THE BOUNDARY OF UNIVERSAL DISCRETE QUANTUM GROUPS, EXACTNESS, AND FACTORIALITY

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Abstract

We study the C^*-algebras and von Neumann algebras associated with the universal discrete quantum groups. They give rise to full prime factors and simple exact C^*-algebras. The main tool in our work is the study of an amenable boundary action, yielding the Akemann-Ostrand property. Finally, this boundary can be identified with the Martin or the Poisson boundary of a quantum random walk.

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0. Introduction

Since Murray and von Neumann introduced *von Neumann algebras*, the ones associated with discrete groups played a prominent role. The main aim of this article is to show how concrete examples of discrete quantum groups give rise to interesting C^*-algebras and von Neumann algebras.

In the 1980s, Woronowicz [29] introduced the notion of a *compact quantum group* and generalized the classical Peter-Weyl representation theory. Many fascinating

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examples of compact quantum groups are available by now. Drinfel'd [10] and Jimbo [15] introduced the *q-deformations of compact semisimple Lie groups,* and Rosso [19] showed that they fit into the theory of Woronowicz. The *universal orthogonal and unitary* quantum groups were introduced by Van Daele and Wang [25] and studied in detail by Banica [3], [4]. Large classes of compact quantum groups arise as symmetry groups (see, e.g., [28]).

This article mainly deals with the universal orthogonal quantum group $\mathbb{G} = A_o(F)$, defined from a matrix $F \in \mathrm{GL}(n, \mathbb{C})$ satisfying $F\overline{F} = \pm 1$. Its underlying C^*-algebra $C(\mathbb{G})$ is the universal C^*-algebra generated by the entries of a unitary $(n \times n)$-matrix (U_{ij}) satisfying $(U_{ij}) = F(U_{ij}^*)F^{-1}$. Using the Gel'fand-Naĭmark-Segal (GNS) construction of the (unique) Haar state of $\mathbb{G}$, we obtain the reduced C^*-algebra $C(\mathbb{G})_{\mathrm{red}}$ and the von Neumann algebra $C(\mathbb{G})''_{\mathrm{red}}$. This article deals with a detailed study of these operator algebras. Note that for $n \geq 3$, $C(\mathbb{G})_{\mathrm{red}}$ is a nontrivial quotient of $C(\mathbb{G})$ by nonamenability of the discrete quantum group $\widehat{\mathbb{G}}$.

In Section 3, we construct a *boundary* for the dual $\widehat{\mathbb{G}}$ of $\mathbb{G} = A_o(F)$. This boundary $\mathscr{B}_\infty$ is a unital C^*-algebra that admits a natural action of $\widehat{\mathbb{G}}$. In Section 4, we introduce the notion of an *amenable action* of a discrete quantum group on a unital C^*-algebra. This definition involves a nontrivial algebraic condition, which is the proper generalization of Anantharaman-Delaroche's centrality condition (see [1, Théorème 3.3]). We then prove that the boundary action of $\widehat{\mathbb{G}}$ is amenable. The construction of the boundary $\mathscr{B}_\infty$ and the proof of the amenability of the boundary action involve precise estimates on the representation theory of $A_o(F)$. These estimates are dealt with in the appendix.

From the amenability of the boundary action of the dual of $\mathbb{G} = A_o(F)$, we deduce that the reduced C^*-algebra $C(\mathbb{G})_{\mathrm{red}}$ is *exact* and satisfies the *Akemann-Ostrand property*. In the setting of finite von Neumann algebras, Ozawa [18] showed that the Akemann-Ostrand property implies solidity of the associated von Neumann algebra. Since, in general, $C(\mathbb{G})''_{\mathrm{red}}$ is of type III, we need a generalization of Ozawa's definition (see Section 2), and we deduce that for $\mathbb{G} = A_o(F)$, the von Neumann algebras $C(\mathbb{G})''_{\mathrm{red}}$ are *generalized solid*. In particular, for F the $n \times n$ identity matrix, we get a solid von Neumann algebra.

In Section 5, we make the link between our boundary $\mathscr{B}_\infty$ for the dual of $\mathbb{G} = A_o(F)$ and boundaries arising from quantum random walks on $\widehat{\mathbb{G}}$. First, we construct a *harmonic state* ω_∞ on $\mathscr{B}_\infty$ and identify the von Neumann algebra $(\mathscr{B}_\infty, \omega_\infty)''$ with the *Poisson boundary* of a random walk on $\widehat{\mathbb{G}}$. Note that Poisson boundaries of discrete quantum groups were defined by Izumi [13], who computed them for the dual of $\mathrm{SU}_q(2)$. This computation was then extended to the dual of $\mathrm{SU}_q(n)$ in [14]. A recent preprint of Tomatsu [21] provides a computation for the Poisson boundary of the dual of an arbitrary q-deformed compact Lie group. Second, we identify the C^*-algebra $\mathscr{B}_\infty$ with the *Martin boundary* of a random walk on $\widehat{\mathbb{G}}$.

Note that Martin boundaries of discrete quantum groups were defined by Neshveyev and Tuset [17] and computed there for the dual of $\mathrm{SU}_q(2)$. Based on the results of our article and [13], the Poisson and Martin boundaries for $\widehat{\mathbb{G}}$ are identified in [23] with concrete von Neumann and C^*-algebras, given by generators and relations.

In the short Section 6, we provide a general exactness result for quantum group C^*-algebras $C(\mathbb{G})_{\mathrm{red}}$. We show that for *monoidally equivalent* quantum groups $\mathbb{G}$ and $\mathbb{G}_1$ (see [6]), $C(\mathbb{G})_{\mathrm{red}}$ is exact if and only if $C(\mathbb{G}_1)_{\mathrm{red}}$ is exact. This provides an alternative proof for the exactness of $C(\mathbb{G})_{\mathrm{red}}$ when $\mathbb{G} = A_o(F)$ and proves exactness in other examples as well.

In Section 7, we deal with *factoriality* of the von Neumann algebra $C(\mathbb{G})''_{\mathrm{red}}$ and *simplicity* of the C^*-algebra $C(\mathbb{G})_{\mathrm{red}}$ whenever $\mathbb{G} = A_o(F)$ and F is at least a (3×3)-matrix. We were only able to settle factoriality and simplicity assuming an extra condition on the norm of F. If $\sqrt{5}\|F\|^2 \leq \mathrm{Tr}(F^*F)$, the von Neumann algebra $C(\mathbb{G})''_{\mathrm{red}}$ is a *full factor* and we compute its Connes invariants. If $8\|F\|^8 \leq 3\,\mathrm{Tr}(F^*F)$, the C^*-algebra $C(\mathbb{G})_{\mathrm{red}}$ is simple. Both conditions are satisfied when F is sufficiently close to the $n \times n$ identity matrix for $n \geq 3$. Moreover, it is our belief that they are superfluous. Note that simplicity of the reduced C^*-algebra of the universal unitary quantum groups $A_u(F)$ was proved by Banica [4]. For $\mathbb{G} = A_u(F)$, the fusion algebra can be described using the free monoid $\mathbb{N} * \mathbb{N}$, while for $\mathbb{G} = A_o(F)$, the fusion algebra is the same as that of SU(2) and is, in particular, abelian. For that reason, Banica's approach is closer to Powers's proof of the simplicity of $C^*_r(\mathbb{F}_n)$.

For the convenience of the reader, we included a rather extensive section of preliminaries, dealing with the general theory of compact/discrete quantum groups, their actions on C^*-algebras, and exactness.

1. Preliminaries

We use the symbol $\otimes$ to denote several types of tensor products. In particular, $\otimes$ denotes the *minimal* tensor product of C^*-algebras. The *maximal* tensor product is denoted by $\otimes_{\max}$. If we want to stress the difference with the minimal tensor product, we write $\otimes_{\min}$. We also make use of the leg numbering notation in multiple tensor products. If $a \in A \otimes A$, then a_{12}, a_{13}, a_{23} denote the obvious elements in $A \otimes A \otimes A$ (e.g., $a_{12} = a \otimes 1$).

Compact quantum groups

We briefly overview the theory of compact quantum groups developed by Woronowicz [29]. We refer to the survey article [16] for a smooth approach to these results.

Definition 1.1

A *compact quantum group* $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$, where

- $C(\mathbb{G})$ is a unital C^*-algebra;

- $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital *-homomorphism satisfying the *coassociativity* relation

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta;$$

- $\mathbb{G}$ satisfies the *left and right cancellation properties* expressed by

$$\Delta\big(C(\mathbb{G})\big)\big(1 \otimes C(\mathbb{G})\big) \text{ and } \Delta\big(C(\mathbb{G})\big)\big(C(\mathbb{G}) \otimes 1\big) \text{ are total in } C(\mathbb{G}) \otimes C(\mathbb{G}).$$

Remark 1.2
We use the fancy notation $C(\mathbb{G})$ to suggest the analogy with the basic example given by continuous functions on a compact group. In the quantum case, however, there is no underlying space $\mathbb{G}$, and $C(\mathbb{G})$ is a nonabelian C^*-algebra.

The two major aspects of the general theory of compact quantum groups are the existence and uniqueness of a Haar measure and the Peter-Weyl representation theory.

THEOREM 1.3
Let $\mathbb{G}$ be a compact quantum group. There exists a unique state h on $C(\mathbb{G})$ satisfying $(\mathrm{id} \otimes h)\Delta(a) = h(a)1 = (h \otimes \mathrm{id})\Delta(a)$ *for all $a \in C(\mathbb{G})$. The state h is called the* Haar state *of* $\mathbb{G}$.

Definition 1.4
A *unitary representation U* of a compact quantum group $\mathbb{G}$ on a Hilbert space H is a unitary element $U \in \mathrm{M}(\mathcal{K}(H) \otimes C(\mathbb{G}))$ satisfying

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}. \tag{1.1}$$

Whenever U^1 and U^2 are unitary representations of $\mathbb{G}$ on the respective Hilbert spaces H_1 and H_2, we define

$$\mathrm{Mor}(U^1, U^2) := \big\{T \in \mathrm{B}(H_2, H_1) \,\big|\, U_1(T \otimes 1) = (T \otimes 1)U_2\big\}.$$

The elements of $\mathrm{Mor}(U^1, U^2)$ are called *intertwiners*. We use the notation $\mathrm{End}(U) := \mathrm{Mor}(U, U)$. A unitary representation U is said to be *irreducible* if $\mathrm{End}(U) = \mathbb{C}1$. Two unitary representations U^1 and U^2 are said to be *unitarily equivalent* when $\mathrm{Mor}(U^1, U^2)$ contains a unitary operator.

The following result is crucial.

THEOREM 1.5
Every irreducible representation of a compact quantum group is finite dimensional. Every unitary representation is unitarily equivalent to a direct sum of irreducibles.

Because of this theorem, we almost exclusively deal with finite-dimensional representations, the regular representation being the exception. By choosing an orthonormal basis of the Hilbert space H, a finite-dimensional unitary representation of $\mathbb{G}$ can be considered as a unitary matrix (U_{ij}) with entries in $C(\mathbb{G})$, and (1.1) becomes

$$\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}.$$

The product in the C^*-algebra $C(\mathbb{G})$ yields a tensor product on the level of unitary representations.

Definition 1.6
Let U^1 and U^2 be unitary representations of $\mathbb{G}$ on the respective Hilbert spaces H_1 and H_2. We define the tensor product

$$U^1 \textcircled{\scriptsize T}\, U^2 := U^1_{13} U^2_{23} \in \mathrm{M}\big(\mathcal{K}(H_1 \otimes H_2) \otimes C(\mathbb{G})\big).$$

Notation 1.7
We denote by $\mathrm{Irred}(\mathbb{G})$ the set of (equivalence classes) of irreducible unitary representations of a compact quantum group $\mathbb{G}$. We choose representatives U^x on the Hilbert space H_x for every $x \in \mathrm{Irred}(\mathbb{G})$. Whenever $x, y \in \mathrm{Irred}(\mathbb{G})$, we use $x \otimes y$ to denote the unitary representation $U^x \textcircled{\scriptsize T}\, U^y$. The class of the trivial unitary representation is denoted by ε.

The set $\mathrm{Irred}(\mathbb{G})$ is equipped with a natural involution $x \mapsto \overline{x}$ such that $U^{\overline{x}}$ is the unique (up to unitary equivalence) irreducible unitary representations such that

$$\mathrm{Mor}(x \otimes \overline{x}, \varepsilon) \neq 0 \neq \mathrm{Mor}(\overline{x} \otimes x, \varepsilon).$$

The unitary representation $U^{\overline{x}}$ is called the contragredient of U^x.

The irreducible representations of $\mathbb{G}$ and the Haar state h are connected by the *orthogonality relations*. For every $x \in \mathrm{Irred}(\mathbb{G})$, we have a unique invertible positive self-adjoint element $Q_x \in \mathrm{B}(H_x)$ satisfying $\mathrm{Tr}(Q_x) = \mathrm{Tr}(Q_x^{-1})$ and

$$\begin{aligned} (\mathrm{id} \otimes h)\big(U^x (A \otimes 1)(U^x)^*\big) &= \frac{\mathrm{Tr}(Q_x A)}{\mathrm{Tr}(Q_x)} 1, \\ (\mathrm{id} \otimes h)\big((U^x)^* (A \otimes 1) U^x\big) &= \frac{\mathrm{Tr}(Q_x^{-1} A)}{\mathrm{Tr}(Q_x^{-1})} 1, \end{aligned} \tag{1.2}$$

for all $A \in \mathrm{B}(H_x)$.

Definition 1.8
For $x \in \mathrm{Irred}(\mathbb{G})$, the value $\mathrm{Tr}(Q_x)$ is called the *quantum dimension* of x and is denoted by $\dim_q(x)$. Note that $\dim_q(x) \geq \dim(x)$ with equality holding if and only if $Q_x = 1$.

Discrete quantum groups and duality
A discrete quantum group is defined as the dual of a compact quantum group, putting together all irreducible representations.

Definition 1.9
Let $\mathbb{G}$ be a compact quantum group. We define the dual (discrete) quantum group $\widehat{\mathbb{G}}$ as follows:

$$c_0(\widehat{\mathbb{G}}) = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} \mathrm{B}(H_x),$$

$$\ell^\infty(\widehat{\mathbb{G}}) = \prod_{x \in \mathrm{Irred}(\mathbb{G})} \mathrm{B}(H_x).$$

We denote the minimal central projections of $\ell^\infty(\widehat{\mathbb{G}})$ by p_x, $x \in \mathrm{Irred}(\mathbb{G})$.

We have a natural unitary $\mathbb{V} \in \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ given by

$$\mathbb{V} = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} U^x.$$

The unitary $\mathbb{V}$ implements the duality between $\mathbb{G}$ and $\widehat{\mathbb{G}}$. We have a natural comultiplication

$$\hat{\Delta} : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}}) \mathbin{\overline{\otimes}} \ell^\infty(\widehat{\mathbb{G}}) : (\hat{\Delta} \otimes \mathrm{id})(\mathbb{V}) = \mathbb{V}_{13}\mathbb{V}_{23}.$$

The notation introduced above is aimed to suggest the basic example where $\mathbb{G}$ is the dual of a discrete group Γ, given by $C(\mathbb{G}) = C^*(\Gamma)$ and $\Delta(\lambda_x) = \lambda_x \otimes \lambda_x$ for all $x \in \Gamma$. The map $x \mapsto \lambda_x$ yields an identification of Γ and $\mathrm{Irred}(\mathbb{G})$, and then $\ell^\infty(\widehat{\mathbb{G}}) = \ell^\infty(\Gamma)$.

Remark 1.10
It is, of course, possible to give an intrinsic definition of a discrete quantum group (not as the dual of a compact quantum group). This was already implicitly clear in Woronowicz's work and was explicitly done in [11] and [24]. For our purposes, it is most convenient to take the compact quantum group as a starting point. Indeed, all

interesting examples of concrete discrete quantum groups are defined as the dual of certain compact quantum groups. We present one particular class at the end of this section.

The discrete quantum group $\ell^\infty(\widehat{\mathbb{G}})$ comes equipped with a natural modular structure.

Notation 1.11
We choose *unit* vectors $t_x \in H_x \otimes H_{\overline{x}}$ invariant under $U^x \mathbin{\textcircled{\scriptsize T}} U^{\overline{x}}$. The vectors t_x are unique up to multiplication by $\mathbb{T}$. We then have canonically defined states φ_x and ψ_x on $\mathrm{B}(H_x)$ related to (1.2) as follows:

$$\psi_x(A) = t_x^*(A \otimes 1)t_x = \frac{\mathrm{Tr}(Q_x A)}{\mathrm{Tr}(Q_x)} = (\mathrm{id} \otimes h)\big(U^x(A \otimes 1)(U^x)^*\big),$$

and

$$\varphi_x(A) = t_{\overline{x}}^*(1 \otimes A)t_{\overline{x}} = \frac{\mathrm{Tr}(Q_x^{-1} A)}{\mathrm{Tr}(Q_x^{-1})} = (\mathrm{id} \otimes h)\big((U^x)^*(A \otimes 1)U^x\big),$$

for all $A \in \mathrm{B}(H_x)$. As a complement to the vectors t_x, we also choose *unit* vectors $s_x \in \mathrm{Mor}(x \otimes \overline{x}, \varepsilon)$ normalized such that $(s_x^* \otimes 1)(1 \otimes t_{\overline{x}}) = 1/\dim_q(x)$ for all $x \in \mathrm{Irred}(\mathbb{G})$. In certain examples, one can consistently choose $s_x = t_x$, but this is not always the case.

The states φ_x and ψ_x are significant since they provide a formula for the invariant weights on $\ell^\infty(\widehat{\mathbb{G}})$.

PROPOSITION 1.12
The left-invariant weight $\widehat{h}_L$ and the right-invariant weight $\widehat{h}_R$ on $\widehat{\mathbb{G}}$ are given by

$$\widehat{h}_L = \sum_{x \in \mathrm{Irred}(\mathbb{G})} \dim_q(x)^2 \psi_x \qquad \textit{and} \qquad \widehat{h}_R = \sum_{x \in \mathrm{Irred}(\mathbb{G})} \dim_q(x)^2 \varphi_x.$$

The following formula is used several times in the article.

