a quantum Cayley tree, and choose $\gamma \in \mathscr{D}$ with $\dim\gamma > 1$. We put $m_k = M_{\alpha_k}$ and $m_{-1} = 0$. Let $\epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2' \in \{+,-\}$, with $\epsilon_2' = -\epsilon_1$. For any $k \geqq 1$ and $l \in [\![1,k]\!]$ the operator $p_{\epsilon_2,\epsilon_2'}\Theta p_{\epsilon_1,\epsilon_1'}$ is a multiple of an isometry on $(p_{k-\epsilon_1} \otimes \mathrm{id})p_{\epsilon_1,\epsilon_1'}q_l P_\gamma K$ and*

$$\|(p_k \otimes p_1)p_{\epsilon_2,\epsilon_2'}\Theta p_{\epsilon_1,\epsilon_1'}q_l P_\gamma\| = \begin{cases} \sqrt{\dfrac{m_l m_{l-1}}{m_k m_{k-1}}} & \text{if } \epsilon_1\epsilon_1' \neq \epsilon_2\epsilon_2', \\ \sqrt{1 - \dfrac{m_l m_{l-1}}{m_k m_{k-1}}} & \text{if } \epsilon_1\epsilon_1' = \epsilon_2\epsilon_2'. \end{cases}$$

Proof. We can assume here that $S = A_u(Q)$ or $A_o(Q)$ because $\dim\gamma > 1$: see the proof of Proposition 4.5. We start with $p_{+-}\Theta p_{+-}$, by reorganizing the terms of the product and composing on the left by Θ^*:

$$\|(p_k \otimes \mathrm{id})p_{+-}\Theta p_{+-}q_l P_\gamma\| = \|(\Theta^* p_{+\star}\Theta)p_{\star-}q_l p_{+\star}(p_{\alpha_{k-1}} \otimes p_{\gamma_k})\|.$$

We know from the proof of Proposition 4.7 that the space $p_{+\star}(p_{\alpha_{k-1}} \otimes p_{\gamma_k})K$ is irreducible for the representation $\hat\pi_4 \circ (\mathrm{id} \otimes \mathrm{id} \otimes \hat\delta)$ of $\hat S \otimes \hat S \otimes \hat S$ and identifies with $\alpha_{k-1} \otimes \gamma_k \otimes \bar\alpha_k$. Let us study how $\Theta^* p_{+\star}\Theta, p_{\star-}$ and q_l act in this identification.

— We have $p_{+\star} = \hat\pi_4(\hat\delta \otimes \mathrm{id} \otimes \mathrm{id})(1 \otimes p)$, where $p = \sum \hat\delta(p_{n+1})(p_1 \otimes p_n)$. Lemma 3.7 shows that $\Theta^* p_{+\star}\Theta = \hat\pi_4(\mathrm{id} \otimes \mathrm{id} \otimes \hat\delta)(1 \otimes p)$, which hence acts on $\alpha_{k-1} \otimes \gamma_k \otimes \bar\alpha_k$ as $1 \otimes p$, i.e. as the projection onto $\alpha_{k-1} \otimes \bar\alpha_{k+1}$.

— We know again from the proof of Proposition 4.7 that $p_{\star-}$ acts in the identification like the projection of $\alpha_{k-1} \otimes \gamma_k \otimes \bar\alpha_k$ onto $\alpha_{k-2} \otimes \bar\alpha_k$.

— Finally $q_l = \hat\pi_4\hat\delta^3(p_{2l})$ corresponds to the projection of $\alpha_{k-1} \otimes \gamma_k \otimes \alpha_k$ onto the sum of its subspaces that are equivalent to α_{2l}.

Therefore Lemma 2.4 gives exactly the desired result for $\|(p_k \otimes \mathrm{id})p_{+-}\Theta p_{+-}q_l P_\gamma\|$. We get then the norm of $(p_k \otimes \mathrm{id})p_{+-}\Theta p_{++}q_l P_\gamma$ by noticing that the sum of the squares of both norms equals 1, and we proceed in the same way for the other cases. $\square$

Remarks 6.4. (1) When $l = 0$—and this includes the cases when $k = 0$ or $\dim \gamma = 1$—, we automatically have $p_{+-}q_0 = p_{-+}q_0 = 0$. In particular

$$p_{++}\Theta p_{--}q_0 = \Theta p_{--}q_0,$$

and therefore Lemma 6.3 is replaced in this case by the statement that $p_{++}\Theta p_{--}$ and $p_{--}\Theta p_{++}$ are isometric on $p_{--}q_0K$ and $p_{++}q_0K$. In fact the subspace q_0K, the analogous subspace $\hat{\pi}_2\hat{\delta}(p_0)H$ and the corresponding restrictions of Θ and E are exactly the hilbertian objects associated to the classical Cayley graph $\mathfrak{g}$.

(2) Lemma 6.3 only concerns the "subtrees" $P_\gamma K$, but this is enough to get results about the whole of K, thanks to a "cut-and-paste" process that we explain now. Let $\mathscr{I}$ be the set of ordered pairs $(\beta, \gamma) \in \operatorname{Irr}\mathscr{C} \times \mathscr{D}$ such that the last direction followed by the geodesic from $1_\mathscr{C}$ to β is different from $\bar{\gamma}$—including $(1_\mathscr{C}, \gamma)$ for all $\gamma \in \mathscr{D}$. For such a (β, γ) we denote by β_k the vertices on the ascending path starting from β and taking the directions $\gamma, \bar{\gamma}, \dots$, and we call $P_{\beta,\gamma}$ the sum of the $p_{\beta_k} \otimes p_{\bar{\gamma}_k}$. Because the edges of the classical Cayley graph $\mathfrak{g}$ are walked through once by exactly one of these paths, we see that K is the orthogonal direct sum over $\mathscr{I}$ of the $P_{\beta,\gamma}K$. Notice that $P_\gamma = P_{1_\mathscr{C},\gamma}$.

Now we use the "extended target operator" $\mathscr{E}_2 : \mathscr{H} \otimes H \to H$, i.e. the operator induced in the GNS construction of the Haar state by the multiplication of S. Take $(\beta, \gamma) \in \mathscr{I}$ and denote by α_k the objects constructed from $(1_\mathscr{C}, \gamma)$ as above. By definition of $\mathscr{I}$ we have $(\beta \otimes \gamma)_- = 0$ hence $\beta_1 = \beta \otimes \gamma$. More generally, $\beta \otimes \alpha_k$ is irreducible and equivalent to β_k for every k, so that the restriction of $\mathscr{E}_2 \otimes \mathrm{id}$ to $p_\beta H \otimes P_\gamma K$ is an isometry onto $P_{\beta,\gamma}K$: this is a trivial case of Proposition 4.7 and Remark 4.8. For the same reason one has $(\beta_k \otimes \bar{\gamma}_k)_+ \simeq \beta \otimes (\alpha_k \otimes \bar{\gamma}_k)_+$, which implies that $p_{\star+}(\mathscr{E}_2 \otimes \mathrm{id}) = (\mathscr{E}_2 \otimes \mathrm{id})(\mathrm{id} \otimes p_{\star+})$, and the similar relations for $p_{+\star}$. Moreover we also have $\Theta(\mathscr{E}_2 \otimes \mathrm{id}) = (\mathscr{E}_2 \otimes \mathrm{id})(\mathrm{id} \otimes \Theta)$ because $S \otimes 1$ commutes to Θ. □

Theorem 6.5. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree.*

(1) *The maps R_k are injective. As a result, the space K_∞ is infinite-dimensional whenever S is not co-commutative.*

(2) *If we have $M_\gamma \neq 2$ for all $\gamma \in \mathscr{D}$, then $p_{++}K_g = \operatorname{Ker} Rs$. Otherwise $p_{++}K_g$ is a strict, dense subspace of* $\operatorname{Ker} Rs$.

Proof. (1) Thanks to the preceding Remark 6.4.2, it is enough to study the restrictions of the considered objects to the subspaces $P_\gamma K$ with $\gamma \in \mathscr{D}$. Let $l \in \mathbb{N}$, we can suppose that $l \in [\![1, k]\!]$, and in particular that $\dim \gamma > 1$: otherwise $p_{+-}(p_k \otimes p_1)q_lP_\gamma = 0$ hence $q_lP_\gamma K$ doesn't meet the definition set of R_k. Because the subspaces $(p_k \otimes \mathrm{id})p_{+-}q_lP_\gamma$ are irreducible, and by definition of the norm of K_∞, we have

$$\|R_kq_lP_\gamma\| = \lim \|r^{k+i}(p_k \otimes \mathrm{id})q_lP_\gamma\| = \prod_{i=0}^{\infty} \|p_{+-}\Theta p_{+-}(p_{k+i} \otimes \mathrm{id})q_lP_\gamma\|.$$

To prove that this infinite product is non-zero we use the quantitative result of

Lemma 6.3. Recall from Lemma 2.1 that the sequence (m_k) satisfies the induction equation $m_{i-1} - m_1 m_i + m_{i+1} = 0$, so that we can write $m_i = (a^{i+1} - a^{-i-1})/(a - a^{-1})$ for some $a > 1$ when $m_1 > 2$, and $m_i = i + 1$ when $m_1 = 2$. It is now very easy to check that the following infinite sum is finite:

$$\operatorname{Log} \prod_{i=k}^{\infty} \| p_{+-} \Theta p_{+-} (p_i \otimes \mathrm{id}) q_l P_\gamma \| = \frac{1}{2} \sum_{i=k}^{\infty} \operatorname{Log}\left(1 - \frac{m_l m_{l-1}}{m_{i+1} m_i}\right) > -\infty.$$

Note that we have $\|R_k q_l P_\gamma\| \to 1$ when $k \to \infty$, and in particular the norm $\|q_l P_\gamma R_k^{-1}\|$ is bounded with respect to k. We will need to know for the second point that it is even bounded with respect to k and l, when $m_1 > 2$. To see this, check that $m_l/m_i \leqq a^{-(i-l)}$ when $l \leqq i$ and conclude that

$$\forall k \geqq 1,\ l \in [\![1, k]\!] \quad \operatorname{Log}\|R_k q_l P_\gamma\| \geqq \frac{1}{2} \sum_{i=1}^{\infty} \operatorname{Log}(1 - a^{-2i}).$$

Now if there indeed exists a direction $\gamma \in \mathscr{D}$ with $\dim \gamma > 1$, the injectivity of R_k implies that $\dim K_\infty > \dim(p_k \otimes p_1) p_{+-} P_\gamma K = \dim \alpha_{k-1} \dim \alpha_{k+1}$, which tends to infinity with k according to Lemma 4.4.

(2) We will use the decomposition given by the $q_l P_\gamma$ to study the expression of $p_{++} K_g$ obtained in Proposition 6.2, and in particular the operator $Rsp_{\geqq k}$ restricted to $\operatorname{Ker} Rs$. If $l = 0$ we have $p_{++} P_\gamma K_g = P_\gamma(\operatorname{Ker} Rs) = P_\gamma K_{++}$ since we are considering a classical graph. Now we assume that $l \geqq 1$. In particular the map $s : (p_k \otimes \mathrm{id}) q_l P_\gamma K_{++} \to (p_{k+1} \otimes \mathrm{id}) q_l P_\gamma K_{+-}$ is bijective for any $k \geqq l$ according to Lemma 6.3, and hence $R_{k+1} s : (p_k \otimes \mathrm{id}) q_l P_\gamma K_{++} \to K_\infty$ is injective. Therefore it is possible to unitarily identify all the subspaces $(p_k \otimes \mathrm{id}) q_l P_\gamma K_{++}$ to their common image $G_l \subset K_\infty$ in such a way that $R_{k+1} s q_l P_\gamma$ identifies with $\lambda_{k,l} \, \mathrm{id}_{G_l}$, where

$$\lambda_{k,l} = \|R_{k+1} s q_l P_\gamma\| = \|R_{k+1} q_l P_\gamma\| \sqrt{\frac{m_l m_{l-1}}{m_{k+1} m_k}}.$$

In particular the operator $\big((p_k \otimes \mathrm{id})\zeta\big)_k \mapsto (Rsp_{\geqq k}\zeta)_k$ from $q_l P_\gamma K_{++}$ to $G_l^{\mathbb{N}}$ identifies then with the augmentation by G_l of the matrix $\Lambda_l = (\lambda_{j,l} \delta_{j \geqq i \geqq l})_{i,j}$.

(2a) We start with the case $m_1 = M_\gamma > 2$, which is particularly simple. As a matter of fact, Λ_l is then bounded, even as an operator from $\ell^\infty(\mathbb{N})$ to $\ell^2(\mathbb{N})$, and uniformly with respect to l: we have

$$\sum_{i \geqq l} \Big(\sum_j |\lambda_{j,l} \delta_{j \geqq i}| \Big)^2 = \sum_{i \geqq l} \left(\sum_{j \geqq i} \|R_{j+1} q_l P_\gamma\| \sqrt{\frac{m_l m_{l-1}}{m_{j+1} m_j}} \right)^2$$

$$\leqq \sum_{i \geqq l} \Big(\sum_{j \geqq i} a^{-(j+1-l)} \Big)^2 = \frac{a^2}{(a^2 - 1)(a - 1)^2},$$

using the same estimate for m_l/m_j as in the first point. As a result, for any vector $\zeta \in P_\gamma K_{++}$ the sequence $(Rsp_{\geqq k}\zeta)_k$ is square-summable. The operators R_k^{-1} being uniformly bounded in our case, the sequence $(R_k^{-1} Rsp_{\geqq k}\zeta)_k$ is also square-summable. Therefore the condition of Proposition 6.2 is satisfied by any vector in $P_\gamma(\operatorname{Ker} Rs)$.

(2b) Now we address the case $m_1 = 2$. Let $\varepsilon > 0$, there exists $I \geqq l$ such that $\|R_i q_l P_\gamma\| \geqq 1 - \varepsilon$ for every $i \geqq I$. We have then the following inequalities:

$$\delta_{j\geqq i\geqq I}(1-\varepsilon)\sqrt{\frac{(l+1)l}{(j+2)(j+1)}} \leqq \lambda_{j,l}\delta_{j\geqq i\geqq l} \leqq \delta_{j\geqq i}\sqrt{\frac{(l+1)l}{(j+2)(j+1)}}$$

$$\Rightarrow \delta_{j\geqq i\geqq I}(1-\varepsilon)\frac{l}{i+j+2} \leqq \lambda_{j,l}\delta_{j\geqq i\geqq l} \leqq \frac{2l+2}{i+j+2}$$

$$\Rightarrow l(1-\varepsilon)[\delta_{i,j\geqq I}\mu_{i+1,j+1}] \leqq \Lambda_l + \Lambda_l^*$$

$$\text{and} \qquad \Lambda_l \leqq (2l+2)[\mu_{i,j}],$$

where we put $\mu_{i,j} = (i+j+1)^{-1}$. The last two inequalities are understood in the coefficientwise meaning, but it is well known that this implies norm inequalities, because all the coefficients are non-negative. Hence we have

$$\frac{l(1-\varepsilon)}{2}\|[\delta_{i,j\geqq I}\mu_{i+1,j+1}]\| \leqq \frac{1}{2}\|\Lambda_l + \Lambda_l^*\| \leqq \|\Lambda_l\| \leqq (2l+2)\|[\mu_{i,j}]\|.$$

Now we have in the left-hand (resp. right-hand) side a compact perturbation of (resp. exactly) the Hilbert matrix $M = [\mu_{i,j}]$, which is known from the theory of Hankel operators to have a norm and an essential norm both equal to $\pi/2$ (cf. [11], th. 5.3.1). Hence we obtain, letting furthermore ε go to zero, the estimate $l\pi/4 \leqq \|\Lambda_l\| \leqq (l+1)\pi$.