PROPOSITION 1.13
Let $x, y \in \mathrm{Irred}(\mathbb{G})$, and suppose that $p_z^{x \otimes y} \in \mathrm{End}(x \otimes y)$ is an orthogonal projection onto a subrepresentation equivalent with $z \in \mathrm{Irred}(\mathbb{G})$. Then

$$(\mathrm{id} \otimes \psi_y)(p_z^{x \otimes y}) = \frac{\dim_q(z)}{\dim_q(x)\dim_q(y)} 1$$

and

$$(\varphi_x \otimes \mathrm{id})(p_z^{x \otimes y}) = \frac{\dim_q(z)}{\dim_q(x)\dim_q(y)} 1.$$

Proof
Since $(\mathrm{id} \otimes \psi_y)(p_z^{x \otimes y}) = (1 \otimes t_y^*)(p_z^{x \otimes y} \otimes 1)(1 \otimes t_y) \in \mathrm{End}(x) = \mathbb{C}1$, it suffices to check that $(\psi_x \otimes \psi_y)(p_z^{x \otimes y}) = \dim_q(z)/(\dim_q(x) \dim_q(y))$, which immediately follows from the formula $(Q_x \otimes Q_y)T = T Q_z$ for all $T \in \mathrm{Mor}(x \otimes y, z)$. □

Regular representations
Both the algebras $C(\mathbb{G})$ and $c_0(\widehat{\mathbb{G}})$ have two natural representations on the same Hilbert space.

Using (1.2), we canonically identify the GNS Hilbert space $L^2(C(\mathbb{G}), h)$ with

$$L^2(\mathbb{G}) := \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} (H_x \otimes H_{\overline{x}})$$

by taking

$$\begin{aligned} \rho : C(\mathbb{G}) \to \mathrm{B}\big(L^2(\mathbb{G})\big) : \rho\big((\omega_{\eta,\xi} \otimes \mathrm{id})(U^x)\big)\xi_0 = \xi \otimes (1 \otimes \eta^*)t_{\overline{x}}, \\ \lambda : C(\mathbb{G}) \to \mathrm{B}\big(L^2(\mathbb{G})\big) : \lambda\big((\omega_{\eta,\xi} \otimes \mathrm{id})(U^x)\big)\xi_0 = (1 \otimes \eta^*)t_{\overline{x}} \otimes \xi, \end{aligned} \tag{1.3}$$

for all $x \in \mathrm{Irred}(\mathbb{G})$ and all $\xi, \eta \in H_x$. Here, ξ_0 denotes the canonical unit vector in $H_\varepsilon \otimes H_\varepsilon = \mathbb{C}$. We use the notation $\omega_{\eta,\xi}(a) = \langle \eta, a\xi \rangle$, and we use inner products that are linear in the second variable.

Notation 1.14
Let $\mathbb{G}$ be a compact quantum group with Haar state h. We denote by $C(\mathbb{G})_{\mathrm{red}}$ the C^*-algebra $\rho(C(\mathbb{G}))$ given by the GNS construction for h. We denote by $C(\mathbb{G})''_{\mathrm{red}}$ the generated von Neumann algebra.

The aim of this article is a careful study of $C(\mathbb{G})_{\mathrm{red}}$ and $C(\mathbb{G})''_{\mathrm{red}}$ for certain concrete examples of compact quantum groups. The previous paragraph clearly suggests that these algebras are the natural counterparts of $C_r^*(\Gamma)$ and $\mathscr{L}(\Gamma)$ for a discrete group Γ.

We introduce the left- and right-regular representations for $\mathbb{G}$ and $\widehat{\mathbb{G}}$. For the convenience of the reader, we provide several explicit formulas.

Definition 1.15
The right-regular representation $\mathscr{V} \in \mathscr{L}(L^2(\mathbb{G}) \otimes C(\mathbb{G}))$ of $\mathbb{G}$ and the left-regular representation $\mathscr{W} \in \mathscr{L}(C(\mathbb{G}) \otimes L^2(\mathbb{G}))$ are defined as

$$\mathscr{V}\big(\rho(a)\xi_0 \otimes 1\big) = \big((\rho \otimes \mathrm{id})\Delta(a)\big)(\xi_0 \otimes 1),$$

$$\mathscr{W}^*\big(1 \otimes \rho(a)\xi_0\big) = \big((\mathrm{id} \otimes \rho)\Delta(a)\big)(1 \otimes \xi_0).$$

Recall that $\mathbb{V} \in \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ is given by $\mathbb{V} = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} U^x$.

Notation 1.16
We define

$$\widehat{\lambda} : \ell^\infty(\widehat{\mathbb{G}}) \to \mathrm{B}\big(L^2(\mathbb{G})\big) : \widehat{\lambda}(a)\xi_x = (a_x \otimes 1)\xi_x \quad \text{for all } a \in \ell^\infty(\widehat{\mathbb{G}}), \xi_x \in H_x \otimes H_{\overline{x}},$$

$$\widehat{\rho} : \ell^\infty(\widehat{\mathbb{G}}) \to \mathrm{B}\big(L^2(\mathbb{G})\big) : \widehat{\rho}(a)\xi_x = (1 \otimes a_{\overline{x}})\xi_x \quad \text{for all } a \in \ell^\infty(\widehat{\mathbb{G}}), \xi_x \in H_x \otimes H_{\overline{x}}.$$

We define the unitary $u \in \mathrm{B}(L^2(\mathbb{G}))$ by $u(\xi \otimes \eta) = \eta \otimes \xi$ for $\xi \in H_x, \eta \in H_{\overline{x}}$. Note that $\widehat{\rho} = (\mathrm{Ad}\, u)\widehat{\lambda}$ and $u^2 = 1$.

PROPOSITION 1.17
The left- and right-regular representations of $\mathbb{G}$ are given by

$$\mathscr{V} = (\widehat{\lambda} \otimes \mathrm{id})(\mathbb{V}) \qquad \textit{and} \qquad \mathscr{W} = (\mathrm{id} \otimes \widehat{\rho})(\mathbb{V}_{21}).$$

So, as it should be, the left- and right-regular representations of $\mathbb{G}$ give rise to two commuting representations of $\ell^\infty(\widehat{\mathbb{G}})$. We now symmetrically and explicitly write down how the left- and right-regular representations of $\widehat{\mathbb{G}}$ give rise to two commuting representations of $C(\mathbb{G})$ (see Proposition 1.20).

We explicitly perform the GNS construction for the weights $\widehat{h}_L$ and $\widehat{h}_R$ (see Proposition 1.12) in order to give formulas for the left- and right-regular representations of $\widehat{\mathbb{G}}$. Recall the choice of unit vectors s_x made in Notation 1.11.

Notation 1.18
Let $a \in \ell^\infty(\widehat{\mathbb{G}})$. We define, whenever the right-hand side makes sense,

$$\widehat{\Lambda}_L(a) = \sum_{x \in \mathrm{Irred}(\mathbb{G})} \dim_q(x)(ap_x \otimes 1)s_x$$

and

$$\widehat{\Lambda}_R(a) = \sum_{x \in \mathrm{Irred}(\mathbb{G})} \dim_q(x)u(1 \otimes ap_x)s_{\overline{x}}.$$

The maps $\widehat{\Lambda}_L$ and $\widehat{\Lambda}_R$, together with the representation $\widehat{\lambda} : \ell^\infty(\widehat{\mathbb{G}}) \to \mathrm{B}(L^2(\mathbb{G}))$, provide a GNS construction for $\widehat{h}_L$ and $\widehat{h}_R$, respectively.

Definition 1.19
The left-regular representation $\widehat{\mathscr{W}} \in \mathscr{L}(c_0(\widehat{\mathbb{G}}) \otimes L^2(\mathbb{G}))$ and the right-regular representation $\widehat{\mathscr{V}} \in \mathscr{L}(L^2(\mathbb{G}) \otimes c_0(\mathbb{G}))$ are defined as

$$\widehat{\mathscr{W}}^*\big(1 \otimes \widehat{\Lambda}_L(a)\big) = (\mathrm{id} \otimes \widehat{\Lambda}_L)\hat{\Delta}(a) \qquad \text{and} \qquad \widehat{\mathscr{V}}\big(\widehat{\Lambda}_R(a) \otimes 1\big) = (\widehat{\Lambda}_R \otimes \mathrm{id})\hat{\Delta}(a),$$

for all $a \in \ell^\infty(\widehat{\mathbb{G}})$ where $\widehat{\Lambda}_L(a)$ (resp., $\widehat{\Lambda}_R(a)$) makes sense.

PROPOSITION 1.20
The left- and right-regular representations of $\widehat{\mathbb{G}}$ *are given, respectively, by*

$$\widehat{\mathscr{W}} = (\mathrm{id} \otimes \rho)(\mathbb{V}) \qquad \text{and} \qquad \widehat{\mathscr{V}} = (\lambda \otimes \mathrm{id})(\mathbb{V}_{21}),$$

and $\rho = (\mathrm{Ad}\, u)\lambda$. *In particular, the* C^**-algebras* $\lambda(C(\mathbb{G}))$ *and* $\rho(C(\mathbb{G}))$ *commute with each other.*

Remark 1.21
Our notation and conventions agree with those of Baaj and Skandalis [2] in the following way. We consider ρ as the canonical representation of $C(\mathbb{G})$ and $\widehat{\lambda}$ as the canonical one for $c_0(\widehat{\mathbb{G}})$. If we then write $V := (\widehat{\lambda} \otimes \rho)(\mathbb{V}) = (\mathrm{id} \otimes \rho)(\mathscr{V})$, the operator V is a *multiplicative unitary* on $L^2(\mathbb{G})$. Together with the unitary u, V is *irreducible* in the sense of [2, Définition 6.2], and the corresponding multiplicative unitaries of [2] are given by

$$\widehat{V} = (\rho \otimes \widehat{\rho})(\mathbb{V}_{21}) \qquad \text{and} \qquad \widetilde{V} = (\lambda \otimes \widehat{\lambda})(\mathbb{V}_{21}).$$

Actions and crossed products
We provide a brief introduction to the theory of actions of compact and discrete quantum groups on C^*-algebras (for details and proofs, see [2]).

Definition 1.22
A (right) *action of a compact quantum group* $\mathbb{G}$ on a C^*-algebra A is a nondegenerate *-homomorphism

$$\alpha : A \to \mathrm{M}\big(A \otimes C(\mathbb{G})\big) \quad \text{satisfying} \; (\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta)\alpha$$

and such that $\alpha(A)(1 \otimes C(\mathbb{G}))$ is total in $A \otimes C(\mathbb{G})$.

The *crossed product* $A \rtimes \mathbb{G}$ is defined as the closed linear span of $(\mathrm{id} \otimes \rho)\alpha(A)\ \big(1 \otimes \widehat{\lambda}(c_0(\mathbb{G}))\big)$ and is a C^*-algebra. Observe that $A \rtimes \mathbb{G}$ is realized as a subalgebra of $\mathscr{L}(A \otimes L^2(\mathbb{G}))$.

Note that because of amenability of $\mathbb{G}$, there is no need to define full and reduced crossed products.

Remark 1.23
The action α of a compact quantum group on a unital C^*-algebra A is said to be *ergodic* if

$$\alpha(a) = a \otimes 1 \quad \text{if and only if } a \in \mathbb{C}1.$$

For ergodic actions of compact quantum groups, the usual theory of spectral subspaces is available. In particular, one defines the multiplicity with which an irreducible representation $x \in \mathrm{Irred}(\mathbb{G})$ appears in an ergodic action. We need this notion at one place in the article and refer to [6, Introduction] for details.

Definition 1.24
A (left) *action of a discrete quantum group* $\widehat{\mathbb{G}}$ on a C^*-algebra A is a nondegenerate *-homomorphism

$$\alpha : A \to \mathrm{M}\left(c_0(\widehat{\mathbb{G}}) \otimes A\right) \quad \text{satisfying } (\mathrm{id} \otimes \alpha)\alpha = (\hat{\Delta} \otimes \mathrm{id})\alpha$$

and such that $(\hat{\varepsilon} \otimes \mathrm{id})\alpha(a) = a$ for all $a \in A$.

Since $\widehat{\mathbb{G}}$ need not be amenable, we introduce the notions of a covariant representation, full crossed product, and reduced crossed product.

Definition 1.25
Let $\alpha : A \to \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$ be an action of a discrete quantum group $\widehat{\mathbb{G}}$ on a C^*-algebra A. A *covariant representation* of (A, α) into a C^*-algebra B is a pair (θ, X), where $\theta : A \to \mathrm{M}(B)$ is a nondegenerate *-homomorphism and $X \in \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes B)$ is a unitary representation of $\widehat{\mathbb{G}}$ satisfying the covariance relation

$$(\mathrm{id} \otimes \theta)\alpha(a) = X^*\big(1 \otimes \theta(a)\big)X \quad \text{for all } a \in A.$$

PROPOSITION 1.26
Let $\alpha : A \to \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$ be an action of a discrete quantum group $\widehat{\mathbb{G}}$ on a C^*-algebra A.

- *For any covariant representation (θ, X) of (A, α), the closed linear span of*

  $$\theta(A)\big\{(\omega \otimes \mathrm{id})(X) \,\big|\, \omega \in \ell^\infty(\widehat{\mathbb{G}})_*\big\}$$

  is a C^*-algebra: the C^*-algebra generated by (θ, X).
- *The* reduced crossed product $\widehat{\mathbb{G}}_{\mathrm{r}} \ltimes A$ *is the C^*-algebra generated by the* regular covariant representation *into $\mathscr{L}(L^2(\mathbb{G}) \otimes A)$ given by $((\widehat{\lambda} \otimes \mathrm{id})\alpha, \widehat{\mathscr{W}}_{12})$.*
- *The* full crossed product $\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes A$ *is the unique (up to isomorphism) C^*-algebra B generated by a covariant representation (θ, X) into B satisfying the following universal property: for any covariant representation (θ_1, X_1) into a C^*-algebra B_1, there exists a nondegenerate *-homomorphism $\pi : B \to \mathrm{M}(B_1)$ satisfying $\theta_1 = \pi\theta$ and $X_1 = (\mathrm{id} \otimes \pi)(X)$.*

Exactness

Recall that a C^*-algebra A is said to be *exact* if the operation $A \otimes_{\min} \cdot$ transforms short exact sequences into short exact sequences.

Definition 1.27

A discrete quantum group $\widehat{\mathbb{G}}$ is said to be *exact* if the operation $\widehat{\mathbb{G}}_{\mathrm{r}} \ltimes \cdot$ transforms $\widehat{\mathbb{G}}$-equivariant short exact sequences into short exact sequences.

The following proposition is proved using a classical trick.

PROPOSITION 1.28

A discrete quantum group $\widehat{\mathbb{G}}$ is exact if and only if $C(\mathbb{G})_{\mathrm{red}}$ is an exact C^*-algebra.

Proof

One implication is obvious by applying the definition of exactness of $\widehat{\mathbb{G}}$ to short exact sequences equivariant with respect to the trivial action of $\widehat{\mathbb{G}}$.

So, suppose that $C(\mathbb{G})_{\mathrm{red}}$ is an exact C^*-algebra, and suppose that $0 \to J \to A \to A/J \to 0$ is a $\widehat{\mathbb{G}}$-equivariant short exact sequence. Denote by δ_J (resp., δ_A) the actions of $\widehat{\mathbb{G}}$ on J (resp., A). For any C^*-algebra B with an action δ of $\widehat{\mathbb{G}}$ on B, we have a canonical injective *-homomorphism

$$\widehat{\delta} : \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes B \to C(\mathbb{G})_{\mathrm{red}} \underset{\min}{\otimes} (\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes B),$$

which is a form of the dual action of $\mathbb{G}$ on the crossed product. Observe that at the right-hand side, the full crossed product appears. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes J & \longrightarrow & \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes A & \longrightarrow & \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes \frac{A}{J} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widehat{\delta}_J} & & \downarrow{\scriptstyle \widehat{\delta}_A} & & \downarrow & & \\
0 & \longrightarrow & C(\mathbb{G})_{\mathrm{red}} \underset{\min}{\otimes} (\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes J) & \longrightarrow & C(\mathbb{G})_{\mathrm{red}} \underset{\min}{\otimes} (\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes A) & \longrightarrow & C(\mathbb{G})_{\mathrm{red}} \underset{\min}{\otimes} (\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes \frac{A}{J}) & \longrightarrow & 0
\end{array}$$

By universality, the sequence $0 \to \widehat{\mathbb{G}}_{\mathrm{f}} \ltimes J \to \widehat{\mathbb{G}}_{\mathrm{f}} \ltimes A \to \widehat{\mathbb{G}}_{\mathrm{f}} \ltimes A/J \to 0$ is exact. So, by exactness of $C(\mathbb{G})_{\mathrm{red}}$, the bottom row of the commutative diagram is exact. Suppose that $a \in \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes A$ becomes zero in $\widehat{\mathbb{G}}_{\mathrm{r}} \ltimes A/J$. Since the bottom row of the diagram is exact, $\widehat{\delta_A}(a) \in C(\mathbb{G})_{\mathrm{red}} \otimes_{\min} (\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes J)$. Using an approximate identity (e_α) for J, it is easy to check that $\widehat{\delta_J}(e_\alpha)\widehat{\delta_A}(a) \to \delta_A(a)$. Since $\widehat{\delta_A}$ is isometric, it follows that $e_\alpha a \to a$, and hence, $a \in \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes J$. This proves the exactness of the top row in the diagram. $\square$

Examples: The universal compact quantum groups

The universal compact quantum groups were introduced by Van Daele and Wang [25]. They are defined as follows.

Definition 1.29
Let $F \in \mathrm{GL}(n, \mathbb{C})$. We define the compact quantum group $\mathbb{G} = A_u(F)$ as follows:

- $C(\mathbb{G})$ is the universal C^*-algebra with generators (U_{ij}) and relations making $U = (U_{ij})$ and $F\overline{U}F^{-1}$ unitary elements of $\mathrm{M}_n(\mathbb{C}) \otimes C(\mathbb{G})$, where $(\overline{U})_{ij} = U_{ij}^*$;
- $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

Definition 1.30
Let $F \in \mathrm{GL}(n, \mathbb{C})$ satisfying $F\overline{F} = \pm 1$. We define the compact quantum group $\mathbb{G} = A_o(F)$ as follows:

- $C(\mathbb{G})$ is the universal C^*-algebra with generators (U_{ij}) and relations making $U = (U_{ij})$ a unitary element of $\mathrm{M}_n(\mathbb{C}) \otimes C(\mathbb{G})$ and $U = F\overline{U}F^{-1}$, where $(\overline{U})_{ij} = U_{ij}^*$;
- $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

In both examples, the unitary matrix U is a representation, called the *fundamental representation*. The definition of $\mathbb{G} = A_o(F)$ makes sense without the requirement $F\overline{F} = \pm 1$, but the fundamental representation is irreducible if and only if $F\overline{F} \in \mathbb{R}1$.