From this we conclude that every vector of $q_l P_\gamma K_{++}$ satisfies the condition of Proposition 6.2—recall that the operators $q_l P_\gamma R_k^{-1}$ are uniformly bounded with respect to k. As a result, $p_{++}K_g$ is dense in $\bigoplus q_l P_\gamma(\operatorname{Ker} Rs) = \operatorname{Ker} Rs$. However, $p_{++}K_g$ is not equal to $\operatorname{Ker} Rs$. As a matter of fact, the lower estimate we have obtained proves that there exist vectors $\zeta_l \in q_l P_\gamma K_{++}$ such that $\|\zeta_l\| = 1/l$ and $\|(Rsp_{\geqq k}\zeta_l)_k\| \geqq \pi/4$. Moreover one can assume that $Rs(\zeta_l) = 0$: this only corresponds to composing Λ_l on the right by a co-rank 1 projection, which is a compact perturbation. One has then $\zeta = \sum_l \zeta_l \in P_\gamma(\operatorname{Ker} Rs)$, but $(Rsp_{\geqq k}\zeta)_k$ is not square-summable. □

7. Edges at infinity: the action

In the previous section, the interest of the Hilbert space K_∞ mainly lays in its relation with the Hilbert space of geometric edges, via the projection p_{++}. The aim of this section is to endow K_∞ with a representation of S_{red}, which will turn it into an interesting geometric object on its own. On the way, we will be led to study certain aspects of the regular representation $S_{\text{red}} \subset L(H)$ which can be of independent use: see Lemma 7.1 and the remarks after it.

A first step however will be to notice that K_∞ can easily be equipped with a representation of $\hat{S}$, namely the inductive limit $\hat{\pi}_\infty$ of the representation $\hat{\pi}_4\hat{\delta}^3$. As a matter of fact the image of $\hat{\pi}_4\hat{\delta}^3$ commutes to p_{+-}, $p_k \otimes \mathrm{id}$ and r. Recall from the preceding section that the decomposition of $(p_k \otimes \mathrm{id})K_{+-}$ into irreducible subspaces with respect to $\hat{\pi}_4\hat{\delta}^3$ are given by the projections $P_{\beta,\gamma}q_l$. The subspace $P_{\beta,\gamma}q_l(p_k \otimes \mathrm{id})K_{+-}$ is non-zero iff $\dim\gamma > 1$ and $|\beta| + 1 \leqq l \leqq k$ and is then equivalent to $\beta \otimes \gamma\bar{\gamma}\cdots\gamma_{2l-2|\beta|} \otimes \bar{\beta}$. As a result $\hat{\pi}_\infty(p_\alpha)K_\infty$ is irreducible if $\alpha \neq 1_{\mathscr{C}}$ and $\alpha \subset \delta \otimes \bar{\delta}$ for some $\delta \in \mathrm{Irr}\,\mathscr{C}$, and vanishes else.

Lemma 7.1. *Let $p_{\star+}$ be the left ascending projection of a quantum Cayley tree. Choose $\gamma \in \mathscr{D}$ with $\dim\gamma > 1$ and let $m_k = M_{\alpha_k}$ be the corresponding sequence of quantum dimensions. Let $a \in S_{\mathrm{red}}$ be a coefficient of γ. Then the commutator $[a \otimes 1, p_{\star+}]$ vanishes on $(1 - P_{\bar{\gamma}})K$, and there exists a real number $C_a > 0$ such that*

$$\forall k \in \mathbb{N} \quad \|[a \otimes 1, p_{\star+}](p_k \otimes \mathrm{id})\| \leqq C_a m_k^{-1}.$$

Proof. For this proof we can of course assume that k is greater than 2. It is enough to study $p_{\star+}[a \otimes 1, p_{\star+}](p_k \otimes \mathrm{id})$ because $[a \otimes 1, p_{\star+}] = [a \otimes 1, p_{\star+}]p_{\star+} - p_{\star+}[a \otimes 1, p_{\star+}]$. We will use the "extended target operator" $\mathscr{E}_2 : p_\gamma H \otimes H \to H$ given by the product of S. Denoting by $\tilde{a}$ the map $\big(\mathbb{C} \to H, 1 \mapsto \Lambda_h(a)\big)$ we have

$$\begin{aligned} p_{\star+}[a \otimes 1, p_{\star+}] &= p_{\star+}(\mathscr{E}_2 \otimes \mathrm{id})(\tilde{a} \otimes \mathrm{id}_K)p_{\star+} - p_{\star+}(\mathscr{E}_2 \otimes \mathrm{id})(\tilde{a} \otimes \mathrm{id}_K) \\ &= (\mathscr{E}_2 \otimes \mathrm{id})(\hat{\delta} \otimes \mathrm{id})(p_{\star+})(\mathrm{id} \otimes p_{\star+} - 1)(\tilde{a} \otimes \mathrm{id}_K). \end{aligned}$$

Hence it is enough to show that $\|P_1(1 - P_2)\| \leqq m_k^{-1}$, where P_1 and P_2 are the respective restrictions to $p_\gamma H \otimes p_k H \otimes p_1 H$ of $(\hat{\delta} \otimes \mathrm{id})(p_{\star+})$ and $(\mathrm{id} \otimes p_{\star+})$. These projections act through the left representation of $\hat{S}^{\otimes 3}$ on $H^{\otimes 3}$, so that it suffices to look at their action on $L = H_\gamma \otimes H_\alpha \otimes H_{\gamma'}$, with $\gamma' \in \mathscr{D}$ and $|\alpha| = k$.

Let $H_{(\gamma\otimes\alpha)_+}$ and $H_{(\gamma\otimes\alpha)_-}$ be the irreducible subspaces of $H_\gamma \otimes H_\alpha$, the latter being possibly vanishing. We let $p_{\star+} \in M(\hat{S} \otimes \hat{S})$ act on any representation space of $\hat{S} \otimes \hat{S}$. The image of P_2 is then $H_\gamma \otimes p_{\star+}(H_\alpha \otimes H_{\gamma'})$, whereas the image of P_1 is the sum of $L_+ = p_{\star+}(H_{(\gamma\otimes\alpha)_+} \otimes H_{\gamma'})$ and $L_- = p_{\star+}(H_{(\gamma\otimes\alpha)_-} \otimes H_{\gamma'})$. Let us first consider the case when α is not of the form $\delta\bar{\delta}\cdots\delta_k$ for any $\delta \in \mathscr{D}$. Notice that we are automatically in this case when $\dim\gamma = 1$, because we restricted ourselves to the values $k \geqq 2$. The corepresentation α can then be written as an irreducible tensor product $\alpha_1 \otimes \alpha_2$, so that one has

$$\begin{aligned} L_+ &= p_{\star+}\big((H_{(\gamma\otimes\alpha_1)_+} \otimes H_{\alpha_2}) \otimes H_{\gamma'}\big) \\ &= (\mathrm{id} \otimes p_{\star+})\big(H_{(\gamma\otimes\alpha_1)_+} \otimes (H_{\alpha_2} \otimes H_{\gamma'})\big) \\ &\subset H_\gamma \otimes H_{\alpha_1} \otimes p_{\star+}(H_{\alpha_2} \otimes H_{\gamma'}) = H_\gamma \otimes p_{\star+}(H_\alpha \otimes H_{\gamma'}), \end{aligned}$$

and similarly $L_- \subset \mathrm{Im}\,P_2$. In this case we therefore have $(1 - P_2)P_1 = 0$. One can check in the same way that it is also the case when the geodesic from $1_{\mathscr{C}}$ to α does not start in the direction $\bar{\gamma}$ or does not end with the direction $\bar{\gamma}'$.