Remark 1.31
It is easy to classify the quantum groups $A_o(F)$. For $F_1, F_2 \in \mathrm{GL}(2, \mathbb{C})$ with $F_i\overline{F}_i = \pm 1$, we write $F_1 \sim F_2$ if there exists a unitary matrix v such that $F_1 = vF_2v^t$, where v^t is the transpose of v. Then $A_o(F_1) \cong A_o(F_2)$ if and only if $F_1 \sim F_2$. It follows that the $A_o(F)$ are classified up to isomorphism by n, the sign $F\overline{F}$, and the eigenvalue list of F^*F (see, e.g., [6, Section 5], where an explicit fundamental domain for the relation $\sim$ is described).

If $F \in \mathrm{GL}(2, \mathbb{C})$, we get, up to isomorphism, the matrices

$$F_q^{\pm} = \begin{pmatrix} 0 & \sqrt{q} \\ \mp\frac{1}{\sqrt{q}} & 0 \end{pmatrix}$$

for $0 < q \leq 1$ with corresponding quantum groups $A_o(F_q^{\pm}) \cong \mathrm{SU}_{\pm q}(2)$.

For the rest of the article, we assume that $F \not\sim F_1^{\pm}$, which means that we deal neither with the classical group SU(2) *nor with* $\mathrm{SU}_{-1}(2)$. Generally speaking, our interest lies in $A_o(F)$ with $\dim F \geq 3$.

The quantum groups $A_u(F)$ and $A_o(F)$ have been studied extensively by Banica [3], [4]. In particular, he gave a complete description of their representation theory. In the rest of the article, we focus on $A_o(F)$. The following result is proved in [3]. It tells us that $A_o(F)$ has the same fusion rules as the classical compact group SU(2). Observe, however, that the dimension of the fundamental representation U is n. Conversely, it is easy to see that any compact quantum group with the same fusion algebra as SU(2) is isomorphic to an $A_o(F)$.

THEOREM 1.32
Let $F \in \mathrm{GL}(n, \mathbb{C})$*, and let* $F\overline{F} = \pm 1$. *Let* $\mathbb{G} = A_o(F)$. *One can identify* $\mathrm{Irred}(\mathbb{G})$ *with* $\mathbb{N}$ *in such a way that*

$$x \otimes y \cong |x-y| \oplus (|x-y|+2) \oplus \cdots \oplus (x+y)$$

for all $x, y \in \mathbb{N}$.

It is easy to check that $\dim_q(1) = \mathrm{Tr}(F^*F)$. Take $0 < q < 1$ such that $\mathrm{Tr}(F^*F) = q + 1/q$. Then

$$\dim_q(n) = [n+1]_q, \quad \text{where } [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

When there is no confusion, we do not write the index q in the q-number $[n]_q$.

Notation 1.33
Let $\mathbb{G} = A_o(F)$, and let $x, y \in \mathrm{Irred}(\mathbb{G})$. Whenever $z \in x \otimes y$, we denote by $V(x \otimes y, z)$ an *isometric* element in $\mathrm{Mor}(x \otimes y, z)$. Note that $V(x \otimes y, z)$ is defined up to a number of modulus 1.

Whenever $z \in x \otimes y$, we denote by $p_z^{x \otimes y}$ the unique orthogonal projection in $\mathrm{End}(x \otimes y)$ projecting onto the irreducible representation equivalent with z.

Throughout the article, the letters n, x, y, z, r, s are reserved to denote irreducible representations of $A_o(F)$ (i.e., natural numbers). The letters $a, b, c, \ldots$ are used to denote elements of C^*-algebras. The capital letters A and B denote matrices.

2. Solidity and the Akemann-Ostrand property

In [18], Ozawa introduced the following remarkable definition. Recall that a von Neumann algebra is said to be diffuse if it does not contain minimal projections.

Definition 2.1 (*N. Ozawa [17, Section 1]*)
A von Neumann algebra M is said to be *solid* if $M \cap A'$ is injective for any *diffuse* subalgebra $A \subset M$.

A solid von Neumann algebra is necessarily finite. The following definition is a straightforward adaptation of solidity to arbitrary von Neumann algebras and has been observed independently by D. Shlyakhtenko [20].

From now on, we assume that von Neumann algebras have separable predual.

Definition 2.2
A von Neumann algebra M is said to be *generalized solid* if $M \cap A'$ is injective for any *diffuse* subalgebra $A \subset M$ for which there exists a *faithful normal conditional expectation* $E : M \to A$.

Several results in [18] can now be easily generalized. For the convenience of the reader, we give an overview of what we need in this article. The first result is immediate.

PROPOSITION 2.3
We have the following:

- *a finite von Neumann algebra is generalized solid if and only if it is solid;*
- *a subalgebra $M_1 \subset M$ of a generalized solid von Neumann algebra which admits a faithful normal conditional expectation $M \to M_1$ is again generalized solid;*
- *a noninjective generalized solid factor M is prime: if $M \cong M_1 \otimes M_2$, then either M_1 or M_2 is a type I factor.*

The main result of [18] consists in deducing solidity from the *Akemann-Ostrand property*. Recall the following definition from [18].

Definition 2.4 ([18, Theorem 3])
A von Neumann algebra $M \subset \mathrm{B}(H)$ is said to satisfy the *Akemann-Ostrand property* if there exist unital weakly dense C^*-subalgebras $B \subset M$, $C \subset M'$ such that B is locally reflexive and the *-homomorphism

$$B \otimes_{\mathrm{alg}} C \to \frac{\mathrm{B}(H)}{\mathscr{K}(H)} : \sum_{i=1}^{n} b_i \otimes c_i \mapsto \sum_{i=1}^{n} \pi(b_i c_i)$$

extends continuously to $B \otimes_{\min} C$. Here, π denotes the quotient map $\mathrm{B}(H) \to \mathrm{B}(H)/\mathscr{K}(H)$.

The following generalization appears in [18, Theorem 6].

THEOREM 2.5
A von Neumann algebra $M \subset \mathrm{B}(H)$ satisfying the Akemann-Ostrand property is generalized solid.

Proof
One follows almost line by line [18, proof of Theorem 6], paying attention only to the fact that there are conditional expectations everywhere since they do not exist

automatically on von Neumann subalgebras (contrary to the finite case). Suppose that $A \subset M$ is diffuse, and suppose that $E : M \to A$ is a faithful normal conditional expectation. Choose on A a faithful state φ so that the centralizer algebra A^φ has a diffuse abelian subalgebra $A_0 \subset A^\varphi$. This is indeed possible, the most difficult case of A a type III_1 factor being dealt with in [9, Corollary 8].

Write $\psi = \varphi E$. Since there is a ψ-preserving conditional expectation $M \cap A_0' \to M \cap A'$, it is sufficient to show that $M \cap A_0'$ is injective. Because there is a unique ψ-preserving conditional expectation $M \to M \cap A_0'$, [18, proof of Theorem 6] applies literally. □

On the level of compact quantum group C^*-algebras, we have the following version of the Akemann-Ostrand property.

Definition 2.6
Let $\mathbb{G}$ be a compact quantum group. We say that $\mathbb{G}$ satisfies the *Akemann-Ostrand property* if the *-homomorphism

$$C(\mathbb{G})_{\mathrm{red}} \otimes_{\mathrm{alg}} C(\mathbb{G})_{\mathrm{red}} \to \frac{\mathrm{B}(L^2(\mathbb{G}))}{\mathcal{K}(L^2(\mathbb{G}))} : \sum_{i=1}^n a_i \otimes b_i \mapsto \sum_{i=1}^n \pi\big(\lambda(a_i)\rho(b_i)\big)$$

extends continuously to $C(\mathbb{G})_{\mathrm{red}} \otimes_{\min} C(\mathbb{G})_{\mathrm{red}}$. Here, π denotes the quotient map $\mathrm{B}(L^2(\mathbb{G})) \to \mathrm{B}(L^2(\mathbb{G}))/\mathcal{K}(L^2(\mathbb{G}))$.

Obviously, if $\mathbb{G}$ satisfies the Akemann-Ostrand property and $C(\mathbb{G})_{\mathrm{red}}$ is locally reflexive, the von Neumann algebra $C(\mathbb{G})_{\mathrm{red}}''$ satisfies the Akemann-Ostrand property as well and is, by Theorem 2.5, a generalized solid von Neumann algebra.

3. Boundary and boundary action for the dual of $A_o(F)$

Fix $F \in \mathrm{GL}(n, \mathbb{C})$ with $F\overline{F} = \pm 1$. Put $\mathbb{G} = A_o(F)$. Recall that we assume that $\mathbb{G} \not\cong \mathrm{SU}_{\pm 1}(2)$. Recall that we identify $\mathrm{Irred}(\mathbb{G}) = \mathbb{N}$, and recall that we use the letters n, x, y, z, r, s to denote irreducible representations of $\mathbb{G}$.

We introduce a boundary for $\widehat{\mathbb{G}}$, inspired by the construction of the boundary of a free group by adding infinite reduced words. So, we first define a *compactification* of $\widehat{\mathbb{G}}$, which is a unital C^*-algebra $\mathcal{B}$ such that

$$c_0(\widehat{\mathbb{G}}) \subset \mathcal{B} \subset \ell^\infty(\widehat{\mathbb{G}}).$$

The boundary $\mathcal{B}_\infty$ is then defined as $\mathcal{B}_\infty = \mathcal{B}/c_0(\widehat{\mathbb{G}})$.

We show that the comultiplication $\hat{\Delta}$ yields, by restriction and passage to the quotient $\mathcal{B}_\infty = \mathcal{B}/c_0(\widehat{\mathbb{G}})$,

- an action of $\widehat{\mathbb{G}}$ on $\mathcal{B}_\infty$ on the left-hand side;
- the trivial action of $\widehat{\mathbb{G}}$ on $\mathcal{B}_\infty$ on the right-hand side.

In the next section, we introduce the notion of an amenable action and prove that the boundary action is amenable.

If one compactifies a free group Γ by adding infinite words, a continuous function on this compactification is an element of $\ell^\infty(\Gamma)$ whose value in a long word of Γ essentially depends only on the beginning part of that word. In order to give, somehow, the same kind of definition for $\widehat{\mathbb{G}}$ the dual of $A_o(F)$, we need to compare the values that an element of $\ell^\infty(\widehat{\mathbb{G}})$ takes in two different irreducible representations. So, we compare matrices in $\mathrm{B}(H_x)$ and $\mathrm{B}(H_y)$ for $x, y \in \mathrm{Irred}(\widehat{\mathbb{G}})$. To do so, we use the following linear maps.

Definition 3.1
Let $x, y \in \mathbb{N}$. We define unital completely positive maps

$$\psi_{x+y,x} : \mathrm{B}(H_x) \to \mathrm{B}(H_{x+y}) : \psi_{x+y,x}(A) = V(x \otimes y, x+y)^*(A \otimes 1)V(x \otimes y, x+y).$$

Recall that we have chosen isometric intertwiners $V(x \otimes y, z) \in \mathrm{Mor}(x \otimes y, z)$.

PROPOSITION 3.2
The maps $\psi_{x+y,x}$ form an inductive system of completely positive maps.

Proof
Since $\mathrm{Mor}(x + y + z, x \otimes y \otimes z)$ is one-dimensional, we have

$$\begin{aligned}&\big(V(x \otimes y, x + y) \otimes 1\big)V\big((x + y) \otimes z, x + y + z\big)\\ &\quad = \big(1 \otimes V(y \otimes z, y + z)\big)V\big(x \otimes (y + z), x + y + z\big) \quad \operatorname{mod} \mathbb{T}.\end{aligned}$$

So, we are done. □

Notation 3.3
We define

$$\psi_{\infty,x} : \mathrm{B}(H_x) \to \ell^\infty(\widehat{\mathbb{G}}) : \psi_{\infty,x}(A)p_y = \begin{cases} \psi_{y,x}(A) & \text{if } y \geq x, \\ 0 & \text{otherwise.} \end{cases}$$

We use the same notation for the map $\psi_{\infty,x} : \mathrm{B}(H_x) \to \ell^\infty(\widehat{\mathbb{G}})/c_0(\widehat{\mathbb{G}})$. Recall that p_x denotes the minimal central projection in $\ell^\infty(\widehat{\mathbb{G}})$ associated with $x \in \mathrm{Irred}(\mathbb{G})$.

PROPOSITION 3.4
Define

$$\mathscr{B}_0 = \big\{a \in \ell^\infty(\widehat{\mathbb{G}}) \;\big|\; \textit{there exists } x \textit{ such that } ap_y = \psi_{y,x}(ap_x) \textit{ for all } y \geq x\big\}.$$

The norm closure of $\mathscr{B}_0$ is a unital C^*-subalgebra of $\ell^\infty(\widehat{\mathbb{G}})$ containing $c_0(\widehat{\mathbb{G}})$. We denote it by $\mathscr{B}$. The C^*-algebra $\mathscr{B}$ is nuclear.

Proof
It follows from (A.1) that there exists a constant C such that

$$\left\| \left[\left(\psi_{x+y,x}(A) \otimes 1 \right), \, p_{x+y+z}^{(x+y)\otimes z} \right] \right\| \leq C q^y \|A\|$$

for all x, y, z and $A \in \mathrm{B}(H_x)$. Hence,

$$\left\| \psi_{x+y+z,x+y}\left(\psi_{x+y,x}(A) B \right) - \psi_{x+y+z,x}(A) \psi_{x+y+z,x+y}(B) \right\| \leq C q^y \|A\| \, \|B\|$$

for all $x, y, z, A \in \mathrm{B}(H_x)$ and $B \in \mathrm{B}(H_{x+y})$. We easily conclude that the norm closure $\mathscr{B}$ is a unital C^*-subalgebra of $\ell^\infty(\widehat{\mathbb{G}})$. It obviously contains $c_0(\widehat{\mathbb{G}})$.

Define $B_n = \bigoplus_{x=0}^n \mathrm{B}(H_x)$, define $\mu_n : \mathscr{B} \to B_n$ by restriction, and define $\gamma_n : B_n \to \mathscr{B}$ by the formulas $\gamma_n(a) p_x = a p_x$ if $x \leq n$ and $\gamma_n(a) p_x = \psi_{x,n}(a p_n)$ if $x \geq n$. Then $\gamma_n(\mu_n(a)) \to a$ for all $a \in \mathscr{B}$, and the nuclearity of $\mathscr{B}$ is proved. □

Notation 3.5
The comultiplication $\hat{\Delta}$ yields a (left) action of $\widehat{\mathbb{G}}$ on the C^*-algebra $\ell^\infty(\widehat{\mathbb{G}})$ which we denote by

$$\beta : \ell^\infty(\widehat{\mathbb{G}}) \to \mathrm{M}\left(c_0(\widehat{\mathbb{G}}) \otimes \ell^\infty(\widehat{\mathbb{G}}) \right).$$

PROPOSITION 3.6
We have $\beta(\mathscr{B}) \subset \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes \mathscr{B})$, and as such, β is an action of $\widehat{\mathbb{G}}$ on $\mathscr{B}$.

Proof
It suffices to show that $(p_x \otimes 1)\beta(a) \in \mathrm{B}(H_x) \otimes \mathscr{B}$ for all $a \in \mathscr{B}$ and $x \in \mathbb{N}$. Take $a = \psi_{\infty,r}(A)$. Take $y \geq x + r$, and take z. Then

$$(p_x \otimes p_{y+z})\beta(x) = \sum_{s \in x \otimes y} V\left(x \otimes (y+z), s+z \right) (a p_{s+z}) \, V\left(x \otimes (y+z), s+z \right)^*.$$

Fix an $s \in x \otimes y$. Observe that $a p_{s+z} = \psi_{s+z,s}(\psi_{s,r}(A))$. Using (A.3), we get

$$\begin{aligned} &\Big\| V\left(x \otimes (y+z), s+z \right) (a p_{s+z}) \, V\left(x \otimes (y+z), s+z \right)^* \\ &\quad - (\mathrm{id} \otimes \psi_{y+z,y})\left(V(x \otimes y, s) \psi_{s,r}(A) V(x \otimes y, s)^* \right) \Big\| \leq C q^{-x+y} \|A\|. \end{aligned}$$

We keep x and $A \in \mathrm{B}(H_r)$ fixed. Choose $\varepsilon > 0$. Take y such that $(x+1) C q^{-x+y} \|A\| < \varepsilon$. Since there are less than $x+1$ irreducible components of $x \otimes y$, the computation above shows that $(p_x \otimes 1)\beta(a)$ is at distance at most ε of

$$(\mathrm{id} \otimes \psi_{\infty,y})\Big(\sum_{s\in x\otimes y} V(x\otimes y, s)\psi_{s,r}(A)V(x\otimes y, s)^*\Big).$$

Hence, $(p_x \otimes 1)\beta(a) \in \mathrm{B}(H_x) \otimes \mathscr{B}$. $\square$

Definition 3.7
We define $\mathscr{B}_\infty := \mathscr{B}/c_0(\widehat{\mathbb{G}})$, and we still denote by β the action of $\widehat{\mathbb{G}}$ on $\mathscr{B}_\infty$.

As it is the case for the action of a free group on its boundary, we prove that the action by right translation on $c_0(\widehat{\mathbb{G}})$ extends to an action on $\mathscr{B}$ which becomes the trivial action on $\mathscr{B}_\infty$. The precise statement is as follows.

PROPOSITION 3.8
Consider the (right) action $\gamma : \ell^\infty(\widehat{\mathbb{G}}) \to \mathrm{M}(\ell^\infty(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}))$ of $\widehat{\mathbb{G}}$ on the C^*-algebra $\ell^\infty(\widehat{\mathbb{G}})$ by right translation. For all $a \in \mathscr{B}$ and all x, we have

$$\big(\gamma(a) - a\otimes 1\big)(1\otimes p_x) \in c_0(\widehat{\mathbb{G}}) \otimes \mathrm{B}(H_x).$$

Hence, γ becomes the trivial action on $\mathscr{B}_\infty$.

Proof
Suppose that $a = \psi_{\infty,x}(A)$ for $A \in \mathrm{B}(H_x)$. Fix a z, and take $y \geq z$. Using (A.6), we get a constant C such that

$$\begin{aligned}
&\hat{\Delta}\big(\psi_{\infty,x}(A)\big)(p_{x+y} \otimes p_z)\\
&\quad = \sum_{s\in y\otimes z} V\big((x+y)\otimes z, x+s\big)\psi_{x+s,x}(A)V\big((x+y)\otimes z, x+s\big)^*\\
&\quad \approx \sum_{s\in y\otimes z} \big(V(x\otimes y, x+y)^* \otimes 1\big)(A\otimes p_s^{y\otimes z})\big(V(x\otimes y, x+y)\otimes 1\big)\\
&\qquad \text{(with error at most } (z+1)Cq^{-z+y})\\
&\quad = \psi_{x+y,x}(A) \otimes p_z.
\end{aligned}$$

If we keep fixed A and let $y \to \infty$, the conclusion follows. $\square$

For later use, we prove the following lemma. The only interest at this point is that it shows that $\mathscr{B}_\infty$ is nontrivial because it follows that the maps $\psi_{\infty,x} : \mathrm{B}(H_x) \to \mathscr{B}_\infty$ are injective.

LEMMA 3.9
There exists a constant $D > 0$ depending only on q such that

$$D\|A\|_{\psi_x} \leq \|\psi_{x+y,x}(A)\|_{\psi_{x+y}} \leq \|A\|_{\psi_x}$$

for all x, y and $A \in \mathrm{B}(H_x)$.

Proof
Consider $\mathrm{B}(H_x)$ as a Hilbert space using the state ψ_x. Then

$$B(H_x) \to H_x \otimes H_x : A \mapsto (A \otimes 1)t_x$$

is a unitary operator. Using the notation $D(x, y) = [x+1][y+1][x+y+1]^{-1}$ and (7.2), we know that t_{x+y} equals, up to a number of modulus one,

$$D(x, y)^{1/2}\big(V(x \otimes y, x+y)^* \otimes V(y \otimes x, x+y)^*\big)(1 \otimes t_y \otimes 1)t_x.$$

Using Lemma A.6, we get a constant $D > 0$ such that

$$\begin{aligned}
\|\psi_{x+y,x}(A)\|_{\psi_{x+y}} &= \big\|\big(\psi_{x+y,x}(A) \otimes 1\big)t_{x+y}\big\| \\
&= D(x, y)^{1/2}\big\|\big(V(x \otimes y, x+y)^* \otimes 1\big)(A \otimes 1 \otimes 1) \\
&\qquad\qquad \times \big(p^{x\otimes y}_{x+y} \otimes V(y \otimes x, x+y)^*\big)(1 \otimes t_y \otimes 1)t_x\big\| \\
&= D(x, y)^{1/2}\big\|\big(V(x \otimes y, x+y)^* \otimes 1\big)(A \otimes 1 \otimes 1) \\
&\qquad\qquad \times \big(1 \otimes 1 \otimes V(y \otimes x, x+y)^*\big)(1 \otimes t_y \otimes 1)t_x\big\| \\
&= D(x, y)^{1/2}\big\|\big(V(x \otimes y, x+y)^* \otimes V(y \otimes x, x+y)^*\big) \\
&\qquad\qquad \times (1 \otimes t_y \otimes 1)\,(A \otimes 1)t_x\big\| \\
&\geq D\|(A \otimes 1)t_x\| = D\|A\|_{\psi_x}.
\end{aligned}$$

So, we are done. □

4. Amenability of the boundary action and the Akemann-Ostrand property

We introduce the notion of an *amenable action* of a discrete quantum group on a unital C^*-algebra. We prove that for $\mathbb{G} = A_o(F)$, the action of $\widehat{\mathbb{G}}$ on its boundary $\mathscr{B}_\infty$, as introduced in Section 3, is amenable. We then deduce the exactness of $C(\mathbb{G})_{\mathrm{red}}$ and the Akemann-Ostrand property.

In the following definition, we make use of the representation

$$\begin{aligned}
\widehat{\rho\lambda} : \ell^\infty(\widehat{\mathbb{G}}) \mathbin{\overline{\otimes}} \ell^\infty(\widehat{\mathbb{G}}) \to \mathrm{B}\big(L^2(\mathbb{G})\big) &: (\widehat{\rho\lambda})(a \otimes b)\xi_x \\
&= (bp_x \otimes ap_{\overline{x}})\xi_x \quad \text{for all } \xi_x \in H_x \otimes H_{\overline{x}}.
\end{aligned}$$

Recall that $\widehat{\mathscr{V}}$ denotes the right-regular representation of $\widehat{\mathbb{G}}$.

Definition 4.1

Let $\beta : \mathscr{B} \to \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes \mathscr{B})$ be an action of the discrete quantum group $\widehat{\mathbb{G}}$ on a unital C^*-algebra $\mathscr{B}$. We say that β is *amenable* if there exists a sequence $\xi_n \in L^2(\mathbb{G}) \otimes \mathscr{B}$ satisfying

- $\xi_n^* \xi_n \to 1$ in $\mathscr{B}$;
- for all x, $\|((\mathrm{id} \otimes \beta)(\xi_n) - \widehat{\mathscr{V}}_{12}(\xi_n)_{13})(1 \otimes p_x \otimes 1)\| \to 0$;
- $(\widehat{\rho\lambda}\hat{\Delta} \otimes \mathrm{id})\beta(a)\xi_n = \xi_n a$ for all n and all $a \in \mathscr{B}$.

Remark 4.2

It is clear that $\widehat{\mathbb{G}}$ is amenable if and only if the trivial action on $\mathbb{C}$ is amenable if and only if every action is amenable.

The first two conditions in Definition 4.1 are natural. The ξ_n are approximately equivariant unit vectors. The third condition may seem mysterious, but already the definition of an amenable action of a discrete group on a unital C^*-algebra involves an extra condition (see [1, Théorème 3.3], where positive-definite functions take values in the center). In the quantum setting, this centrality condition is replaced by the third condition in Definition 4.1, and it reduces to centrality in the case where $\widehat{\mathbb{G}}$ is a discrete group. Indeed, in that case, $(\widehat{\rho\lambda})\hat{\Delta}$ is the counit $\hat{\varepsilon}$, and the condition above reads $(1 \otimes a)\xi_n = \xi_n a$ for all $a \in \mathscr{B}$ and all n.

Notation 4.3

In this section, we write H for the Hilbert space $L^2(\mathbb{G})$.

PROPOSITION 4.4

Let $\beta : \mathscr{B} \to \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes \mathscr{B})$ be an amenable action of a discrete quantum group $\widehat{\mathbb{G}}$ on a unital C^*-algebra $\mathscr{B}$. Then the natural homomorphism $\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes \mathscr{B} \to \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes \mathscr{B}$ is an isomorphism. If, moreover, $\mathscr{B}$ is nuclear, then $\widehat{\mathbb{G}} \ltimes \mathscr{B}$ is nuclear, the reduced C^*-algebra $C(\mathbb{G})_{\mathrm{red}}$ is an exact C^*-algebra, and $\widehat{\mathbb{G}}$ is an exact quantum group.

Proof

Let (θ, X) be a covariant representation of β on the Hilbert space K. Define bounded linear maps

$$v_n : K \to H \otimes K : v_n \eta = (\hat{\lambda} \otimes \mathrm{id})(X)(\mathrm{id} \otimes \theta)(\xi_n)\eta.$$

We prove that the v_n approximately intertwine the covariant representation (θ, X) with a regular covariant representation. First, observe that for all $a \in \mathscr{B}$,

$$v_n \theta(a) = (\hat{\lambda} \otimes \mathrm{id})(X)\big((\widehat{\rho\lambda} \otimes \theta)(\mathrm{id} \otimes \beta)\beta(a)\big)(\mathrm{id} \otimes \theta)(\xi_n).$$

For all $a \in c_0(\widehat{\mathbb{G}})$ and $b \in \mathscr{B}$, we have

$$\begin{aligned}(\widehat{\lambda} \otimes \mathrm{id})(X)\big((\widehat{\rho}\widehat{\lambda} \otimes \theta)(\mathrm{id} \otimes \beta)(a \otimes b)\big) &= (\widehat{\lambda} \otimes \mathrm{id})(X)\big(\widehat{\rho}(a) \otimes 1\big)(\widehat{\lambda} \otimes \mathrm{id})\beta(b) \\ &= \big(\widehat{\rho}(a) \otimes 1\big)\big(1 \otimes \theta(b)\big)(\widehat{\lambda} \otimes \mathrm{id})(X) \\ &= (\widehat{\rho} \otimes \theta)(a \otimes b)(\widehat{\lambda} \otimes \mathrm{id})(X).\end{aligned}$$

Hence,

$$v_n\theta(a) = (\widehat{\rho} \otimes \theta)\beta(a)v_n$$

for all $a \in \mathscr{B}$.

Next, observe that

$$\big((1 \otimes v_n)X(p_x \otimes 1)\big)_{213} = (\widehat{\lambda} \otimes \mathrm{id})(X)_{13}X_{23}(\mathrm{id} \otimes \mathrm{id} \otimes \theta)\big((\mathrm{id} \otimes \beta)(\xi_n)(1 \otimes p_x \otimes 1)\big),$$

while

$$\big(\widehat{\mathscr{V}}_{21}(1 \otimes v_n)(p_x \otimes 1)\big)_{213} = (\widehat{\lambda} \otimes \mathrm{id})(X)_{13}X_{23}(\mathrm{id} \otimes \mathrm{id} \otimes \theta)\big(\widehat{\mathscr{V}}_{12}(\xi_n)_{13}(1 \otimes p_x \otimes 1)\big).$$

The condition in Definition 4.1 yields

$$(p_x \otimes 1 \otimes 1)\big((1 \otimes v_n)X - \widehat{\mathscr{V}}_{21}(1 \otimes v_n)\big) \to 0$$

for all $x \in \mathrm{Irred}(\mathbb{G})$. So, we have shown that the v_n approximately intertwine (θ, X) with the regular covariant representation $((\widehat{\rho} \otimes \theta)\beta, \widehat{\mathscr{V}}_{21})$. Since $v_n^* v_n \to 1$ in the norm topology, this shows that $\widehat{\mathbb{G}}_{\mathrm{f}} \ltimes \mathscr{B} \to \widehat{\mathbb{G}}_{\mathrm{r}} \ltimes \mathscr{B}$ is an isomorphism.

In order to show that $\widehat{\mathbb{G}} \ltimes \mathscr{B}$ is nuclear when $\mathscr{B}$ is nuclear, it suffices to observe that the action $\beta \otimes \mathrm{id}$ of $\widehat{\mathbb{G}}$ on $\mathscr{B} \otimes D$ is amenable when β is amenable. As a subalgebra of a nuclear C^*-algebra, the reduced C^*-algebra $C(\mathbb{G})_{\mathrm{red}}$ is exact. The exactness of $\widehat{\mathbb{G}}$ follows from Proposition 1.28. □

We now fix $F \in \mathrm{GL}(n, \mathbb{C})$ with $F\overline{F} = \pm 1$ and take for the rest of this section $\mathbb{G} = A_o(F)$. We still have our standing assumption that $\mathbb{G} \ncong \mathrm{SU}_{\pm 1}(2)$.

THEOREM 4.5

Let $\mathbb{G} = A_o(F)$. The boundary action of $\widehat{\mathbb{G}}$ on $\mathscr{B}_\infty$, constructed in Section 3, is amenable.

Proof

Consider the unit vector $\mu := \widehat{\mathscr{V}}(\xi_0 \otimes 1) \in \mathscr{L}(c_0(\widehat{\mathbb{G}}), H \otimes c_0(\widehat{\mathbb{G}}))$, as well as the unit vectors

$$\mu_x = \mu p_x \in H \otimes \mathrm{B}(H_x).$$

Note that $\mu = (\widehat{\Lambda}_R \otimes \mathrm{id})\hat{\Delta}(p_\varepsilon)$. Observe that $\mu_x \in \widehat{\lambda}(p_{\bar{x}})H \otimes \mathrm{B}(H_x)$, which implies that the vectors μ_x are mutually orthogonal.

Define $\xi_n \in H \otimes \mathscr{B}_\infty$ by the formula $\xi_n = (1/\sqrt{n+1})\sum_{y=0}^{n}(\mathrm{id} \otimes \psi_{\infty,y})(\mu_y)$. We claim that $\xi_n^*\xi_n \to (1-q^2)1$. Consider, for a fixed x, the vector $\eta_x \in H \otimes \mathscr{B}_\infty$ given by $\eta_x = (\mathrm{id} \otimes \psi_{\infty,x})(\mu_x)$. Since $\mu_x = (\widehat{\Lambda}_R \otimes \mathrm{id})(p_\varepsilon^{\bar{x}\otimes x})$, we get

$$\begin{aligned}(\mu_x^* \otimes 1)(1 \otimes p_{x+y}^{x\otimes y})(\mu_x \otimes 1) &= [x+1]^2(\varphi_{\bar{x}} \otimes \mathrm{id} \otimes \mathrm{id}) \\ &\quad \times \big((p_\varepsilon^{\bar{x}\otimes x} \otimes 1)(1 \otimes p_{x+y}^{x\otimes y})(p_\varepsilon^{\bar{x}\otimes x} \otimes 1)\big) \\ &= [x+1]^2\big((\varphi_{\bar{x}} \otimes \mathrm{id})(p_\varepsilon^{\bar{x}\otimes x})\otimes 1\big)\big(1 \otimes (\varphi_x \otimes \mathrm{id})(p_{x+y}^{x\otimes y})\big) \\ &= \frac{[x+y+1]}{[x+1][y+1]} 1 \otimes 1.\end{aligned}$$

It follows that $\eta_x^*\eta_x = (q^{-x}/[x+1])1$ in $\mathscr{B}_\infty$. Since $q^{-x}/[x+1] \to 1-q^2$ when $x \to \infty$, the claim is proved.

In order to verify that $(\widehat{\rho\lambda}\hat{\Delta} \otimes \mathrm{id})\beta(a)\xi_n = \xi_n a$ for all n and all $a \in \mathscr{B}_\infty$, it is sufficient to check that

$$(\mathrm{id} \otimes \psi_{x+y,x})(\mu_x)(ap_{x+y}) = \big((1 \otimes p_{x+y})(\widehat{\rho\lambda}\hat{\Delta} \otimes \mathrm{id})\beta(a)\big)(\mathrm{id} \otimes \psi_{x+y,x})(\mu_x) \quad (4.1)$$

for all $a \in \ell^\infty(\widehat{\mathbb{G}})$ and all $x, y \in \mathrm{Irred}(\mathbb{G})$. Observe that the right-hand side of (4.1) equals

$$\big(1 \otimes V(x\otimes y, x+y)^*\big)\big((1 \otimes p_x \otimes p_y)(\widehat{\rho\lambda} \otimes \mathrm{id} \otimes \mathrm{id})\hat{\Delta}^{(3)}(a)(\mu \otimes 1)\big)V(x\otimes y, x+y). \quad (4.2)$$

But, for all $a, b \in c_0(\widehat{\mathbb{G}})$, we have

$$\begin{aligned}(\widehat{\rho\lambda} \otimes \mathrm{id})\big(a \otimes \hat{\Delta}(b)\big)\mu &= \big(\widehat{\rho}(a) \otimes 1\big)\,(\widehat{\lambda} \otimes \mathrm{id})\hat{\Delta}(b)\,\widehat{\mathscr{V}}(\xi_0 \otimes 1) \\ &= \big(\widehat{\rho}(a) \otimes 1\big)\widehat{\mathscr{V}}\big(\widehat{\lambda}(b)\xi_0 \otimes 1\big) \\ &= \widehat{\mathscr{V}}(\widehat{\rho} \otimes \mathrm{id})\hat{\Delta}^{\mathrm{op}}(a)(\xi_0 \otimes 1)\hat{\varepsilon}(b) = \mu a\hat{\varepsilon}(b).\end{aligned}$$

It follows that (4.2) equals

$$\begin{aligned}&\big(1 \otimes V(x\otimes y, x+y)^*\big)(\mu_x \otimes 1)\big((p_x \otimes p_y)\hat{\Delta}(a)\big)V(x\otimes y, x+y) \\ &\quad = (\mathrm{id} \otimes \psi_{x+y,x})(\mu_x)(ap_{x+y}).\end{aligned}$$

This proves (4.1). We then come to the crucial approximate equivariance condition for ξ_n. First, observe that $\widehat{\mathscr{V}}_{12}\mu_{13} = \widehat{\mathscr{V}}_{12}\widehat{\mathscr{V}}_{13}(\xi_0 \otimes 1 \otimes 1) = (\mathrm{id} \otimes \hat{\Delta})(\mu)$. Hence, for all x and all $y \geq n$,

$$\widehat{\mathscr{V}}_{12}(\xi_n)_{13}(p_x \otimes p_y) = \frac{1}{\sqrt{n+1}} \sum_{s=0}^{n} (\mathrm{id} \otimes \mathrm{id} \otimes \psi_{y,s})\big(\widehat{\mathscr{V}}_{12}\mu_{13}(p_x \otimes p_s)\big)$$
$$= \frac{1}{\sqrt{n+1}} \sum_{s=0}^{n} (\mathrm{id} \otimes \mathrm{id} \otimes \psi_{y,s})\big((\mathrm{id} \otimes \hat{\Delta})(\mu)(p_x \otimes p_s)\big).$$

Take $n \geq K \geq x$ and $y \geq n + x$. We now write equalities up to an error term, which we estimate using the norm of the C^*-module $H \otimes \mathrm{B}(H_x) \otimes \mathrm{B}(H_y)$,

$$\widehat{\mathscr{V}}_{12}(\xi_n)_{13}(p_x \otimes p_y)$$
$$\approx \frac{1}{\sqrt{n+1}} \sum_{s=K}^{n} (\mathrm{id} \otimes \mathrm{id} \otimes \psi_{y,s})\big((\mathrm{id} \otimes \hat{\Delta})(\mu)(p_x \otimes p_s)\big)$$
$$\Big(\text{with error at most } \frac{K}{\sqrt{n+1}}\Big)$$
$$= \frac{1}{\sqrt{n+1}} \sum_{s=K}^{n} \sum_{z \in x \otimes s} (\mathrm{id} \otimes \mathrm{id} \otimes \psi_{y,s})\big((1 \otimes V(x \otimes s, z))\mu_z V(x \otimes s, z)^*\big) = (\star).$$

It follows from (A.3) that there exists a constant C such that

$$\big\|(\mathrm{id} \otimes \mathrm{id} \otimes \psi_{y,s})\big((1 \otimes V(x \otimes s, z))\mu_z V(x \otimes s, z)^*\big)$$
$$- \big(1 \otimes V(x \otimes y, z + y - s)\big)(\mathrm{id} \otimes \psi_{z+y-s,z})(\mu_z) V(x \otimes y, z + y - s)^* \big\|$$
$$\leq 2Cq^{-x+s} \leq 2Cq^{-x+K}.$$