Therefore it remains to consider the situation when γ is the generator of some copy of $A_o(Q)$ or $A_u(Q)$ in S, and $\alpha = \bar{\gamma}\gamma\cdots\bar{\gamma}_k$, $\gamma' = \gamma_k$. In other words we have $H_\gamma \otimes H_\alpha \otimes H_{\gamma'} = H_{1,k,1}$ with the notation of Lemma 2.5. Let us notice first that L_+ is the unique irreducible subspace of $H_{1,k,1}$ which is at distance $k + 2$ from the origin $1_{\mathscr{C}}$, and is therefore included in $(\mathrm{id} \otimes p_{\star+})(H_{1,k,1})$. Hence it suffices to consider the restriction of P_1 and P_2 to the copies of H_k in $H_{1,k,1}$. We are then exactly in the situation of Lemma 2.5, with $k' = 1$, $G_1 = \mathrm{Im}\, P_1$ and $G_2 = \mathrm{Im}\, P_2$. Because we are looking now at morphisms between irreducible subspaces, we can finally use the lemma to write

$$\|(1 - P_2)P_1\|^2 = 1 - \|P_2 P_1\|^2 = m_k^{-2}. \quad \square$$

Remarks 7.2. (1) Let $(m_k)_k, (m'_k)_k$ be two sequences of quantum dimensions associated to two directions $\gamma, \gamma' \in \mathscr{D}$. If $m'_1 \geqq m_1$, it is easy to check by induction, using Point (4) of Lemma 2.1, that $m'_{k+1}/m'_k \geqq m_{k+1}/m_k$:

$$m'_{k+1} m_k - m_{k+1} m'_k = (m'_1 - m_1) m'_k m_k + (m'_k m_{k-1} - m_k m'_{k-1}) \geqq 0.$$

In particular we have $m'_k \geqq m_k$ for all k. If m_1 is minimal (resp. maximal) amongst the M_γ with $\gamma \in \mathscr{D}$ and $\dim \gamma > 1$, we will call $(m_k)_k$ the minimal (resp. maximal) sequence of quantum dimensions for (S, p_1).

(2) It is clear from the proof of the lemma that $[p_{\star+}, a \otimes 1](p_k \otimes \mathrm{id})$ vanishes as soon as $k \geqq 2$ if a is a coefficient of some $\gamma \in \mathscr{D}$ with $\dim \gamma = 1$. Let us prove now that the result of the lemma holds in fact for any $a \in \mathscr{S} \subset S_{\mathrm{red}}$ if one uses the minimal sequence of quantum dimensions to state it. To see this, assume that a satisfies the inequalities of the lemma and let u be a coefficient of a corepresentation $\gamma \in \mathscr{D}$. Because the algebra $\mathscr{S}$ is spanned by such coefficients, it is enough to prove that au also satisfies the same inequalities for some other constant C_{au}. We remark that $(u \otimes 1)(p_k \otimes \mathrm{id})K$ is included in $(p_{k-1} \otimes \mathrm{id})K + (p_{k+1} \otimes \mathrm{id})K$, so that one can write, using the inequalities $m_k \leqq m_{k+1} \leqq m_1 m_k$:

$$\begin{aligned}
\|[au \otimes 1, p_{\star+}](p_k \otimes \mathrm{id})\| &\leqq \|(a \otimes 1)[u \otimes 1, p_{\star+}](p_k \otimes \mathrm{id})\| \\
&\quad + \|[a \otimes 1, p_{\star+}](u \otimes 1)(p_k \otimes \mathrm{id})\| \\
&\leqq \|a\| C_u m_k^{-1} + \|u\| C_a (m_{k-1}^{-1} + m_{k+1}^{-1}) \\
&\leqq \big(\|a\| C_u + (m_1 + 1)\|u\| C_a\big) m_k^{-1}.
\end{aligned}$$

(3) The lemma also admits the following generalization. If we put $\mathscr{P}_{\star+} = \sum (p_k \otimes p_{k'}) \hat{\delta}(p_{k+k'})$ as in Remark 4.8, we have for any coefficient a of any $\gamma \in \mathscr{D}$ and for any $k, k' \in \mathbb{N}^*$:

$$\|[a \otimes 1, \mathscr{P}_{\star+}](p_k \otimes p_{k'})\| \leqq C_a \sqrt{\frac{m_{k'-1}}{m_{k+k'-1} m_k}}, \tag{16}$$

where (m_k) is the sequence of quantum dimensions associated with γ. Moreover $[a \otimes 1, \mathscr{P}_{\star+}](p_\alpha \otimes p_\beta)$ can only be non-zero when $\alpha = \bar{\gamma}\gamma \cdots \bar{\gamma}_k$ and $\beta = \gamma_k \cdots \gamma_{k+k''-1} \otimes \beta'$ with $k'' \geqq 1$—we have then $k' = k'' + |\beta'|$. Notice that in this case $M_\alpha = m_{|\alpha|}$ and the subobject $\delta \subset \alpha \otimes \beta$ with maximal length is $\bar{\gamma} \cdots \bar{\gamma}_{k+k''} \otimes \beta'$, so that M_β / M_δ equals $m_{k''}/m_{k+k''}$, which is less than $m_{|\beta|}/m_{|\delta|}$. We will use these facts in the proof of Theorem 8.3.

To prove (16), one considers like in the proof of the lemma intertwining projections in $H_\gamma \otimes H_\alpha \otimes H_\beta$: the complete statement of Lemma 2.5 gives then the result with

$m_{k''-1}/m_{k+k''-1}$. But this quotient is less than $m_{k'-1}/m_{k+k'-1}$, because the sequence $(m_{k'-1}/m_{k+k'-1})_{k'}$ is non-decreasing for every k: compare $m_{k'-1}m_{k+k'}$ and $m_{k+k'-1}m_{k'}$ by considering the irreducible decompositions of $H_{k'-1,k+k'}$ and $H_{k+k'-1,k'}$ relative to the appropriate $A_o(Q)$ or $A_u(Q)$.

(4) Using the same starting point as in the proof of Lemma 7.1, we can prove the following result: if $a \in \mathscr{S} \subset S_{\text{red}}$ is a coefficient of the corepresentation $\alpha \in \operatorname{Irr}\mathscr{C}$, and for any $\beta \in \operatorname{Irr}\mathscr{C}$, we have $ap_\beta H \subset \sum\{p_\delta H \mid \delta \subset \alpha \otimes \beta\}$. As a matter of fact one can write, using the notation of the proof, $p_\delta a p_\beta = p_\delta \mathscr{E}_2(\tilde{a} \otimes p_\beta) = \mathscr{E}_2 \hat{\hat{\delta}}(p_\delta)(\tilde{a} \otimes p_\beta)$. But $\tilde{a}$ lies in $p_\alpha H$ by assumption, hence the considered product vanishes if $\delta \not\subset \alpha \otimes \beta$. Similarly, one can check that $(a \otimes 1)\hat{\pi}_4 \hat{\delta}^3(p_\beta)K$ is included in the sum of the $\hat{\pi}_4 \hat{\delta}^3(p_\delta)K$ with $\delta \subset \alpha \otimes \beta \otimes \alpha$. These "propagation properties" will in particular be used in relation with the following elementary fact: if $H = \bigoplus pH = \bigoplus qH$ are orthogonal decompositions of H, and if $f \in L(H)$ is an operator such that $\operatorname{Card}\{q \mid pfq \neq 0\} \leqq N$ for all p, one has $\|f\| \leqq \sqrt{N} \sup \|fq\|$. □

Theorem 7.3. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0 p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree. Let us denote by $\varphi_{+-}(a)$ the operator $p_{+-}(a \otimes 1)p_{+-}$, for any $a \in S_{\text{red}}$.*

(1) *Let $\zeta \in (p_k \otimes p_1)K_{+-}$ and $a \in \mathscr{S} \subset S_{\text{red}}$. The sequence $\left(R\varphi_{+-}(a)r^n\zeta\right)_n$ converges in K_∞ to a vector which only depends on $R\zeta$ and which we denote by $\pi_\infty(a)(R\zeta)$.*

(2) *This defines a $*$-algebra morphism $\pi_\infty : \mathscr{S} \to L(K_\infty)$ which extends by continuity to S_{red}.*

Proof. Let a be an element of $\mathscr{S} \subset S_{\text{red}}$. There exists an integer p such that a can be expressed as a sum of coefficients of corepresentations $\beta \in \operatorname{Irr}\mathscr{C}$ with $|\beta| \leqq p$. We will use in this proof the finite propagation properties of a, see Remark 7.2.4, with respect to two decompositions of K. The first one is simply given by the projections $(p_k \otimes \mathrm{id})$, but the second one is a little bit more subtle. Using the notation of Remark 6.4.2, for $k_0 \in \mathbb{N}$ and $l \in \mathbb{N}^*$ we denote by $Q_{k_0,l}$ the sum of the projections $P_{\beta,\gamma} q_{k_0+l} p_{+-}$ with $(\beta, \gamma) \in \mathscr{I}$ and $|\beta| = k_0$. In other words, the $\hat{S}$-subspace $(p_{k+k_0} \otimes \mathrm{id})Q_{k_0,l}K$ is the sum over $(\beta, \gamma) \subset \mathscr{I}$, $|\beta| = k_0$, of the irreducible subspaces $\beta \otimes \alpha_{2l} \otimes \bar{\beta} \subset (p_{\beta \otimes \alpha_k} \otimes p_{\bar{\gamma}_k})K_{+-}$, where $\alpha_k = \gamma \cdots \gamma_k$ as usual. In particular $Q_{k_0,l}$ commutes to r and we have $RQ_{k_0,l} = Q_{k_0,l}^\infty R$, if $Q_{k_0,l}^\infty$ is the sum of the projections $\hat{\pi}_\infty(p_{\beta \otimes \alpha_{2l} \otimes \bar{\beta}})$.

We first want to bound from above the norm of the commutator $\mathfrak{C}_a = [\varphi_{+-}(a), r]$ on each subspace $Q_{k_0,l}K$. Because $S \otimes 1$ commutes to Θ, and using Proposition 4.3, we see that the operator $p_{+-}(a \otimes 1)p_{\star -}\Theta p_{+-}$ equals $p_{+-}\Theta p_{+\star}(a \otimes 1)p_{+-}$. We subtract and add this quantity from $\mathfrak{C}_a$ and force the apparition of the commutators of Lemma 7.1:

$$\begin{aligned}
\mathfrak{C}_a &= p_{+-}(a \otimes 1)p_{+-}\Theta p_{+-} - p_{+-}\Theta p_{+-}(a \otimes 1)p_{+-} \\
&= -p_{+-}(a \otimes 1)p_{--}\Theta p_{+-} + p_{+-}\Theta p_{++}(a \otimes 1)p_{+-} \\
&= p_{+-}[a \otimes 1, p_{+\star}]p_{--}\Theta p_{+-} - p_{+-}\Theta p_{++}[a \otimes 1, p_{\star +}]p_{+-}. \qquad (17)
\end{aligned}$$

Thanks to Remark 6.4.2, the norm of $p_{--}\Theta p_{+-}$ on $(p_k \otimes \mathrm{id})Q_{k_0,l}K$ is the same as the one on $(p_{k-k_0} \otimes \mathrm{id})Q_{0,l}K$, which is given by Lemma 6.3:

$$(18) \qquad \|p_{--}\Theta p_{+-}(p_k \otimes \mathrm{id})Q_{k_0,l}\|^2 \leqq \frac{M_l M_{l-1}}{m_{k-k_0+1} m_{k-k_0}},$$

where $(m_k)_k$ and $(M_k)_k$ are the minimal and maximal sequences of quantum dimensions from Remark 7.2.1. We proceed in the same way for the second term of (17), but this time we have to consider the restriction of $p_{+-}\Theta p_{++}$ to the subspaces $(p_{k'} \otimes \mathrm{id})Q_{k_0',l'}K$ that meet the image of $(a \otimes 1)(p_k \otimes \mathrm{id})Q_{k_0,l}$. Remark 7.2.4 provides control over the set of indices (k', k_0', l') to be considered, and the fact that quantum dimensions are increasing with the distance to the origin shows that the greatest value of the quantity (18) is obtained when $(k', k_0', l') = (k-p, k_0+p, l+p)$. Putting this together with the estimate of Lemma 7.1 we get

$$\|\mathfrak{C}_a(p_k \otimes \mathrm{id})Q_{k_0,l}\| \leqq 2\frac{C_a}{m_k}\sqrt{\frac{M_{l+p}M_{l+p-1}}{m_{k-k_0-2p+1}m_{k-k_0-2p}}} \leqq \frac{C_{a,k_0,l}}{m_k^2}.$$

Notice that we have used the inequality $m_{k-i} \geqq m_k m_1^{-i}$ and introduced a new constant $C_{a,k_0,l}$ to obtain the estimate order m_k^{-2}.

Now we consider a vector $\zeta \in (p_k \otimes \mathrm{id})Q_{k_0,l}K$, for fixed integers k_0 and l. To prove that the sequence $\left(R\varphi_{+-}(a)r^n\zeta\right)$ converges, it is enough to study the series $\left(\sum R\varphi_{+-}(a)r^{n+1}\zeta - R\varphi_{+-}(a)r^n\zeta\right)$, which can be written as $\left(\sum R\mathfrak{C}_a r^n\zeta\right)$. Because the vector $\mathfrak{C}_a r^n\zeta = [\varphi_{+-}(a), r]r^n\zeta$ belongs to the direct sum of the subspaces $(p_{k+n+i+1} \otimes \mathrm{id})K_{+-}$ with $i \in [\![-p, p]\!]$, we have

$$\|R\mathfrak{C}_a r^n\zeta\| \leqq (2p+1)\|\mathfrak{C}_a r^n\zeta\| \leqq \frac{2p+1}{m_{k+n}^2}C_{a,k_0,l}\|\zeta\|.$$

Now we have $m_{k+n} \geqq k+n+1$, hence the series $\left(\sum_n m_{k+n}^{-2}\right)$ is convergent and the sequence $\left(R\varphi_{+-}(a)r^n\zeta\right)_n$ indeed converges in K_∞. If $R\zeta = R\zeta'$ with $\zeta' \in (p_{k'} \otimes \mathrm{id})K_{+-}$ and $k' \geqq k$, we have $\zeta' = r^{k'-k}\zeta$ by injectivity of $R_{k'}$, hence the associated sequences are equal up to an index shift.