Observe now that in the sum $(\star)$, for a given z there are less than $x+1$ corepresentations s such that $z \in x \otimes s$. Moreover, in the sum $(\star)$, z ranges from $K - x$ to $n + x$, and we have $\mu_z \in \widehat{\lambda}(p_z)H \otimes \mathrm{B}(H_z)$ with the $\widehat{\lambda}(p_z)H$ orthogonal for different z. We also observe that $z \in x \otimes s$ if and only if $z + y - s \in x \otimes y$ and conclude that

$$(\star) \approx \frac{1}{\sqrt{n+1}} \sum_{s=K}^{n} \sum_{z \in x \otimes s} \big(1 \otimes V(x \otimes y, z + y - s)\big)(\mathrm{id} \otimes \psi_{z+y-s,z})(\mu_z)$$
$$\times V(x \otimes y, z + y - s)^* \quad (\text{with error at most } 4(x+1)Cq^{-x+K})$$
$$= \frac{1}{\sqrt{n+1}} \sum_{s=K}^{n} \sum_{r \in x \otimes y} \big(1 \otimes V(x \otimes y, r)\big)(\mathrm{id} \otimes \psi_{r,r-y+s})(\mu_{r-y+s}) V(x \otimes y, r)^*$$
$$= (\mathrm{id} \otimes \hat{\Delta})(\eta)(p_x \otimes p_y),$$

where $\eta \in H \otimes c_0(\widehat{\mathbb{G}})$ is given by $\eta p_r = (1/\sqrt{n+1}) \sum_{s=K}^{n} (\mathrm{id} \otimes \psi_{r,r-y+s})(\mu_{r-y+s})$ whenever $y - x \leq r \leq y + x$ and $\eta p_r = 0$ elsewhere. When $y - x \leq r \leq y + x$,

$$\|\eta p_r - \xi_n p_r\| \leq \frac{2x + K}{\sqrt{n+1}}.$$

Since the expression $(\mathrm{id} \otimes \hat{\Delta})(\eta)(p_x \otimes p_y)$ takes into account only the values of ηp_r for $y - x \leq r \leq y + x$, it follows that

$$(\mathrm{id} \otimes \hat{\Delta})(\eta)(p_x \otimes p_y) \approx (\mathrm{id} \otimes \hat{\Delta})(\xi_n)(p_x \otimes p_y) \quad \Big(\text{with error at most } \frac{2x + K}{\sqrt{n+1}}\Big).$$

We finally conclude that

$$\big\| \big(\widehat{\mathscr{V}}_{12}(\xi_n)_{13} - (\mathrm{id} \otimes \beta)(\xi_n)\big)(1 \otimes p_x \otimes 1) \big\| \leq \frac{2x + 2K}{\sqrt{n+1}} + 4C(x+1)q^{-x+K}.$$

Given x, we first take K such that $4C(x+1)q^{-x+K}$ is small. We then take n such that $(2x + 2K)/\sqrt{n+1}$ is small. As such, we have shown the amenability of the action β. Indeed, it suffices to replace ξ_n by $(1 - q^2)^{-1/2}\xi_n$. □

Remark 4.6
The same proof shows that the action of $\widehat{\mathbb{G}}$ on $\ell^\infty(\widehat{\mathbb{G}})/c_0(\widehat{\mathbb{G}})$ by left translation is an amenable action. But since $\ell^\infty(\widehat{\mathbb{G}})/c_0(\widehat{\mathbb{G}})$ is nonnuclear (even nonexact), we really need the amenability of the action on the nuclear C^*-algebra $\mathscr{B}_\infty$ to show, for example, the exactness of $C(\mathbb{G})_{\mathrm{red}}$.

We deduce exactness and the Akemann-Ostrand property from the amenability of the boundary action. Note that an independent proof of the Akemann-Ostrand property has been given by Vergnioux in [26].

COROLLARY 4.7
Let $\mathbb{G} = A_o(F)$. Then $C(\mathbb{G})_{\mathrm{red}}$ is exact, and $\mathbb{G}$ satisfies the Akemann-Ostrand property.

Proof
The exactness of $C(\mathbb{G})_{\mathrm{red}}$ follows from Proposition 4.4. Put $H = L^2(\mathbb{G})$. Consider the left-right representation

$$\lambda\rho : C(\mathbb{G})_{\mathrm{red}} \underset{\max}{\otimes} C(\mathbb{G})_{\mathrm{red}} \to \frac{\mathrm{B}(H)}{\mathscr{K}(H)}.$$

We have to show that this homomorphism factorizes through $C(\mathbb{G})_{\mathrm{red}} \otimes_{\min} C(\mathbb{G})_{\mathrm{red}}$. But we also have the homomorphism $\mathscr{B}_\infty \to \mathrm{B}(H)/\mathscr{K}(H)$. It follows from

Proposition 3.8 that $\rho(C(\mathbb{G})_{\mathrm{red}})$ and $\mathscr{B}_\infty$ commute in $\mathrm{B}(H)/\mathscr{K}(H)$. Hence, we get a homomorphism

$$(\widehat{\mathbb{G}} \ltimes \mathscr{B}_\infty) \underset{\max}{\otimes} C(\mathbb{G})_{\mathrm{red}} \to \frac{\mathrm{B}(H)}{\mathscr{K}(H)}.$$

Since $\widehat{\mathbb{G}} \ltimes \mathscr{B}_\infty$ is nuclear, the left-hand side equals $(\widehat{\mathbb{G}} \ltimes \mathscr{B}_\infty) \otimes_{\min} C(\mathbb{G})_{\mathrm{red}}$, and we are done. □

Combining with Theorem 2.5, we get the following result.

COROLLARY 4.8

Let $\mathbb{G} = A_o(F)$, and denote $M = C(\mathbb{G})''_{\mathrm{red}}$. Then M is a generalized solid von Neumann algebra.

5. Probabilistic interpretations of the boundary $\mathscr{B}_\infty$

A natural setting where boundaries of discrete groups appear comes from considering (invariant) random walks on the group. One associates to such a random walk a Poisson boundary, which is a probability space, and a Martin boundary, which comes from a bona fide compactification of the group.

Both notions of Poisson boundary and Martin boundary have been generalized to random walks on discrete quantum groups (see [5], [8], [13], [14], [17], [21], [23]).

In this section, we show that the Martin boundary for the dual of $A_o(F)$ is naturally isomorphic with the boundary $\mathscr{B}_\infty$ constructed above. Moreover, the Poisson boundary is isomorphic with the von Neumann algebra generated by $\mathscr{B}_\infty$ in the GNS construction of a natural harmonic state on $\mathscr{B}_\infty$. Bounded harmonic elements of $\ell^\infty(\widehat{\mathbb{G}})$ are written with a Poisson integral formula. Note, in this respect, that a theorem establishing the link between Martin boundary and Poisson boundary for general discrete quantum groups has not yet been established (see [17]).

Notation 5.1

Recall the states φ_x and ψ_x introduced in Notation 1.11. For every probability measure μ on $\mathrm{Irred}(\mathbb{G})$, we consider the states

$$\psi_\mu = \sum_x \mu(x)\psi_x \qquad \text{and} \qquad \varphi_\mu = \sum_x \mu(x)\varphi_x .$$

Associated with these states are the Markov operators

$$P_\mu = (\varphi_\mu \otimes \mathrm{id})\hat{\Delta} \qquad \text{and} \qquad Q_\mu = (\mathrm{id} \otimes \psi_\mu)\hat{\Delta}.$$

Note that a state ω is of the form ψ_μ if and only if the Markov operator $(\mathrm{id} \otimes \omega)\hat{\Delta}$ preserves the center of $\ell^\infty(\widehat{\mathbb{G}})$ (see, e.g., [17, Proposition 2.1]). Also, note that we have a convolution product $\mu \star \nu$ on the measures on $\mathrm{Irred}(G)$ such that $\psi_{\mu\star\nu} = \psi_\mu \star \psi_\nu$ and $\varphi_{\mu\star\nu} = \varphi_\mu \star \varphi_\nu$.

The operators P_μ and Q_μ are the Markov operators associated with a *quantum random walk*. Their restriction to the center of $\ell^\infty(\widehat{\mathbb{G}})$ yields a Markov operator for a classical random walk on the state space $\mathrm{Irred}(\mathbb{G})$ with transition probabilities $p(x, y)$ and n-step transition probabilities $p_n(x, y)$ given by

$$p_x p(x, y) = p_x Q_\mu(p_y), \qquad p_x p_n(x, y) = p_x Q_\mu^n(p_y). \tag{5.1}$$

Note that $p_n(e, y) = \mu^{\star n}(y) = \psi_\mu^{\star n}(p_y)$.

Definition 5.2
The probability measure μ on $\mathrm{Irred}(\mathbb{G})$ is said to be transient if $\sum_{n=0}^\infty p_n(x, y) < \infty$ for all $x, y \in \mathrm{Irred}(\mathbb{G})$.

Contrary to the case of random walks on discrete groups, probability measures on $\mathrm{Irred}(\mathbb{G})$ are very often transient (see [17, Proposition 2.6]). In particular, if $\mathbb{G} = A_o(F)$ with $\mathbb{G} \ncong \mathrm{SU}(2), \mathrm{SU}_{-1}(2)$, every probability measure not concentrated in zero is transient.

Poisson boundary
Definition 5.3
For any probability measure μ on $\mathrm{Irred}(\mathbb{G})$, we define

$$H^\infty(\widehat{\mathbb{G}}, \mu) = \big\{a \in \ell^\infty(\widehat{\mathbb{G}}) \,\big|\, Q_\mu(a) = a\big\}.$$

The weakly closed linear space $H^\infty(\widehat{\mathbb{G}}, \mu)$ is in fact a von Neumann algebra with product given by

$$a \cdot b = \mathrm{s}^*\text{-}\lim_{n\to\infty} Q_\mu^n(ab).$$

Note that the Poisson boundary has a natural interpretation as a relative commutant in the study of infinite tensor-product actions (see [13], [22]).

Terminology 5.4
The support of a measure μ on $\mathrm{Irred}(\mathbb{G})$ is denoted by $\mathrm{supp}\,\mu$. We say that μ is generating if

$$\mathrm{Irred}(\mathbb{G}) = \bigcup_{n=1}^\infty \mathrm{supp}(\mu^{\star n}).$$

The restriction of $\hat{\varepsilon}$ to $H^\infty(\widehat{\mathbb{G}}, \mu)$ defines a normal state on $H^\infty(\widehat{\mathbb{G}}, \mu)$. This state is faithful when μ is generating.

From now on, we fix $\mathbb{G} = A_o(F)$ for a given matrix F satisfying $F\overline{F} = \pm 1$.

Since the fusion rules of $A_o(F)$ are abelian, we know from [14, Proposition 1.1] that $H^\infty(\widehat{\mathbb{G}}, \mu)$ does not depend on the choice of a *generating* measure μ. Moreover, a measure μ on $\mathbb{N} = \mathrm{Irred}(\mathbb{G})$ is generating if and only if its support contains an odd number.

The aim of this section is to define a *harmonic measure* (i.e., a state) on the boundary $\mathscr{B}_\infty$ and to write every harmonic function (i.e., element of $H^\infty(\widehat{\mathbb{G}}, \mu)$) as an integral with respect to the harmonic measure.

PROPOSITION 5.5
The formula

$$\omega(a) = \lim_{n\to\infty} \psi_n(a)$$

yields a well-defined state on $\mathscr{B}$, *and* $\omega(c_0(\widehat{\mathbb{G}})) = \{0\}$. *The resulting state on* $\mathscr{B}_\infty = \mathscr{B}/c_0(\widehat{\mathbb{G}})$ *is denoted by* ω_∞.

Proof
It suffices to observe that $\psi_{x+y}\psi_{x+y,x} = \psi_x$ for all $x, y \in \mathrm{Irred}(\mathbb{G})$. □

Denote by $(\mathscr{B}_\infty, \omega_\infty)''$ the von Neumann algebra generated by $\mathscr{B}_\infty$ in the GNS construction for the state ω_∞. It is easy to check that ω_∞ is a Kubo-Martin-Schwinger (KMS) state on $\mathscr{B}_\infty$ with modular group given by $\sigma_t^{\omega_\infty}(\psi_{\infty,x}(A)) = \psi_{\infty,x}(Q_x^{it} A Q_x^{-it})$ for all $t \in \mathbb{R}$, $x \in \mathrm{Irred}(\mathbb{G})$, and $A \in \mathrm{B}(H_x)$. In particular, ω_∞ induces a normal faithful state on $(\mathscr{B}_\infty, \omega_\infty)''$.

THEOREM 5.6
Denote $\mathbb{G} = A_o(F)$, *and suppose that* $\mathbb{G} \ncong \mathrm{SU}(2), \mathrm{SU}_{-1}(2)$. *Let* μ *be any generating measure on* $\mathrm{Irred}(\mathbb{G})$. *The linear map*

$$T : \mathscr{B}_\infty \to \ell^\infty(\widehat{\mathbb{G}}) : T(a) = (\mathrm{id} \otimes \omega_\infty)\beta_\infty(a)$$

yields a **-homomorphism* $T : \mathscr{B}_\infty \to H^\infty(\widehat{\mathbb{G}}, \mu)$ *satisfying* $\hat{\varepsilon} T = \omega_\infty$. *Moreover,* T *yields a* **-isomorphism*

$$(\mathscr{B}_\infty, \omega_\infty)'' \cong \big(H^\infty(\widehat{\mathbb{G}}, \mu), \hat{\varepsilon}\big).$$

Proof
We first claim that $(\psi_x \otimes \omega_\infty)\beta_\infty(a) = \omega_\infty(a)$ for all $a \in \mathscr{B}_\infty$. It then follows that $T(a) \in H^\infty(\widehat{\mathbb{G}}, \mu)$ for all $a \in \mathscr{B}_\infty$. To prove our claim, observe that for all x and

$y \geq x$, $(\psi_x \otimes \psi_y)\hat{\Delta}$ is a convex combination of $\psi_{y-x}, \ldots, \psi_{y+x}$. It follows that $(\psi_x \otimes \omega)\beta(a) = \omega(a)$ for all $a \in \mathscr{B}$.

To show that T is multiplicative, it suffices to show that for all fixed $a, b \in \mathscr{B}_\infty$,

$$\big\| \big(T(a)T(b) - T(ab)\big) p_n \big\| \to 0 \tag{5.2}$$

when $n \to \infty$. Indeed, for fixed $a, b \in \mathscr{B}_\infty$ and a fixed x,

$$\big(T(a) \cdot T(b)\big) p_x = \operatorname*{s^*\text{-}lim}_{n\to\infty} (\mathrm{id} \otimes \psi_\mu^{\star n})\hat{\Delta}\big(T(a)T(b)\big) p_x.$$

By the transience of the state ψ_μ, the expression on the right-hand side for n big, takes into account only $T(a)T(b)p_m$ for m big. This last expression is close to $T(ab)p_m$. But $(\mathrm{id} \otimes \psi_\mu^{\star n})\hat{\Delta}(T(ab)) = T(ab)$, and we are done.

Hence, to prove the multiplicativity of T, it remains to show (5.2). It suffices to show that for all x and $A \in \mathrm{B}(H_x)$ fixed,

$$\big\| (\mathrm{id} \otimes \omega_\infty)\beta_\infty\big(\psi_{\infty,x}(A)\big) p_n - \psi_{n,x}(A) \big\| \to 0 \tag{5.3}$$

when $n \to \infty$. Fix x and $A \in \mathrm{B}(H_x)$. Take $y \geq x$ and $z \geq y$. Then

$$\begin{aligned}(\mathrm{id} \otimes \psi_{x+z})\hat{\Delta}\big(\psi_{\infty,x}(A)\big) p_y &= \sum_{s \in y \otimes z} (\mathrm{id} \otimes \psi_{x+z})\big(V(y \otimes (x+z), x+s)\psi_{x+s,x}(A) \\ &\qquad \times V(y \otimes (x+z), x+s)^*\big) \\ &= \sum_{s \in y\otimes z} \frac{[x+s+1]}{[y+1][x+z+1]} V\big((x+s)\otimes(x+z), y\big)^* \\ &\qquad \times \big(\psi_{x+s,x}(A) \otimes 1\big) V\big((x+s)\otimes(x+z), y\big).\end{aligned}$$

From Lemma A. 2, we get a constant C depending only on q such that

$$\begin{aligned}&d_{\mathbb{T}}\big((V(x \otimes s, x+s) \otimes 1)V((x+s)\otimes(x+z), y), \\ &\quad (1 \otimes V(s \otimes (x+z), y-x))V(x \otimes (y-x), y)\big) \leq Cq^{-x-z+s}.\end{aligned}$$

Note, however, that this statement makes sense only when $y - x \in s \otimes (x+z)$. This is the case for $s \geq z - y + 2x$, and so, we can safely go on because our estimate is greater than 1 if $s < z - y + 2x$. Hence, we get a constant D such that

$$(\mathrm{id} \otimes \psi_{x+z})\hat{\Delta}\big(\psi_{\infty,x}(A)\big) p_y \approx \sum_{s \in y \otimes z} \frac{[x+s+1]}{[y+1][x+z+1]} \psi_{y,x}(A) = \psi_{y,x}(A)$$

with error at most

$$\sum_{s\in y\otimes z} 2\|A\|C\frac{[x+s+1]q^{x+s}}{[y+1][x+z+1]q^{x+z}}q^{-x} \leq Dq^{-x}\|A\|\frac{2y+1}{[y+1]}.$$

Since this estimate holds for all $z \geq y$, we find that

$$\left\|(\mathrm{id}\otimes\omega_\infty)\beta_\infty\big(\psi_{\infty,x}(A)\big)p_y - \psi_{y,x}(A)\right\| \leq Dq^{-x}\|A\|\frac{2y+1}{[y+1]}.$$

So, (5.3) follows, and the multiplicativity of T has been proved.

It is obvious that $\hat{\varepsilon}T = \omega_\infty$. Consider the adjoint action of $\mathbb{G}$ on $\ell^\infty(\widehat{\mathbb{G}})$ given by $\Phi(a) = \mathbb{V}(a\otimes 1)\mathbb{V}^*$ for all $a \in \ell^\infty(\widehat{\mathbb{G}})$. Since

$$U^{x+y}\big(\psi_{x+y,x}(A)\otimes 1\big)(U^{x+y})^* = (\psi_{x+y,x}\otimes\mathrm{id})\big(U^x(A\otimes 1)(U^x)^*\big),$$

the action Φ restricts to an action of $\mathbb{G}$ on $\mathscr{B}$. Moreover, the action Φ preserves the ideal $c_0(\widehat{\mathbb{G}}) \subset \mathscr{B}$, yielding an action Φ_∞ of $\mathbb{G}$ on $\mathscr{B}_\infty$. We have $(\mathrm{id}\otimes h)\Phi_\infty(a) = \omega_\infty(a)1$ for all $a \in \mathscr{B}_\infty$. So, Φ_∞ is an ergodic action, and ω_∞ is the unique invariant state. By definition,

$$\Phi_\infty\big(\psi_{\infty,x}(A)\big) = (\psi_{\infty,x}\otimes\mathrm{id})\big(U^x(A\otimes 1)(U^x)^*\big).$$

From Lemma 3.9, we know that $\psi_{\infty,x} : \mathrm{B}(H_x) \to \mathscr{B}_\infty$ is an injective linear map. So, we conclude that the irreducible representation U^x appears with multiplicity one in Φ_∞ when x is even and with multiplicity zero when x is odd. Moreover, the *-homomorphism T intertwines Φ_∞ with the adjoint action of $\mathbb{G}$ on $H^\infty(\widehat{\mathbb{G}},\mu)$.