We moreover get an estimate on the norm of $\|\pi_\infty(a)RQ_{k_0,l}\|$: denoting by $(\rho_i)_i$ the sequence of remainders of the series $\left(\sum m_i^{-2}\right)$, we have

$$(19) \qquad \|\pi_\infty(a)(R\zeta) - R\varphi_{+-}(a)\zeta\| \leqq C_{a,k_0,l}(2p+1)\rho_k\|\zeta\|,$$

$$\text{hence} \quad \|\pi_\infty(a)(R\zeta)\| \leqq (2p+1)(\|a\| + C_{a,k_0,l}\rho_k)\|\zeta\|.$$

If we let k go to infinity without changing $R\zeta$, the norm of ζ converges to $\|R\zeta\|$ and we get the upper bound $\|\pi_\infty(a)Q_{k_0,l}^\infty\| \leqq (2p+1)\|a\|$. We finally use Remark 7.2.4 to notice that $\varphi_{+-}(a)Q_{k_0,l}K$ is included in the sum of the $(2p+1)^2$ subspaces $\{Q_{k_0+i_0,l+j}K \mid i_0$ and $i_0 + j \in [\![-p, p]\!]\}$. As a result the same property of "finite propagation" is true for $\pi_\infty(a)$ in the decomposition $K_\infty = \bigoplus Q_{k_0,l}^\infty K_\infty$ and we obtain the inequality $\|\pi_\infty(a)\| \leqq (2p+1)^2\|a\|$.

Let $a, a' \in \mathscr{S} \subset S_{\mathrm{red}}$ and $R_k\zeta \in K_\infty$. By Remark 7.2.2, the norm $\left\|\left(\varphi_{+-}(a)\varphi_{+-}(a') - \varphi_{+-}(aa')\right)r^n\zeta\right\|$ tends to zero as n goes to infinity. By definition of π_∞, the norm $\left\|\left(\pi_\infty(b)R - R\varphi_{+-}(b)\right)r^n\zeta\right\|$, with $b = a, a'$ or aa', also tends to zero. As a result, we see that $\|(\pi_\infty(a)\pi_\infty(a') - \pi_\infty(aa'))Rr^n\zeta\|$ converges to zero with respect to n. But this quantity does not depend on n, and hence we have proved that π_∞ is a morphism of algebras.

In particular, it is enough to prove the identity $\pi_\infty(a^*) = \pi_\infty(a)^*$ for the coefficients a of any $\gamma \in \mathscr{D}$. We have then $\varphi_{+-}(a)(p_{k+n}\otimes \mathrm{id})K \subset (p_{k+n+1}\otimes \mathrm{id})K + (p_{k+n-1}\otimes \mathrm{id})K$. Let $R\zeta, R\xi \in K_\infty$, we can assume that ζ and ξ both lie in some $(p_k \otimes \mathrm{id})K_{+-}$ and we write

$$\begin{aligned}(\pi_\infty(a^*)R\zeta \mid R\xi) &= \lim(R\varphi_{+-}(a)^* r^n\zeta \mid R\xi)\\ &= \lim(\varphi_{+-}(a)^* r^n\zeta \mid r^{n+1}\xi) + \lim(\varphi_{+-}(a)^* r^n\zeta \mid r^{n-1}\xi)\\ &= \lim(r^{n-1}\zeta \mid \varphi_{+-}(a) r^n\xi) + \lim(r^{n+1}\zeta \mid \varphi_{+-}(a) r^n\xi)\\ &= \lim(R\zeta \mid R\varphi_{+-}(a) r^n\xi) = (R\zeta \mid \pi_\infty(a)R\xi).\end{aligned}$$

Finally, let us notice that if the "propagation length" of $a \in \mathscr{S}$ is p, the one of a^n is at most np, so that $\|\pi_\infty(a)^n\| \leqq (2np+1)^2\|a\|^n$ for any n. In particular when $\|a\| < 1$ in S_{red} this proves that $\left(\sum \|\pi_\infty(a)^n\|\right)$ converges, so that the spectral radius of $\pi_\infty(a)$ is less than or equal to 1. If a is moreover hermitian, so is $\pi_\infty(a)$ and we get $\|\pi_\infty(a)\| \leqq 1$. Hence $\pi_\infty : \mathscr{S} \to L(K_\infty)$ is continuous when $\mathscr{S}$ is equipped with the norm of S_{red}. □

Remark 7.4. In the case when $M_\gamma \neq 2$ for all $\gamma \in \mathscr{D}$, the proof of the theorem can be simplified. More precisely, it is enough to use in (17) the evident upper bound 1 for the norms of $p_{--}\Theta p_{+-}$ and $p_{+-}\Theta p_{++}$—and in particular there is no need to introduce the projections $Q_{k_0,l}$ anymore. As a matter of fact, the inequality $\|\mathfrak{C}_a(p_k\otimes \mathrm{id})\| \leqq 2C_a m_k^{-1}$ is sufficient for the rest of the proof because the series $\left(\sum m_k^{-1}\right)$ is geometrically convergent in this case. □

8. Applications

8.1. Property AO. In this section we will denote by λ and $\rho : S \to L(H)$ the left and right regular representations of a Woronowicz C^*-algebra (S,δ), i.e. $\rho(x) = U\lambda(x)U$. They commute and therefore define a representation (λ,ρ) of $S_{\mathrm{red}} \otimes_{\max} S_{\mathrm{red}}$ on H. Besides, we will call $\lambda\otimes\rho$ the natural representation of $S_{\mathrm{red}} \otimes_{\max} S_{\mathrm{red}}$ on $H\otimes H$, so that $(\lambda\otimes\rho)(S_{\mathrm{red}} \otimes_{\max} S_{\mathrm{red}}) = S_{\mathrm{red}}\otimes S_{\mathrm{red}}$. Let $\pi : L(H) \to L(H)/K(H)$ be the quotient map. We say that (S,δ) has Property AO, after Akemann and Ostrand, if $\pi\circ(\lambda,\rho)$ factorizes through $S_{\mathrm{red}}\otimes S_{\mathrm{red}}$.

When the antipode of (S,δ) is involutive, it is easy to see that $(\lambda,\rho)\circ\delta$ contains the trivial representation ε. Hence in this case (λ,ρ) factorizes through $S_{\mathrm{red}}\otimes S_{\mathrm{red}}$ iff (S,δ) is amenable. Consequently, Property AO is only interesting for non-amenable Kac-C^*-algebras and can be seen as a restriction on their non-amenability.

Property AO was first introduced in [1] to study the non-nuclear C^*-algebra $S = C^*(\mathbb{F}_2)$: it was used in this case to show that $S_{\mathrm{red}}\otimes_{\max} S_{\mathrm{red}}/S_{\mathrm{red}}\otimes S_{\mathrm{red}} \simeq K(H)$. This result was generalized to reduced C^*-algebras of ICC discrete groups in [15], where Property AO was also used in conjunction with Property T of Kazhdan to produce non-K-nuclear C^*-algebras. More recently, Property AO was used in [12] in conjunction with local reflexivity to produce solid factors.

The aim of this section is to prove Property AO for the free quantum groups (S, δ) studied in this article. We will use the original method of [1]: the factorization of $\pi \circ (\lambda, \rho)$ arises from an isometry $F : H \to H \otimes H$ such that $F^*(\lambda \otimes \rho)(x)F \equiv (\lambda, \rho)(x) \bmod K(H)$, for any $x \in S_{\mathrm{red}} \otimes_{\max} S_{\mathrm{red}}$. In the case of $\mathbb{F}_2$, the isometry F is the polar part of the closable operator which maps each characteristic function $\mathbb{1}_\alpha \in H$ to the sum of the $\mathbb{1}_{\beta_1} \otimes \mathbb{1}_{\beta_2}$ with $\beta_1\beta_2 = \alpha$ and $|\beta_1| + |\beta_2| = |\alpha|$. In particular, the adjoint of this operator coincides on $\mathscr{H} \otimes \mathscr{H}$ with the natural extension $\mathscr{E}_2\mathscr{P}_{\star+}$ of $E_2 p_{\star+}$. In the quantum case, we also define F from this extension.

Definition 8.1. Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree. Let $\mathscr{E}_2 : \mathscr{H} \otimes \mathscr{H} \to H$ be the operator induced in the GNS construction by the multiplication of S, and $\mathscr{P}_{\star+} = \sum \in \mathbb{N}\hat{\delta}(p_{k+k'})(p_k \otimes p_{k'})$. We define the closed operator F_0 by $F_0^* = \mathscr{E}_2\mathscr{P}_{\star+}$ and we denote by F its polar part.

Lemma 8.2. *We use the hypotheses and notation of Definition* 8.1. *We suppose that* $M_\gamma \neq 2$ *for all* $\gamma \in \mathscr{D}$. *For every* $k \in \mathbb{N}^*$ *we have then* $\|F_0p_k\|^2 \geqq k + 1$, *and there exists a constant* $C > 0$ *such that*

$$\forall \alpha, \alpha' \in \operatorname{Irr}\mathscr{C} \quad \alpha' \subset \mathscr{D} \otimes \alpha \Rightarrow \big|\, \|F_0p_{\alpha'}\|^2 - \|F_0p_\alpha\|^2 \big| \leqq C.$$

Proof. We know from Proposition 4.7 and Remark 4.8 that F_0 is a multiple of an isometry on each subspace $p_\alpha H$, the corresponding norm being given by

$$(20) \qquad \|F_0p_\alpha\|^2 = \sum \left\{ \frac{M_{\beta_1}M_{\beta_2}}{M_\alpha} \,\middle|\, \alpha \subset \beta_1 \otimes \beta_2, |\beta_1| + |\beta_2| = |\alpha| \right\}.$$

Each term of the sum is clearly greater than or equal to 1, and there are $|\alpha| + 1$ terms in the sum: one obtains the admissible pairs (β_1, β_2) by following the geodesic from $1_\mathscr{C}$ to α until an arbitrary point β_1, and then using the remaining sequence of directions to go up from $1_\mathscr{C}$ to β_2. Recall from Lemma 4.4 that the conditions for a sequence of directions to define an ascending path are only local.

To get the second estimate, let us consider an inclusion $\alpha' \subset \gamma \otimes \alpha$ with $\gamma \in \mathscr{D}$. By exchanging α and α' if necessary, one can assume that $|\alpha'| > |\alpha|$. As a first step, we will assume that $\alpha = \bar{\gamma}\gamma \cdots \bar{\gamma}_i$ and $\alpha' = \gamma\bar{\gamma}\gamma \cdots \gamma_i$. We can moreover suppose then that $i \geqq 2$, and hence $\dim \gamma > 1$. Let (m_k) be the sequence of quantum dimensions associated to γ, by hypothesis we have $m_k \sim a^{k+1}/(a - a^{-1})$ for some $a > 1$. We write then

$$f_i := \|F_0p_\alpha\|^2 = \sum_{k+k'=i} \frac{m_k m_{k'}}{m_i} = \frac{a^i}{m_i} \sum_{k+k'=i} \frac{m_k}{a^k} \frac{m_{k'}}{a^{k'}},$$

and similarly $\|F_0p_{\alpha'}\|^2 = f_{i+1}$. But by a variant of Cesaro's Lemma f_i is equivalent to $a(i+1)/(a - a^{-1})$, and in particular $(f_{i+1} - f_i)_i$ is bounded. We take C to be a common bound for these sequences when γ varies in $\mathscr{D}$.

We address now the general case and express α as a tensor product $\bar{\gamma}\gamma\cdots\bar{\gamma}_i\otimes\tilde{\alpha}$, where $\tilde{\alpha}$ does not start with $\bar{\gamma}$, and possibly $i=0$ or $\tilde{\alpha}=1_{\mathscr{C}}$. We have then $M_\alpha=m_iM_{\tilde{\alpha}}$ and $M_{\alpha'}=m_{i+1}M_{\tilde{\alpha}}$. Let us first consider the terms in (20) where $|\beta_1|>i$. One has then $M_{\beta_1}=m_iM_{\tilde{\beta}_1}$ for some $\tilde{\beta}_1$, hence $M_{\beta_1}M_{\beta_2}/M_\alpha=M_{\tilde{\beta}_1}M_{\beta_2}/M_{\tilde{\alpha}}$. If we consider similarly in the expression (20) for $\|F_0p_{\alpha'}\|^2$ the terms where $|\beta_1|>i+1$, we see that $M_{\beta_1}=m_{i+1}M_{\tilde{\beta}_1}$ and $M_{\beta_1}M_{\beta_2}/M_{\alpha'}=M_{\tilde{\beta}_1}M_{\beta_2}/M_{\tilde{\alpha}}$. Hence all these terms can be simplified from the difference $\|F_0p_{\alpha'}\|^2-\|F_0p_\alpha\|^2$. We proceed symmetrically with the terms of (20) where $|\beta_1|\leqq i$ (resp. $i+1$): this time β_2 can be expressed as an irreducible tensor product $\tilde{\beta}_2\otimes\tilde{\alpha}$, the factors $M_{\tilde{\alpha}}$ disappear from the quotient $M_{\beta_1}M_{\beta_2}/M_\alpha$ (resp. $M_{\alpha'}$) and one recognizes f_i (resp. f_{i+1}). As a result we have $\|F_0p_{\alpha'}\|^2-\|F_0p_\alpha\|^2=f_{i+1}-f_i$ and the first step gives the desired upper bound. □

Theorem 8.3. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U=p_1$ and $p_0p_1=0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S,p_1) is a directional tree and that $M_\gamma\neq 2$ for all $\gamma\in\mathscr{D}$. Let $F:H\to H\otimes H$ be the isometry of Definition* 8.1. *Then*

$$F^*(\lambda\otimes\rho)(x)F\equiv(\lambda,\rho)(x)\bmod K(H),$$

for any $x\in S_{\mathrm{red}}\otimes_{\max}S_{\mathrm{red}}$. In particular (S,δ) has Property AO.

Proof. By symmetry one can assume that $x=a\otimes 1$ with $a\in S_{\mathrm{red}}$ a coefficient of some $\gamma\in\mathscr{D}$. We put $G=(F_0^*F_0)^{-\frac{1}{2}}$, so that $F=F_0G$. We have then

$$aF^*-F^*(a\otimes 1)=G[G^{-1},a]F^*+G\mathscr{E}_2[\mathscr{P}_{\star+},a\otimes 1].$$

By the first statement of Lemma 8.2, the operator G is compact. Hence it suffices to prove that $[G^{-1},a]$ and $\mathscr{E}_2[\mathscr{P}_{\star+},a\otimes 1]$ are bounded.

For the first commutator, we remark that $ap_\alpha=(p_{\alpha'}+p_{\alpha''})ap_\alpha$ if $\gamma\otimes\alpha=\alpha'\oplus\alpha''$. Moreover we have $Gp_\alpha=\|F_0p_\alpha\|^{-1}p_\alpha$, so that

$$[G^{-1},a]p_\alpha=(\|F_0p_{\alpha'}\|-\|F_0p_\alpha\|)p_{\alpha'}ap_\alpha+(\|F_0p_{\alpha''}\|-\|F_0p_\alpha\|)p_{\alpha''}ap_\alpha.$$

Hence the result follows from the second statement of Lemma 8.2, after factoring out $\|F_0p_{\alpha'}\|+\|F_0p_\alpha\|$ from it. (In fact this even proves that $[G^{-1},a]$ is compact.)