From [14, Corollary 3.5], we know that the multiplicities of the irreducible representations in the adjoint action on $H^\infty(\widehat{\mathbb{G}},\mu)$ are at most the multiplicities in Φ_∞. Since ω_∞ yields a faithful, normal state on $(\mathscr{B}_\infty,\omega_\infty)''$ and since $\hat{\varepsilon}T = \omega_\infty$, the homomorphism $T : (\mathscr{B}_\infty,\omega_\infty)'' \to H^\infty(\widehat{\mathbb{G}},\mu)$ is faithful. It follows that T is a *-isomorphism. □

Martin boundary

The Martin boundary and the Martin compactification of a discrete quantum group have been defined by Neshveyev and Tuset [17]. We first introduce the necessary terminology and notation and then prove that the Martin compactification of the dual of $A_o(F)$ is equal to the compactification $\mathscr{B}$ constructed above.

Let $\widehat{\mathbb{G}}$ be a discrete quantum group, and let μ be a probability measure on $\mathrm{Irred}(\mathbb{G})$. We have an associated Markov operator Q_μ and a classical random walk on $\mathrm{Irred}(\mathbb{G})$ with n-step transition probabilities given by (5.1). We suppose throughout that μ is a generating measure and that μ is transient. It follows that $0 < \sum_{n=1}^\infty p_n(x,y) < \infty$ for all $x, y \in \mathrm{Irred}(\mathbb{G})$.

Denote by $c_c(\widehat{\mathbb{G}}) \subset c_0(\widehat{\mathbb{G}})$ the algebraic direct sum of the algebras $\mathrm{B}(H_x)$. We define, for $a \in c_c(\widehat{\mathbb{G}})$,

$$G_\mu(a) = \sum_{n=0}^{\infty} Q_\mu^n(a).$$

Observe that, usually, $G_\mu(a)$ is unbounded, but it makes sense in the multiplier algebra of $c_c(\widehat{\mathbb{G}})$ (i.e., $G_\mu(a)p_x \in \mathrm{B}(H_x)$ makes sense for every $x \in \mathrm{Irred}(\mathbb{G})$). Moreover, $G_\mu(p_\varepsilon)$ is strictly positive and central. This allows us to define the Martin kernel as follows.

Whenever μ is a measure on $\mathrm{Irred}(\mathbb{G})$, we use the notation $\overline{\mu}$ to denote the measure given by $\overline{\mu}(x) = \mu(\overline{x})$.

Definition 5.7 ([17, Definitions 3.1, 3.2])
Define

$$K_\mu : c_c(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}}) : K_\mu(a) = G_\mu(a)G_\mu(p_\varepsilon)^{-1}.$$

Define the *Martin compactification* $\widetilde{A}_\mu$ as the C^*-subalgebra of $\ell^\infty(\widehat{\mathbb{G}})$ generated by $K_{\overline{\mu}}(c_c(\widehat{\mathbb{G}}))$ and $c_0(\widehat{\mathbb{G}})$. Define the *Martin boundary* A_μ as the quotient $\widetilde{A}_\mu / c_0(\widehat{\mathbb{G}})$.

The aim of this section is to prove the following result.

THEOREM 5.8
Denote $\mathbb{G} = A_o(F)$, and suppose that $\mathbb{G} \ncong \mathrm{SU}(2), \mathrm{SU}_{-1}(2)$. Let μ be a generating measure on $\mathrm{Irred}(\mathbb{G})$ with finite first moment

$$\sum_{x \in \mathbb{N}} x\mu(x) < \infty.$$

Then the Martin compactification $\widetilde{A}_\mu$ equals the compactification $\mathcal{B}$ defined in Proposition 3.4. In particular, the Martin boundary A_μ equals $\mathcal{B}_\infty$.

Proof
Introduce the notation $p_x g_\mu(x, y) = p_x G_\mu(p_y)$. One has $g_\mu(0, x) = g_{\overline{\mu}}(x, 0) \dim_q(x)^2$. So, if $A \in \mathrm{B}(H_x)$ and $y \geq x$, we get

$$G_{\overline{\mu}}(Ap_x)p_y = \sum_{z \in x \otimes y} g_{\overline{\mu}}(0, z)(\mathrm{id} \otimes \psi_z)\big(V(y \otimes z, x)AV(y \otimes z, x)^*\big).$$

An easy computation yields

$$K_{\overline{\mu}}(Ap_x)p_y = \sum_{z=0}^{x} \frac{g_\mu(y-x+2z,0)}{g_\mu(y,0)} \frac{\dim_q(y-x+2z)\dim_q(x)}{\dim_q(y)} \times V\big(x \otimes (y-x+2z)\big)^*(A \otimes 1)V\big(x \otimes (y-x+2z)\big).$$

From [17, Proposition 4.7], we know that

$$\lim_{x\to\infty} \frac{g_\mu(x+1,0)}{g_\mu(x,0)} = q^2.$$

Using Notation 5.9, the previous formula, and the asymptotics for the quantum dimensions, we find that for all $x \in \mathrm{Irred}(\mathbb{G})$ and $A \in \mathrm{B}(H_x)$,

$$\lim_{y\to\infty} \Big\| K_{\overline{\mu}}(Ap_x)p_y - \sum_{z=0}^{x} q^{-x+2z} \dim_q(x) \psi^z_{y,x}(A) \Big\| = 0.$$

Denoting by $[Y]$ the closed linear span of Y, we conclude that

$$\big[c_0(\widehat{\mathbb{G}}) + K_{\overline{\mu}}\big(c_c(\widehat{\mathbb{G}})\big)\big] = \Big[c_0(\widehat{\mathbb{G}}) + \sum_{z=0}^{x} q^{2z} \psi^z_{\infty,x}(A) \,\Big|\, x \in \mathbb{N},\, A \in \mathrm{B}(H_x)\Big]. \quad (5.4)$$

Recall the C^*-algebra $\mathscr{B}$ defined in Proposition 3.4. Combining (5.4) with (5.5) in Lemma 5.10, we conclude that $[c_0(\widehat{\mathbb{G}}) + K_{\overline{\mu}}(c_c(\widehat{\mathbb{G}}))] \subset \mathscr{B}$. The opposite inclusion follows by combining (5.4) with (5.6) in Lemma 5.10. In particular, the C^*-algebra generated by $c_0(\widehat{\mathbb{G}})$ and $K_{\overline{\mu}}(c_c(\widehat{\mathbb{G}}))$ equals $\mathscr{B}$. □

Notation 5.9

Definition 3.1 admits the following natural generalization. For all $y \geq x \geq z$, we write

$$\psi^z_{y,x} : \mathrm{B}(H_x) \to \mathrm{B}(H_y) : \psi^z_{y,x}(A) = V\big(x \otimes (y-x+2z), y\big)^*(A \otimes 1)V\big(x \otimes (y-x+2z), y\big).$$

Note that $\psi_{y,x} = \psi^0_{y,x}$. We define as well $\psi^z_{\infty,x}(A)$ for $x \geq z$ and $A \in \mathrm{B}(H_x)$ by

$$\psi^z_{\infty,x}(A)p_y = \psi^z_{y,x}(A) \quad \text{whenever } y \geq x.$$

LEMMA 5.10

There exists a constant $C > 0$ depending only on q such that for all $x, y, z \in \mathbb{N}$ and

$A \in \mathrm{B}(H_x)$, we have

$$\|\psi_{x+y+z,x+y}\psi^r_{x+y,x}(A) - \psi^r_{x+y+z,x}(A)\| \le Cq^{y+r}\|A\| \quad \textit{for all } 0 \le r \le x, \quad (5.5)$$

$$\|\psi^r_{x+y+z,x+y}\psi_{x+y,x}(A) - \psi_{x+y+z,x}(A)\| \le Cq^{y-r}\|A\| \quad \textit{for all } 0 \le r \le x+y. \quad (5.6)$$

Proof

Inequality (5.5) follows from (A.2) in Lemma A.1, while (5.6) follows from (A.5) in Lemma A.2. □

6. A general exactness result

In [6], a notion of monoidal equivalence of compact quantum groups was introduced. It was shown, in particular, that for all $F \in \mathrm{GL}(n, F)$ with $F\overline{F} = c1$, $c = \pm 1$, $\mathrm{Tr}(F^*F) = q + 1/q$, and $0 < q < 1$, the quantum groups $A_o(F)$ and $\mathrm{SU}_{-cq}(2)$ are monoidally equivalent.

In this section, we prove that the exactness of the reduced C^*-algebra of a compact quantum group is invariant under monoidal equivalence. As a corollary, we obtain an alternative proof for the first half of Corollary 4.7.

THEOREM 6.1

Let $\mathbb{G}$ and $\mathbb{G}_1$ be compact quantum groups. Let $\varphi : \mathbb{G} \to \mathbb{G}_1$ be a monoidal equivalence in the sense of [6, Definition 3.1]. Let B_{red} be the associated reduced C^*-algebra. The following statements are equivalent:

- *$C(\mathbb{G})_{\mathrm{red}}$ is exact;*
- *$C(\mathbb{G}_1)_{\mathrm{red}}$ is exact; and*
- *B_{red} is exact.*

Proof

Following [6, Theorem 3.9], we consider the *-algebra $\mathscr{B}$ generated by the coefficients of unitary elements $X^x \in \mathrm{B}(H_x, H_{\varphi(x)}) \otimes \mathscr{B}$. We consider the canonical invariant state ω on $\mathscr{B}$ and denote by B_{red} the associated reduced C^*-algebra.

By symmetry (i.e., using the inverse $\varphi^{-1} : \mathbb{G}_1 \to \mathbb{G}$), we consider the *-algebra $\widetilde{\mathscr{B}}$ generated by the coefficients of unitary elements $Y^x \in \mathrm{B}(H_{\varphi(x)}, H_x) \otimes \widetilde{\mathscr{B}}$. We denote by $\widetilde{\omega}$ the invariant state on $\widetilde{\mathscr{B}}$ and by $\widetilde{B}_{\mathrm{red}}$ the associated reduced C^*-algebra. Observe that we have a canonical anti-isomorphism $\pi : B_{\mathrm{red}} \to \widetilde{B}_{\mathrm{red}}$ given by $(\mathrm{id} \otimes \pi)(X^x) = (Y^x)^*$ for all $x \in \mathrm{Irred}(\mathbb{G})$. So, $\widetilde{B}_{\mathrm{red}}$ is nothing else than the opposite of B_{red}.

Suppose first that B_{red} is exact. Then $\widetilde{B}_{\mathrm{red}}$ is also exact. Moreover, we get an injective *-homomorphism $\theta : C(\mathbb{G})_{\mathrm{red}} \to \widetilde{B}_{\mathrm{red}} \otimes B_{\mathrm{red}}$ given by $(\mathrm{id} \otimes \theta)(U^x) = Y^x_{12}X^x_{13}$. A priori, θ defines a *-homomorphism $C(\mathbb{G})_{\mathrm{u}} \to \widetilde{B}_{\mathrm{red}} \otimes B_{\mathrm{red}}$, but it is easy to verify that $(\widetilde{\omega} \otimes \omega)\theta$ is the Haar state. So, θ is well defined on $C(\mathbb{G})_{\mathrm{red}}$. It follows that $C(\mathbb{G})_{\mathrm{red}}$ is exact. In a similar way, we deduce that $C(\mathbb{G}_1)_{\mathrm{red}}$ is exact.

Suppose next that $C(\mathbb{G})_{\mathrm{red}}$ is exact. Then $\widehat{\mathbb{G}}$ is an exact quantum group. Since B_{red} is Morita equivalent with a reduced crossed product $\widehat{\mathbb{G}}_{\mathrm{r}} \ltimes \mathscr{K}$, it follows that B_{red} is exact. In a similar way, exactness of $C(\mathbb{G}_1)_{\mathrm{red}}$ implies exactness of B_{red}. □

COROLLARY 6.2
Let $\mathbb{G} = A_o(F)$. Then $C(\mathbb{G})_{\mathrm{red}}$ is exact.

Proof
By amenability of (the dual of) $\mathrm{SU}_q(2)$, the exactness of its reduced (and universal) C^*-algebra is obvious. The result follows since every $A_o(F)$ is monoidally equivalent with some $\mathrm{SU}_q(2)$. □

The same argument admits the following generalization.

COROLLARY 6.3
The reduced C^*-algebra of any compact quantum group monoidally equivalent with a q-deformation of a simple compact Lie group is exact.

Remark 6.4
We can as well give a sledgehammer argument for the exactness of the reduced C^*-algebra of $A_u(F)$. It follows from [6] that any $A_u(F)$ is monoidally equivalent with an $A_u(F)$ with $F \in \mathrm{GL}(2, \mathbb{C})$. But it follows from [4] that the reduced C^*-algebra of such an $A_u(F)$ is a subalgebra of the reduced free product of $\mathrm{SU}_q(2)$ and S^1, and hence, we are done.

7. Factoriality and simplicity

We prove that at least in most cases, the von Neumann algebras associated with $\mathbb{G} = A_o(F)$ are factors. We determine their Connes invariants, and we prove that the reduced C^*-algebras are simple. In combination with the results above on the Akemann-Ostrand property, we obtain new examples of generalized solid (in particular, prime) factors.

On the von Neumann algebra side, we get the following.

THEOREM 7.1
Let $N \geq 3$, and let $F \in \mathrm{GL}(N, \mathbb{C})$ with $F\overline{F} = \pm 1$. Suppose that $\|F\|^2 \leq \mathrm{Tr}(FF^*)/\sqrt{5}$. Write $\mathbb{G} = A_o(F)$ and $M = C(\mathbb{G})''_{\mathrm{red}}$.

- *M is a full generalized solid factor with almost periodic state h. In particular, M is prime.*

- $\mathrm{Sd}(M)$ *is the subgroup* Γ *of* $\mathbb{R}_+^*$ *generated by the eigenvalues of* $Q \otimes Q^{-1}$*. In particular,* M *is of type* II_1 *when* $FF^* = 1$*, of type* III_λ *when* $\Gamma = \lambda^{\mathbb{Z}}$*, and of type* III_1 *in the other cases.*
- *The* C^**-subalgebra of* $\mathrm{B}(L^2(\mathbb{G}))$ *generated by* $\lambda(C(\mathbb{G}))$ *and* $\rho(C(\mathbb{G}))$ *contains the compact operators.*

If $FF^* = 1$ *and* $N \geq 3$*,* M *is a solid (in particular, prime)* II_1*-factor.*

On the C^*-algebra side, we obtain the following.

THEOREM 7.2
Let $N \geq 3$*, and let* $F \in \mathrm{GL}(N, \mathbb{C})$ *with* $F\overline{F} = \pm 1$*. Suppose that* $\|F\|^8 \leq (3/8)\,\mathrm{Tr}(FF^*)$*. Write* $\mathbb{G} = A_o(F)$*. Then* $C(\mathbb{G})_{\mathrm{red}}$ *is a simple exact* C^**-algebra, and* h *is the unique state on* $C(\mathbb{G})_{\mathrm{red}}$ *satisfying the KMS condition with respect to* (σ_t^h)*. In particular, for* $\mathbb{G} = A_o(I_N)$ *and* $N \geq 3$*,* $C(\mathbb{G})_{\mathrm{red}}$ *is a simple exact* C^**-algebra with unique tracial state* h*.*

Theorems 7.1 and 7.2 are proved through a careful analysis of the quantum analogue of the operation of conjugation by the generators in free groups (see Definition 7.3).

Fix a matrix $F \in \mathrm{GL}(n, \mathbb{C})$ satisfying $F\overline{F} = \pm 1$, and put $\mathbb{G} = A_o(F)$. Define $H_1 = \mathbb{C}^n$, and define $U^1 := U$, the fundamental representation on H_1. Recall that the modular theory of the compact quantum group $\mathbb{G}$ is encoded by positive invertible elements $Q_x \in \mathrm{B}(H_x)$. In the case of $\mathbb{G} = A_o(F)$, we write $Q := Q_1$, and we have

$$Q = F^t\overline{F} \qquad \text{and} \qquad Q^{-1} = FF^*.$$

Write $F\overline{F} = c1$ with $c = \pm 1$. Write $t_1 = \mathrm{Tr}(Q)^{-1/2} \sum_{i=1}^n e_i \otimes Fe_i$, which is a *unit* invariant vector for the tensor square $U^{\otimes 2}$.

In order to study factoriality and simplicity, we introduce the following operators, using Notation 1.18. Recall as well the regular representation $\rho : C(\mathbb{G}) \to \mathrm{B}(L^2(\mathbb{G}))$ given by (1.3). We denote the antihomomorphism ρ^{op} defined by

$$\rho^{\mathrm{op}}(a)\rho(b)\xi_0 = \rho(ba)\xi_0 \quad \text{for all } a \in C_{\mathrm{alg}}(\mathbb{G}),\, b \in C(\mathbb{G}),$$

where $C_{\mathrm{alg}}(\mathbb{G}) \subset C(\mathbb{G})$ is the dense *-subalgebra given by the coefficients of finite-dimensional representations of $\mathbb{G}$. Note that ρ^{op} is not involutive. We have $\rho^{\mathrm{op}}(a)^* = \rho^{\mathrm{op}}(\sigma_i^h(a)^*)$, where the modular group (σ_t^h) is given by

$$(\mathrm{id} \otimes \sigma_t^h)(U) = (Q^{it} \otimes 1)U(Q^{it} \otimes 1).$$

Definition 7.3
We define operators T and $\widetilde{P}$ as follows:

$$\begin{aligned} T : L^2(\mathbb{G}) \to H_1 \otimes L^2(\mathbb{G}) \otimes H_1 : \mathrm{flip} \circ T &= \frac{1}{\dim_q(1)}\big((\widehat{\Lambda}_L \otimes \rho)(U) - (\widehat{\Lambda}_L \otimes \rho^{\mathrm{op}})(U)\big), \\ \widetilde{P} : C(\mathbb{G}) \to C(\mathbb{G}) : \widetilde{P}(a) &= \frac{1}{2\,\mathrm{Tr}(Q)}\big((\mathrm{Tr}_Q \otimes \mathrm{id})(U^*(1 \otimes a)U) \\ &\qquad + (\mathrm{Tr}_{Q^{-1}} \otimes \mathrm{id})(U(1 \otimes a)U^*)\big), \end{aligned}$$

where flip denotes the identification

$$\mathrm{flip} : H_1 \otimes H_1 \otimes L^2(\mathbb{G}) \to H_1 \otimes L^2(\mathbb{G}) \otimes H_1 : \mathrm{flip}(\xi \otimes \eta \otimes \mu) = \eta \otimes \mu \otimes \xi.$$

We also use $\widetilde{P}$ on the Hilbert space level, writing

$$P : L^2(\mathbb{G}) \to L^2(\mathbb{G}) : P\rho(a)\xi_0 = \rho\big(\widetilde{P}(a)\big)\xi_0 \quad \text{for all } a \in C(\mathbb{G}).$$

It is straightforward to check that

$$\begin{aligned} T &= \frac{c}{\dim_q(1)^{1/2}} \sum_{i,j=1}^{n} Fe_j \otimes \big(\rho(U_{ij}) - \rho^{\mathrm{op}}(U_{ij})\big) \otimes e_i, \\ T^*T &= 2(1-P). \end{aligned}$$

Remark 7.4
The relevance of the operator T in the study of the factoriality of $C(\mathbb{G})''_{\mathrm{red}}$ is clear. Indeed, $C(\mathbb{G})''_{\mathrm{red}}$ is a factor if and only if $T\eta = 0$ implies that $\eta \in \mathbb{C}\xi_0$.

Together with proving the factoriality of $C(\mathbb{G})''_{\mathrm{red}}$, we compute the Connes invariants for $C(\mathbb{G})''_{\mathrm{red}}$. In order to do so, we introduce the following deformation of T:

$$\begin{aligned} T_t &= \frac{1}{\dim_q(1)}\big((\widehat{\Lambda}_L \otimes \rho\sigma_t^h)(U) - (\widehat{\Lambda}_L \otimes \rho^{\mathrm{op}})(U)\big) \\ &= \frac{c}{\dim_q(1)^{1/2}} \sum_{i,j=1}^{n} Fe_j \otimes \big(\rho\sigma_t^h(U_{ij}) - \rho^{\mathrm{op}}(U_{ij})\big) \otimes e_i. \end{aligned}$$

The following is the major technical result of the section. It follows from a series of lemmas proved at the end of the section. We already deduce factoriality and simplicity results from it.

PROPOSITION 7.5
If $\|Q\| \leq \mathrm{Tr}(Q)/\sqrt{5}$, *there exist* $C_1 > C_2 > 0$ *such that*

$$\|T_s \xi\| \geq \sqrt{C_1^2 \|\mathring{\xi}\|^2 + D_s |\langle \xi_0, \xi\rangle|^2} - C_2 \|\mathring{\xi}\|$$

for all $s \in \mathbb{R}$, $\xi \in L^2(\mathbb{G})$, *where*

$$\xi = \mathring{\xi} + \xi_0 \langle \xi_0, \xi\rangle \qquad \textit{and} \qquad D_s = 2\big(1 - \big|\langle (1 \otimes Q^{is})t, t\rangle\big|^2\big).$$

We have already shown how Theorems 7.1 and 7.2 follow from Proposition 7.5.

Proof of Theorem 7.1
Write $S = C(\mathbb{G})_{\mathrm{red}}$, and consider the operator $P \in \mathrm{B}(L^2(\mathbb{G}))$ introduced in Definition 7.3. Note that $2(1 - P) = T^*T$. From the definition of P, we get the fact $P \in C^*(\lambda(S), \rho(S))$. From Proposition 7.5, we get $0 < C < 1$ such that the spectrum of P is included in $[0, C] \cup \{1\}$ and the spectral projection of $\{1\}$ is precisely the projection onto $\mathbb{C}\xi_0$. It follows that $C^*(\lambda(S), \rho(S))$ contains the compact operators and that $M = C(\mathbb{G})''_{\mathrm{red}}$ is a full factor. From Corollary 4.8, we already know that M is a generalized solid von Neumann algebra. Combining with Proposition 2.3, we get the fact that M is prime.

Denote by Γ the subgroup of $\mathbb{R}^*_+$ generated by the eigenvalues of $Q \otimes Q^{-1}$. In order to show that $\mathrm{Sd}(M) = \Gamma$, it suffices to show that given a sequence (s_n) in $\mathbb{R}$, $\sigma^h_{s_n} \to 1$ in $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ if and only if

$$\big|\langle (1 \otimes Q^{is_n})t, t\rangle\big| \to 1.$$

One implication being obvious, suppose that $\sigma^h_{s_n} \to 1$ in $\mathrm{Out}(M)$. Take unitaries $u_n \in M$ such that $(\mathrm{Ad}\, u_n)\sigma^h_{s_n} \to \mathrm{id}$ in $\mathrm{Aut}(M)$. It follows that

$$\|T_{s_n} \rho(u_n^*)\xi_0\| \to 0.$$

Applying Proposition 7.5, we first get that $\|(\rho(u_n^*)\xi_0)^\circ\| \to 0$, and then we obtain $|\langle (1 \otimes Q^{is_n})t, t\rangle| \to 1$. □

Proof of Theorem 7.2
Consider the operator $\widetilde{P} : C(\mathbb{G})_{\mathrm{red}} \to C(\mathbb{G})_{\mathrm{red}}$, as in Definition 7.3. From Proposition 7.5, we get a constant $0 < C < 1$ such that $\|\widetilde{P}(a)\|_2 \leq C\|a\|_2$ for all $a \in C(\mathbb{G})_{\mathrm{red}}$ with $h(a) = 0$. If $C(\mathbb{G})_n$ denotes the linear span of matrix coefficients of $U^0, \ldots, U^n$, we have $\widetilde{P}^k(C(\mathbb{G})_n) \subset C(\mathbb{G})_{n+2k}$. In particular, for $a \in C(\mathbb{G})_n$ such that $h(a) = 0$, we have

$$\|\widetilde{P}^k(a)\| \leq p(n+2k)\|Q\|^{n+2k}\|\widetilde{P}^k(a)\|_2 \leq p(n+2k)\|Q\|^n (C\|Q\|^2)^k \|a\|_2,$$

where p is a fixed polynomial given by the property of rapid decay (RD) for $A_o(F)$ (see Remark 7.6). Hence, if $C\|Q\|^2 < 1$, we find that $\widetilde{P}^k(a) \to 0$. Since $\widetilde{P}$ is unital, this implies that $\widetilde{P}^k(a) \to h(a)$ for any $a \in C(\mathbb{G})_{\mathrm{red}}$; hence, a cannot be in a nontrivial ideal. Moreover, $\widetilde{P}$ leaves invariant any state φ satisfying the KMS condition with respect to (σ_t^h); hence, h and φ agree on any $a \in C(\mathbb{G})_{\mathrm{red}}$. It follows from (7.9) in Remark 7.12 that the condition $C\|Q\|^2 < 1$ is satisfied whenever $\|Q\|^4 \leq (3/8)\,\mathrm{Tr}(Q)$. $\square$

Remark 7.6
In the proof of Theorem 7.2, we made use of property RD for universal quantum groups as introduced in [27]. This property yields a control over the norm in $C(\mathbb{G})_{\mathrm{red}}$ using the norm in $L^2(\mathbb{G})$.

Denote by $C(\mathbb{G})_n$ the linear span of matrix coefficients of $U^0, \dots, U^n$, and denote by $\|\cdot\|_2$ the GNS norm associated with h. One possible definition of property RD goes as follows:

$$\exists p \in \mathbb{R}[X] \quad \text{such that}\, \forall n \in \mathbb{N},\ a \in C(\mathbb{G})_n,\ \|\rho(a)\| \leq p(n)\|a\|_2.$$

In the case where $\mathbb{G} = A_o(F)$, it is proved in [27, Theorem 3.9] that property RD holds if and only if F is a multiple of a unitary matrix (i.e., $Q = 1$). In fact, the techniques of [27] still work in the nonunimodular case but yield nonpolynomial bounds. We get a polynomial $p \in \mathbb{R}[X]$ such that

$$\|\rho(a)\| \leq \|Q\|^n p(n)\|a\|_2$$

for all $n \in \mathbb{N}$ and all $a \in C(\mathbb{G})_n$.

So, it remains to prove Proposition 7.5, which takes the rest of the section.

The proof is not very hard but somewhat computationally involved. In order to streamline our computations, we choose explicit representatives for the irreducible representations of $\mathbb{G} = A_o(F)$, as well as for the intertwiners $V(x \otimes y, z)$ with tensor products of irreducible representations. We know that $\mathrm{Mor}(1^{\otimes n}, 1^{\otimes n})$ is isomorphic with the Temperley-Lieb algebra. In particular, we have the Jones-Wenzl projection $p_n \in \mathrm{Mor}(1^{\otimes n}, 1^{\otimes n})$, which allows us to define $H_n := p_n H_1^{\otimes n}$ and to take U^n as the restriction of the n-fold tensor product $U^{\otimes n}$ to H_n. We write $1_n = 1^{\otimes n}$.

Using [7, Theorem 3.7.1], we can recursively define the *unit* vectors $t_x \in \mathrm{Mor}(x \otimes x, 0)$ and the *isometries* $V((x+z) \otimes (z+y), x+y) \in \mathrm{Mor}((x+z) \otimes (z+y), x+y)$ using the formulas

$$t_1 = \mathrm{Tr}(F^*F)^{-1/2} \sum_{i=1}^{n} e_i \otimes Fe_i, \tag{7.1}$$

$$t_{x+y} = \Big(\frac{[x+1][y+1]}{[x+y+1]}\Big)^{1/2}(p_{x+y} \otimes p_{y+x})(1_x \otimes t_y \otimes 1_x)t_x, \tag{7.2}$$

$$\begin{aligned} &V\big((x+z) \otimes (z+y), x+y\big) \\ &\quad = \left(\frac{[z+1]\big[{x+z \atop z}\big]\big[{y+z \atop z}\big]}{\big[{x+y+z+1 \atop r}\big]}\right)^{1/2}(p_{x+z} \otimes p_{z+y})(1_x \otimes t_z \otimes 1_y)p_{x+y}. \end{aligned} \tag{7.3}$$

Here, we used the usual notation of q-numbers, q-factorials, and q-binomial coefficients. As before, we take $0 < q < 1$ such that

$$\mathrm{Tr}(F^*F) = q + \frac{1}{q}, \qquad F\overline{F} = c1, \quad \text{where } c = \pm 1.$$

Then we use the following notation.

Notation 7.7
With $0 < q < 1$ fixed, we write the q-numbers, q-factorials, and q-binomial coefficients:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]! = [n][n-1]\cdots[1], \qquad \big[{n \atop r}\big] = \frac{[n]!}{[r]![n-r]!}.$$

Note that $\dim_q(x) = [x+1]$. Wenzl's recursion formula for the projections p_n admits the following generalization (see [12, (3.8), p. 462]):

$$p_n = \Big(1 - \sum_{k=1}^{n-1}(-c)^{n-1-k}\frac{[2][k]}{[n]}(1_{k-1} \otimes t_1 \otimes 1_{n-1-k} \otimes t_1^*)\Big)(p_{n-1} \otimes 1). \tag{7.4}$$

Note that by multiplying on the left-hand side with $p_{n-1} \otimes 1$, we obtain Wenzl's recursion

$$p_n = p_{n-1} \otimes 1 - \frac{[2][n-1]}{[n]}(p_{n-1} \otimes 1)(1_{n-2} \otimes t_1 t_1^*)(p_{n-1} \otimes 1). \tag{7.5}$$

Notation 7.8
The study of T consists of comparing the left action and right action of the coefficients of U, and hence, it has a natural counterpart at the level of representations. More precisely, let us introduce the following shorthand notation:

$$\begin{aligned} \phi_l^+ &:= V\big(1 \otimes (x+1), x\big), \qquad & \phi_r^+ &:= V\big((x+1) \otimes 1, x\big), \\ \phi_l^- &:= V\big(1 \otimes (x-1), x\big), \qquad & \phi_r^- &:= V\big((x-1) \otimes 1, x\big). \end{aligned}$$

For any $x \in \mathbb{N}$, we also define $\sigma : H_x \otimes H_1 \to H_1 \otimes H_x$ by $\sigma(\eta \otimes \mu) = \mu \otimes \eta$.

LEMMA 7.9

We have $T = T^+ + T^-$, where $T^+(H_x \otimes H_x) \subset H_{x+1} \otimes H_{x+1}$ and $T^-(H_x \otimes H_x) \subset H_{x-1} \otimes H_{x-1}$ are defined by the formulas

$$T^+\eta = \sqrt{\frac{\dim_q(x+1)}{\dim_q(1)\dim_q(x)}}\,\big((Q^{-1} \otimes 1)\phi_l^+ \otimes \phi_r^+ \;-\; \sigma\phi_r^+ \otimes \sigma^*(Q^{-1} \otimes 1)\phi_l^+\big)\eta,$$

$$T^-\eta = \sqrt{\frac{\dim_q(x-1)}{\dim_q(1)\dim_q(x)}}\,\big((Q^{-1} \otimes 1)\phi_l^- \otimes \phi_r^- \;-\; \sigma\phi_r^- \otimes \sigma^*(Q^{-1} \otimes 1)\phi_l^-\big)\eta,$$

for all $\eta \in H_x \otimes H_x$. The corresponding formulas for T_s are obtained by composing the first term of each difference with $(Q^{is} \otimes 1) \otimes (1 \otimes Q^{-is})$.

Proof

By definition of the tensor product of representations of $\mathbb{G}$, we have

$$(\omega_{\eta,\xi} \otimes \rho)(U^x)(\omega_{\eta',\xi'} \otimes \rho)(U^{x'}) = \sum_{y \in x \otimes x'} (\omega_{V(x\otimes x',y)^*(\eta\otimes\eta'),V(x\otimes x',y)^*(\xi\otimes\xi')} \otimes \rho)(U^y).$$

Since the definition of T involves multiplication on the left and multiplication on the right by coefficients of U^1 and since $1 \otimes x$ and $x \otimes 1$ split into a direct sum of $x-1$ and $x+1$, clearly, T consists of four terms, say, $T = T^+ + T^-$ and $T^\pm = T_l^\pm - T_r^\pm$. More explicitly,

$$\begin{aligned} &T_l^+ \rho\big((\omega_{\eta,\xi} \otimes \mathrm{id})(U^x)\big)\xi_0 \\ &\quad = \frac{c}{\dim_q(1)^{1/2}} \sum_{i,j=1}^n Fe_j \otimes \rho\big((\omega_{(\phi_l^-)^*(e_i\otimes\eta),(\phi_l^-)^*(e_j\otimes\xi)} \otimes \mathrm{id})(U^{x+1})\big)\xi_0 \otimes e_i, \end{aligned} \tag{7.6}$$

and T_r^+, T_l^-, and T_r^- are defined analogously. From (1.3), we get

$$\rho\big((\omega_{\eta,\xi} \otimes \mathrm{id})(U^x)\big)\xi_0 = \xi \otimes (1 \otimes \eta^*)t_x.$$

To compute the right-hand side of (7.6), observe that

$$\begin{aligned} \sum_{i=1}^n \big(1 \otimes (e_i \otimes \eta)^*\phi_l^-\big)t_{x+1} \otimes e_i &= (1 \otimes 1 \otimes \eta^*)(1 \otimes \phi_l^-)t_{x+1} \\ &= (1 \otimes 1 \otimes \eta^*)(\phi_r^+ \otimes 1)t_x = \phi_r^+(1 \otimes \eta^*)t_x. \end{aligned} \tag{7.7}$$

Using the equality $\sum_{j=1}^{n} Fe_j \otimes e_j = c \dim_q(1)^{1/2}(Q^{-1} \otimes 1)t_1$, we observe as well that,

$$\sum_{j=1}^{n} Fe_j \otimes (\phi_l^-)^*(e_j \otimes \xi) = c \dim_q(1)^{1/2}\big(Q^{-1} \otimes (\phi_l^-)^*\big)(t_1 \otimes \xi)$$

$$= c\sqrt{\frac{\dim_q(x+1)}{\dim_q(x)}}(Q^{-1} \otimes 1)\phi_l^+\xi. \tag{7.8}$$

Combining (7.7) and (7.8), we get the fact that the right-hand side of (7.6) equals

$$\sqrt{\frac{\dim_q(x+1)}{\dim_q(1)\dim_q(x)}}\ \big((Q^{-1} \otimes 1)\phi_l^+ \otimes \phi_r^+\big)\big(\xi \otimes (1 \otimes \eta^*)t_x\big).$$

The formulas for T_r^+, T_l^-, and T_r^- are proved analogously. □

The proof of Proposition 7.5 follows immediately from the following two lemmas.

LEMMA 7.10
We have the following inequalities for a given $x \geq 1$ and using Notation 7.8:

$$(\phi_l^+)^*(Q^{-2} \otimes 1)\phi_l^+, (\phi_r^+)^*(1 \otimes Q^2)\phi_r^+ \geq \frac{\dim_q(1)\dim_q(x)}{\dim_q(x+1)} - \frac{\dim_q(x-1)}{\dim_q(x+1)}\|Q\|^2.$$

Proof
Observe that

$$(\phi_l^+)^*(Q^{-2} \otimes 1)\phi_l^+ = \frac{\dim_q(1)\dim_q(x)}{\dim_q(x+1)}\ (t_1^* \otimes 1_x)(Q^{-2} \otimes p_{x+1})(t_1 \otimes 1_x).$$

From a left-handed version of (7.5), we get

$$p_{x+1} = 1 \otimes p_x - \frac{\dim_q(1)\dim_q(x-1)}{\dim_q(x)}(1 \otimes p_x)(t_1 t_1^* \otimes 1_{x-1})(1 \otimes p_x).$$

Combining with the previous equality and using the facts

$$t_1^*(Q^{-2} \otimes 1)t_1 = 1 \qquad \text{and} \qquad (t_1^* \otimes 1)(Q^{-2} \otimes t_1 t_1^*)(t_1 \otimes 1) = \dim_q(1)^{-2}Q^{-2},$$

we obtain the first inequality of the lemma. The second one is proved analogously. □

Now, we prove a more interesting result, which states that the maps σ are far from being intertwiners in some sense, at least when $Q = 1$. Indeed, observe that the numerical coefficient $(\dim_q(x)+1)/\dim_q(x+1)$ in the next statement is always less than 1.