For the second commutator, we will assume that $\dim\gamma>1$: otherwise the proof is as easy as in the classical case. Denote by $(m_k)_k$ the sequence of quantum dimensions associated with γ. We use the Remarks 4.8 and 7.2.3 to write

$$\begin{aligned}&\|p_\alpha\mathscr{E}_2[\mathscr{P}_{\star+},a\otimes 1]\|^2\\&\leqq\sum\|p_\alpha\mathscr{E}_2(p_{\beta_1}\otimes p_{\beta_2})\|^2\times\|[\mathscr{P}_{\star+},a^*\otimes 1](p_{\beta_1}\otimes p_{\beta_2})\|^2\\&\leqq C_a\sum\frac{m_{|\beta_1|}m_{|\beta_2|}}{m_{|\alpha|}}\,\frac{m_{|\beta_2|-1}}{m_{|\alpha|-1}m_{|\beta_1|}}=\frac{C_a}{m_{|\alpha|}m_{|\alpha|-1}}\sum_{k=0}^{|\alpha|}m_km_{k-1}.\end{aligned}$$

The last upper estimate is bounded because $m_k\sim\dfrac{a^{k+1}}{a-a^{-1}}$ for some $a>1$. □

Remark 8.4. Recall that $A_u(Q)$ is never amenable and that $A_o(Q)$ is amenable iff $n = 2$ [5]. Hence the only case, up to free products, where Property AO is neither trivial nor proved by Theorem 8.3, is the one of $A_u(I_2)$. Property AO may however be fulfilled in this case, too. □

8.2. *KK*-theory. The notion of K-amenability was first introduced by Cuntz [7] for discrete groups: the aim was to give a simpler proof to a result of Pimsner and Voiculescu [13] calculating the K-theory of the reduced C^*-algebras of free groups. Cuntz proves that the K-theory of the reduced and full C^*-algebras of a free group are the same, and gives in [6] a simple way to compute it in the full case.

Julg and Valette extended then the notion of K-amenability to the locally compact case and established the K-amenability of locally compact groups acting on trees with amenable stabilizers [8]. This includes the case of the free groups acting on their Cayley graphs. To prove the K-amenability of a locally compact group G, one has to construct an element $\alpha \in KK_G(\mathbb{C}, \mathbb{C})$ using representations of G that are weakly contained in the regular one, and then to prove that α is homotopic to the unit element of $KK_G(\mathbb{C}, \mathbb{C})$. In [8], both of the steps are carried out in a very geometric way. Moreover, it turns out that α can be interpreted as the γ element used to prove the Baum-Connes conjecture in this context [9].

We refer the reader to [2], [17] for details about equivariant KK-theory with respect to Hopf C^*-algebras, and we just recall the equivalent characterizations of K-amenability for a discrete quantum group defined by its full and reduced Woronowicz C^*-algebras S, S_{red}:

(i) $\mathbb{1} \in KK_{\hat{S}}(\mathbb{C}, \mathbb{C})$ can be represented by a triple (E, π, F) such that the representation of S on E factors through S_{red}.

(ii) For every C^*-algebra A endowed with a coaction of $\hat{S}$, $[\lambda_A] \in KK(A \rtimes S, A \rtimes_{\mathrm{red}} S)$ is invertible.

(iii) $[\lambda] \in KK(S, S_{\mathrm{red}})$ is invertible.

(iv) There exists $\alpha \in KK(S_{\mathrm{red}}, \mathbb{C})$ such that $\lambda^*(\alpha) = [\varepsilon] \in KK(S, \mathbb{C})$.

In this subsection we explain how to construct an element $\alpha \in KK(S_{\mathrm{red}}, \mathbb{C})$ from the quantum Cayley graph of a free quantum group. It is the natural quantum generalization of the Julg-Valette element mentioned above. It has index 1, however further work is needed to determine whether $\lambda^*(\alpha) = [\varepsilon]$.

Theorem 8.5. *Let S be a Woronowicz C^*-algebra and p_1 a central projection of $\hat{S}$ such that $Up_1U = p_1$ and $p_0 p_1 = 0$. Assume that the classical Cayley graph $\mathfrak{G}$ of (S, p_1) is a directional tree and that $M_\gamma \neq 2$ for all $\gamma \in \mathscr{D}$. Then $E_2 p_{++} : K_g \to H$ and $E_2(Rs)^* : K_\infty \to H$ commute to the actions of S_{red} modulo compact operators. In particular $E_2\big(p_{++} + (Rs)^*\big)$ defines an element $\alpha \in KK(S_{\mathrm{red}}, \mathbb{C})$ of index* 1.

Proof. In this proof we will denote by $p_{\geqq k_0}$ the sum of the projections p_k with $k \geqq k_0$. We have $E_2 p_{++} = E_2 p_{\star +}$ and the target operator E_2 intertwines the actions of S_{red}, hence it is enough to prove that $p_{\star +}$ commutes to $S_{\mathrm{red}} \otimes 1$ up to compact operators. If

$a \in S_{\mathrm{red}}$ is a coefficient of some $\gamma \in \mathscr{D}$, this results directly from Lemma 7.1: because (m_k^{-1}) is decreasing and $p_{k'} a p_k$ vanishes as soon as $|k - k'| \neq 1$, we have

$$\|[p_{\star+}, a \otimes 1](p_k \otimes \mathrm{id})\| \leqq C_a m_k^{-1} \Rightarrow \|[p_{\star+}, a \otimes 1](p_{\geqq k_0} \otimes \mathrm{id})\| \leqq 2 C_a m_{k_0}^{-1}.$$

This proves that $[p_{\star+}, a \otimes 1]$ is compact, and the general result follows because the coefficients a of the corepresentations $\gamma \in \mathscr{D}$ span the C^*-algebra S_{red}.

For the case of $(Rs)^*$ we will use the proof of Theorem 7.3. Thanks to the hypothesis we can take into account the simplification of Remark 7.4: we avoid the use of the projections $Q_{k_0,l}$ by taking for ρ_k the remainder of $\left(\sum m_k^{-1}\right)$ instead of $\left(\sum m_k^{-2}\right)$. Equation (19) reads then

$$\left\|\left(\pi_\infty(a) R - R p_{+-}(a \otimes 1)\right)(p_k \otimes \mathrm{id})\right\| \leqq (2p+1) C_a \rho_k.$$

We notice also that ρ_k is again equivalent to a multiple of m_k^{-1} because (m_k) grows geometrically. To conclude we use Lemma 7.1 and the fact that $(a \otimes 1)$ commutes to Θ: up to a change of the constant C_a, we obtain $\left\|\left(\pi_\infty(a) Rs - Rs(a \otimes 1)\right)(p_k \otimes \mathrm{id})\right\| \leqq C_a m_k^{-1}$. Summing over $k \geqq k_0$ we obtain an inequality showing that $\left(\pi_\infty(a) Rs - Rs(a \otimes 1)\right)$ is compact:

$$\left\|\left(\pi_\infty(a) Rs - Rs(a \otimes 1)\right)(p_{\geqq k_0} \otimes \mathrm{id})\right\| \leqq C_a \rho_{k_0}.$$

Finally $E_2\left(p_{++} + (Rs)^*\right)$ defines an element $\alpha \in KK(S_{\mathrm{red}}, \mathbb{C})$ of index 1 because $E_2 : K_{++} \to (1 - p_0) H$ is invertible by Proposition 4.7, as well as

$$p_{++} + (Rs)^* : K_g \oplus K_\infty \to K_{++}$$

by Theorem 5.3, Proposition 6.2 and Theorem 6.5. □

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