LEMMA 7.11
We have the following inequalities for a given $x \geq 1$ and using Notation 7.8:

$$\|(\phi_l^+)^*(Q^{-1-is} \otimes 1)\sigma\phi_r^+\| \leq \|Q\| \frac{\dim_q(x)+1}{\dim_q(x+1)}$$

for all $s \in \mathbb{R}$.

Proof
First, we have

$$(\phi_l^+)^*(Q^{-1-is} \otimes 1)\sigma\phi_r^+ = \frac{\dim_q(1)\dim_q(x)}{\dim_q(x+1)}\,(t_1^* \otimes 1_x)(Q^{-1-is} \otimes p_{x+1})\sigma(1_x \otimes t_1).$$

From (7.4), we get

$$p_{x+1} = (1 \otimes p_x)(p_x \otimes 1) - \frac{(-c)^{x+1}\dim_q(1)}{\dim_q(x)}\,(1 \otimes p_x)(t_1 \otimes 1_{x-1} \otimes t_1^*)(p_x \otimes 1),$$

and we easily conclude that

$$\begin{aligned}(\phi_l^+)^*(Q^{-1-is} \otimes 1)\sigma\phi_r^+ = \frac{\dim_q(x)}{\dim_q(x+1)}\;p_x\Big(&c(1_{x-1} \otimes Q^{1+is})\sigma^* \\ &- \frac{(-c)^{x+1}}{\dim_q(x+1)}(Q^{-1-is} \otimes 1_{x-1})\sigma\Big)p_x.\end{aligned}$$

The lemma follows from this equality. □

We finally prove Proposition 7.5.

Proof of Proposition 7.5
We write $\eta = \sum_x \eta_x$ with $\eta_x \in H_x \otimes H_x$ whenever $\eta \in L^2(\mathbb{G})$. Take $\eta \in L^2(\mathbb{G})$. By Lemmas 7.9, 7.10, and 7.11, we have, for $x \geq 2$,

$$\begin{aligned}\|(T_s^+\eta)_x\|^2 &\geq 2\Big(1 - \|Q\|^2\Big(\frac{[x-1]}{[2][x]} + \frac{(1+[x])^2}{[2][x][x+1]}\Big)\Big)\|\eta_{x-1}\|^2 \\ &= 2\Big(1 - 2\frac{1+[x]}{[2][x+1]}\|Q\|^2\Big)\|\eta_{x-1}\|^2.\end{aligned}$$

We also have

$$\|(T_s^+\eta)_1\|^2 = 2\big(1 - \big|\langle(1\otimes Q^{is})t, t\rangle\big|^2\big)\|\eta_0\|^2$$

and $(T_s^+\eta)_0 = 0$. By Lemma 7.9, we have, for all $x \geq 0$,

$$\|(T_s^-\eta)_x\|^2 \leq 4\|Q\|^2 \frac{[x+1]}{[2][x+2]}\|\eta_{x+1}\|^2.$$

Suppose now that $\|Q\|/[2] \leq 1/\sqrt{5}$. We put

$$C_1 = \sqrt{2}\Big(1 - 2\|Q\|^2\frac{1+[2]}{[2][3]}\Big)^{1/2}, \qquad C_2 = 2\|Q\|\Big(\frac{q}{[2]}\Big)^{1/2},$$

and we observe that C_1 is well defined and $C_2 < C_1$. So, Proposition 7.5 is proved. □

Remark 7.12
In the proof of Theorem 7.2, we need an estimate on the norm of P on $\xi_0^\perp$. With the notation introduced at the end of the proof of Proposition 7.5, we get

$$\|P\xi\| \leq \Big(1 - \frac{(C_1 - C_2)^2}{2}\Big)\|\xi\|$$

whenever $\xi \in L^2(\mathbb{G})$ and $\langle\xi_0, \xi\rangle = 0$. In order to prove Theorem 7.2, we need

$$\|Q\|^2\Big(1 - \frac{(C_1 - C_2)^2}{2}\Big) < 1.$$

If $\|Q\|^4 \leq a\,\mathrm{Tr}(Q)$ with $a \leq 1/\sqrt{5}$, we have

$$\|Q\|^2\Big(1 - \frac{(C_1 - C_2)^2}{2}\Big) \leq 2a\Big(\frac{1+[2]}{[3]} - q\Big) + 2a^{3/4}[2]^{1/4}\sqrt{2q}\sqrt{1 - 2\frac{1+[2]}{[2][3]}}.$$

Taking $a = 3/8$ and realizing that $\mathrm{Tr}(Q) \geq 3$, we conclude that

$$\|Q\|^2\|P\xi\| \leq 0,99\|\xi\| \tag{7.9}$$

for all $\xi \in \xi_0^\perp$ when $\|Q\|^4 \leq (3/8)\,\mathrm{Tr}(Q)$.

Appendix. Approximate commutation of intertwiners

In this appendix, we prove several estimates on the representation theory of $A_o(F)$. Weaker versions of these estimates were proved and used in [26].

We denote $d_{\mathbb{T}}(V, W) = \inf\{\|V - \lambda W\| \mid \lambda \in \mathbb{T}\}$ whenever V, W are in a Banach space.

We fix $F \in \mathrm{GL}(n, \mathbb{C})$ with $F\overline{F} = c1$ and $c = \pm 1$. We take $0 < q < 1$ such that $\mathrm{Tr}(F^*F) = q + 1/q$. We freely use the explicit choices that can be made for the representation theory of $\mathbb{G} = A_o(F)$ (see Section 7).

In this appendix, dealing only with the representation theory of $A_o(F)$, all small letters $a, b, c, x, y, z, r, s, \ldots$ denote elements of $\mathbb{N}$ (i.e., irreducible representations of $A_o(F)$).

LEMMA A.1
There exists a constant $C > 0$ depending only on q such that for all a, b, c and all $z \in a \otimes b$,

$$\bigl\| \bigl(V(a \otimes b, z) \otimes 1\bigr) p^{z\otimes c}_{z+c} - (1 \otimes p^{b\otimes c}_{b+c})\bigl(V(a \otimes b, z) \otimes 1\bigr) \bigr\| \leq Cq^{(z+b-a)/2}, \quad \text{(A.1)}$$

$$\begin{aligned} &d_{\mathbb{T}}\bigl((V(a \otimes b, z) \otimes 1)V(z \otimes c, z+c), \\ &\quad (1 \otimes V(b \otimes c, b+c))V(a \otimes (b+c), z+c)\bigr) \leq Cq^{(z+b-a)/2}, \end{aligned} \quad \text{(A.2)}$$

$$\begin{aligned} &d_{\mathbb{T}}\bigl((1 \otimes V(b \otimes c, b+c)^*)(V(a \otimes b, z) \otimes 1), \\ &\quad V(a \otimes (b+c), z+c)V(z \otimes c, z+c)^*\bigr) \leq Cq^{(z+b-a)/2}. \end{aligned} \quad \text{(A.3)}$$

If we write $z = a + b - 2s$ with $0 \leq s \leq \min\{a, b\}$, we have $(z + b - a)/2 = b - s$, and hence, $q^{(z+b-a)/2} \leq q^{-a+b}$.

It is easy to derive (A.2) and (A.3) from (A.1). Obviously, Lemma A.1 has a left-handed analogue.

LEMMA A.2
There exists a constant $C > 0$ depending only on q such that for all a, b, c and all $z \in b \otimes c$,

$$\bigl\| \bigl(1 \otimes V(b \otimes c, z)\bigr) p^{a\otimes z}_{a+z} - (p^{a\otimes b}_{a+b} \otimes 1)\bigl(1 \otimes V(b \otimes c, z)\bigr) \bigr\| \leq Cq^{(z+b-c)/2}, \quad \text{(A.4)}$$

$$\begin{aligned} &d_{\mathbb{T}}\bigl((1 \otimes V(b \otimes c, z))V(a \otimes z, a+z), \\ &\quad (V(a \otimes b, a+b) \otimes 1)V((a+b) \otimes c, a+z)\bigr) \leq Cq^{(z+b-c)/2}, \end{aligned} \quad \text{(A.5)}$$

$$\begin{aligned} &d_{\mathbb{T}}\bigl((V(a \otimes b, a+b)^* \otimes 1)(1 \otimes V(b \otimes c, z)), \\ &\quad V((a+b) \otimes c, a+z)V(a \otimes z, a+z)^*\bigr) \leq Cq^{(z+b-c)/2}. \end{aligned} \quad \text{(A.6)}$$

Remark A.3
It is possible to prove Lemma A.1 from explicit formulas for the quantum $6j$-symbols of $\mathrm{SU}_q(2)$. We give a more direct approach, for which we need only know the quantum $3j$-symbols (i.e., the coefficient appearing in (7.3)).

Before giving the proof of Lemma A.1, we introduce notation and several lemmas. It follows from [6] (using [3], [4]) that $A_o(F)$ is monoidally equivalent with $\mathrm{SU}_{-cq}(2)$, where q is as before and $F\overline{F} = c1$, $c = \pm 1$. So, we can perform all computations on the intertwiners as if we are dealing with the representation theory of $\mathrm{SU}_{-cq}(2)$.

Recall the explicit choices that can be made for the representation theory of $A_o(F)$ in Section 7.

LEMMA A.4
There exists a constant C depending only on q such that

$$\|(p_{a+b} \otimes 1_c)(1_a \otimes p_{b+c}) - p_{a+b+c}\| \leq Cq^b.$$

Proof
From (7.4), it follows that

$$\begin{aligned} p_{b+c+1} &= (p_b \otimes p_{c+1})p_{b+c+1} \\ &= \Big(1_b \otimes p_{c+1} - \frac{[2][b]}{[b+c+1]}(p_b \otimes p_{c+1})(1_{b-1} \otimes t \otimes 1_c \otimes t^*)\Big)(p_{b+c} \otimes 1). \end{aligned} \tag{A.7}$$

In the same way, it follows that

$$\begin{aligned} p_{a+b+c+1} &= (p_a \otimes p_b \otimes p_{c+1})p_{a+b+c+1} \\ &= \Big(1_{a+b} \otimes p_{c+1} - \frac{[2][a]}{[a+b+c+1]}(p_a \otimes p_b \otimes p_{c+1})(1_{a-1} \otimes t \otimes 1_{b+c} \otimes t^*) \\ &\qquad - \frac{[2][a+b]}{[a+b+c+1]}(p_a \otimes p_b \otimes p_{c+1})(1_{a+b-1} \otimes t \otimes 1_c \otimes t^*)\Big)(p_{a+b+c} \otimes 1). \end{aligned}$$

Since both $[2][a]/[a+b+c+1]$ and the difference $|[2][a+b]/[a+b+c+1] - [2][b]/[b+c+1]| = [2][a][c+1]/([b+c+1][a+b+c+1])$ can be estimated by Cq^{b+c} for a constant C depending only on q, we find

$$\begin{aligned} &p_{a+b+c+1} \\ &\approx \Big(1_{a+b} \otimes p_{c+1} - \frac{[2][b]}{[b+c+1]}(1_a \otimes p_b \otimes p_{c+1})(1_{a+b-1} \otimes t \otimes 1_c \otimes t^*)\Big)(p_{a+b+c} \otimes 1) \end{aligned} \tag{A.8}$$

with error at most Cq^{b+c}.

Now, put $\varepsilon(a,b,c) = \|(1_a \otimes p_{b+c})(p_{a+b} \otimes 1) - p_{a+b+c}\|$. Using (A.7), we find

$$\begin{aligned}
&(1_a \otimes p_{b+c+1})(p_{a+b} \otimes 1_{c+1}) \\
&\quad = \Big(1_{a+b} \otimes p_{c+1} - \frac{[2][b]}{[b+c+1]}(1_a \otimes p_b \otimes p_{c+1})(1_{a+b-1} \otimes t \otimes 1_c \otimes t^*)\Big) \\
&\qquad \times (1_a \otimes p_{b+c} \otimes 1)(p_{a+b} \otimes 1_{c+1}) \\
&\quad \approx \Big(1_{a+b} \otimes p_{c+1} - \frac{[2][b]}{[b+c+1]}(1_a \otimes p_b \otimes p_{c+1})(1_{a+b-1} \otimes t \otimes 1_c \otimes t^*)\Big) \\
&\qquad \times (p_{a+b+c} \otimes 1)
\end{aligned}$$

with error at most $\varepsilon(a,b,c)(1+Dq^c)$ because we can find D such that $[2][b]/[b+c+1] \leq Dq^c$. Combining with (A.8), we find

$$\varepsilon(a,b,c+1) \leq \varepsilon(a,b,c)(1+Dq^c) + Cq^{b+c}.$$

By induction, one concludes that

$$\varepsilon(a,b,c) \leq \sum_{k=0}^{c-1}\Big(Cq^{b+k}\prod_{j=k+1}^{c-1}(1+Dq^j)\Big).$$

It follows that

$$\varepsilon(a,b,c) \leq q^b\Big(\prod_{k=0}^{\infty}(1+Dq^k)\Big)\Big(\sum_{k=0}^{\infty}Cq^k\Big).$$

This concludes the proof of the lemma. $\square$

LEMMA A.5

There exists a constant C depending only on q such that

$$1 \leq \frac{\begin{bmatrix} a+r \\ r \end{bmatrix}\begin{bmatrix} r+b \\ r \end{bmatrix}}{\begin{bmatrix} a+b+r \\ r \end{bmatrix}} \leq C$$

for all a, b, r.

Proof

It is easy to find a constant C such that for all a, b and all $k \geq 1$, we have

$$1 \leq \frac{[a+k][b+k]}{[a+b+k][k]} \leq 1 + Cq^{2k}. \tag{A.9}$$

Because $\prod_{k=0}^{\infty}(1+Cq^k) < +\infty$, taking the product for k running from 1 to r yields the result. □

Proof of Lemma A.1
It suffices to prove (A.1). We introduce the notation

$$C(a,b,r) = \frac{[r+1]\left[\begin{smallmatrix} a+r \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} r+b \\ r \end{smallmatrix}\right]}{\left[\begin{smallmatrix} a+b+r+1 \\ r \end{smallmatrix}\right]}. \tag{A.10}$$

We identify

$$\begin{aligned}
&\big(V((a+s)\otimes(s+b),a+b)^*\otimes 1\big)(1_{a+s}\otimes p_{s+b+c}^{(s+b)\otimes c}) \\
&\quad\times\big(V((a+s)\otimes(s+b),a+b)\otimes 1\big) \\
&= C(a,b,s)(p_{a+b}\otimes 1_c)(1_a\otimes p_{b+c})(1_a\otimes t_s^*\otimes 1_{b+c})(p_{a+s}\otimes p_{s+b+c}) \\
&\quad\times(1_a\otimes t_s\otimes 1_{b+c})(1_a\otimes p_{b+c})(p_{a+b}\otimes 1_c).
\end{aligned}\tag{A.11}$$

But

$$C(a,b,s)(1_a\otimes t_s^*\otimes 1_{b+c})(p_{a+s}\otimes p_{s+b+c})(1_a\otimes t_s\otimes 1_{b+c}) = \sum_{z\in a\otimes(b+c)} \lambda_z p_z^{a\otimes(b+c)}.$$

From (7.3), we get $\lambda_{a+b+c} = C(a,b,s)/C(a,b+c,s)$. Since $C(a,b,s)$ is uniformly bounded from above, we get a constant D depending only on q such that $\lambda_z \leq D$ for all z. Using Lemma A.4, we find a constant E such that for all $z < a+b+c$,

$$\| p_z^{a\otimes b+c}\,(1_a\otimes p_{b+c})(p_{a+b}\otimes 1_c)\| \leq Eq^b.$$

As in (A.9), we find

$$1-q^{2(b+k)} \leq \frac{[b+k][a+b+c+1+k]}{[a+b+1+k][b+c+k]} \leq 1.$$

Taking the product for k running from 1 to s, we find a constant G such that

$$1-Gq^{2b} \leq \frac{C(a,b,s)}{C(a,b+c,s)} \leq 1.$$

Combining all these estimates with (A.11), we have shown the existence of a constant C depending only on q such that

$$\begin{aligned}
&\Big\|\big(V((a+s)\otimes(s+b),a+b)^*\otimes 1\big)(1_{a+s}\otimes p_{s+b+c}^{(s+b)\otimes c}) \\
&\quad\times\big(V((a+s)\otimes(s+b),a+b)\otimes 1\big) - p_{a+b+c}\Big\| \leq (Cq^b)^2.
\end{aligned}$$

It follows that

$$\begin{aligned}&\big\|(1_{a+s}\otimes p^{(s+b)\otimes c}_{s+b+c})\big(V((a+s)\otimes(s+b),a+b)\otimes 1\big)\\&\quad-\big(V((a+s)\otimes(s+b),a+b)\otimes 1\big)p_{a+b+c}\big\|\leq C\sqrt{2}q^b.\end{aligned}$$

This is the formula that we had to prove. □

LEMMA A.6
There exists a constant $D>0$ depending only on q such that

$$D\|\xi\|\leq\|(p_{a+s}\otimes p_{s+b})(1_a\otimes t_s\otimes 1_b)\xi\|\leq\|\xi\|$$

for all a,b,s and $\xi\in H_a\otimes H_b$.

Proof
Use the notation (A.10). Since

$$C(a,b,r)=\frac{\begin{bmatrix}a+b+r\\r\end{bmatrix}\begin{bmatrix}1+r\\r\end{bmatrix}}{\begin{bmatrix}a+b+r+1\\r\end{bmatrix}}\frac{\begin{bmatrix}a+r\\r\end{bmatrix}\begin{bmatrix}r+b\\r\end{bmatrix}}{\begin{bmatrix}a+b+r\\r\end{bmatrix}},$$

it follows from Lemma A.5 that there exist constants C_1 and C_2 such that $C_1\leq C(a,b,r)\leq C_2$. Observe

$$\begin{aligned}&(p_{a+x+s}\otimes p_{s+x+b})(1_{a+x}\otimes t_s\otimes 1_{x+b})V\big((a+x)\otimes(x+b),a+b\big)\\&\quad=C(a,b,x)^{1/2}(p_{a+x+s}\otimes p_{s+x+b})(1_{a+x}\otimes t_s\otimes 1_{x+b})(1_a\otimes t_x\otimes 1_b)p_{a+b}\\&\quad=D(x,s)^{-1/2}C(a,b,x)^{1/2}C(a,b,x+s)^{-1/2}V\big((a+x+s)\otimes(s+x+b),a+b\big),\end{aligned}$$

where $D(x,s)=[x+1][s+1][x+s+1]^{-1}$. Also, since $D(x,s)$ lies between two constants, the lemma is proved. □

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