ORTHOGONAL FREE QUANTUM GROUP FACTORS ARE STRONGLY 1-BOUNDED

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Abstract. We prove that the orthogonal free quantum group factors $\mathcal{L}(\mathbb{F}O_N)$ are strongly 1-bounded in the sense of Jung. In particular, they are not isomorphic to free group factors. This result is obtained by establishing a spectral regularity result for the edge reversing operator on the quantum Cayley tree associated to $\mathbb{F}O_N$, and combining this result with a recent free entropy dimension rank theorem of Jung and Shlyakhtenko.

1. Introduction

The theory of discrete quantum groups provides a rich source of interesting examples of C*-algebras and von Neumann algebras. In addition to ordinary discrete groups, there is a wealth of examples and phenomena arising from genuinely quantum groups [15, 42, 7, 29, 25, 1]. Within the class of non-amenable discrete quantum groups, the so-called *free quantum groups* of Wang and Van Daele [34, 41] somehow form the most prominent examples.

In this paper, our main focus is on the structural theory of a family of II_1-factors associated to a special family of free quantum groups, called the *orthogonal free quantum groups.* Given an integer $N \geq 2$, the orthogonal free quantum group $\mathbb{F}O_N$ is the discrete quantum group defined via the full Woronowicz C*-algebra

$$C^*_{\mathrm{f}}(\mathbb{F}O_N) = \langle u_{ij}, 1 \leq i, j \leq N \mid u = [u_{ij}] \text{ unitary}, \ u_{ij} = u_{ij}^* \ \forall i, j\rangle.$$

The C*-algebra $C^*_{\mathrm{f}}(\mathbb{F}O_N)$ can be interpreted simultaneously as a free analogue of the C*-algebra of continuous functions on the real orthogonal group O_N, and also as a "matricial" analogue of the full free group C*-algebra $C^*_{\mathrm{f}}((\mathbb{Z}/2\mathbb{Z})^{\star N})$. Indeed, by quotienting by the commutator ideal or by setting $u_{ij} = 0$ $(i \neq j)$, respectively, we obtain surjective Woronowicz-C*-morphisms

$$C^*_{\mathrm{f}}(\mathbb{F}O_N) \to C(O_N), \qquad C^*_{\mathrm{f}}(\mathbb{F}O_N) \to C^*_{\mathrm{f}}((\mathbb{Z}/2\mathbb{Z})^{\star N}).$$

Using the (tracial) Haar state $h : C^*_{\mathrm{f}}(\mathbb{F}O_N) \to \mathbb{C}$, the GNS construction yields in the usual way a Hilbert space $\ell^2(\mathbb{F}O_N)$ and a corresponding von Neumann algebra $\mathcal{L}(\mathbb{F}O_N) = \pi_h(C^*_{\mathrm{f}}(\mathbb{F}O_N))'' \subseteq B(\ell^2(\mathbb{F}O_N))$, where π_h denotes the GNS representation. Over the past two decades, the structure of the algebras $\mathcal{L}(\mathbb{F}O_N)$ has been investigated by many hands, and in many respects $\mathbb{F}O_N$ and $\mathcal{L}(\mathbb{F}O_N)$ $(N \geq 3)$ were shown to share many properties with free groups F_n and their von Neumann algebras $\mathcal{L}(F_n)$.

For example, $\mathcal{L}(\mathbb{F}O_N)$ is a full type II_1-factor, it is strongly solid, and in particular prime and has no Cartan subalgebra; it has the Haagerup property (HAP), is weakly amenable with Cowling-Haagerup constant 1 (CMAP), and satisfies the Connes' Embedding conjecture [3, 32, 19, 9, 17, 11, 16]. Moreover, it is known that $\mathcal{L}(\mathbb{F}O_N)$ behaves asymptotically like a free group factor in the sense that the canonical generators of $\mathcal{L}(\mathbb{F}O_N)$ become strongly asymptotically free semicircular systems as $N \to \infty$ [5, 10].

With these many similarities between $\mathcal{L}(\mathbb{F}O_N)$ and $\mathcal{L}(F_n)$ at hand, the following question naturally arises:

Can $\mathcal{L}(\mathbb{F}O_N)$ be isomorphic to a free group factor?

This particular question has been circulating within the operator algebra and quantum group communities ever since the publication of Banica's thesis [3, 4] in the mid 1990's, which first connected the corepresentation theory of free quantum groups to Voiculescu's free probability theory. This deep connection with free independence established by Banica was a direct inspiration for the many structural results for $\mathcal{L}(\mathbb{F}O_N)$ described in the previous paragraph. In this paper, our main objective is to finally answer the above question in the negative.

The first evidence suggesting a negative answer to an isomorphism with a free group factor came from the work of the second author [36], where the L^2-cohomology of $\mathbb{F}O_N$ was investigated. There it was shown that the first L^2-Betti number of $\mathbb{F}O_N$ vanishes for all $N \geq 3$, see also [24]. Combining this result with some deep work of Connes-Shlyakhtenko [13], Jung [20], and Biane-Capitaine-Guionnet [8] on free entropy dimension, it was shown by Collins and the authors [11] that

$$\delta_0(u) = \delta^*(u) = 1 \qquad (N \geq 4)^1, \tag{1}$$

where $u = (u_{ij})_{1 \leq i,j \leq N}$ is the set of canonical self-adjoint generators of $\mathcal{L}(\mathbb{F}O_N)$, and δ_0, δ^* are Voiculescu's (modified) microstates free entropy dimension and non-microstates free entropy dimension, respectively [40, 38, 39].

Recall that if X is a finite set of self-adjoint generators of a finite von Neumann algebra M with faithful normal tracial state τ, $\delta_0(X)$ can be interpreted as an asymptotic Minkowski dimension of the space of microstates of X. The fundamental problem relating to δ_0 is whether or not it is a W*-invariant: If $X, X' \subset M_{sa}$ are finite sets generating the same von Neumann subalgebra, do we have $\delta_0(X) = \delta_0(X')$? If the answer to this question is yes, then this would solve the well-known free group factor isomorphism problem since $\mathcal{L}(F_n)$ admits a finite generating set X with $\delta_0(X) = n$ [40].

In the remarkable work [21], Jung introduced a certain technical strengthening of the condition $\delta_0(X) \leq \alpha$ (see Section 2.3 for details), which he called *α-boundedness* of X. There, Jung proved the remarkable result that if (M, τ) is a finite von Neumann algebra generated by a 1-bounded set $X \subset M_{sa}$ containing at least one element with finite free entropy, then every other self-adjoint generating set X' of M has $\delta_0(X') \leq 1$. In this case, we call M a *strongly 1-bounded von Neumann algebra*, and δ_0 becomes a W^*-invariant for M. Note, in particular, that any strongly 1-bounded von Neumann algebra cannot be isomorphic to any (interpolated) free group factor $\mathcal{L}(F_r)$ $(r \geq 2)$ [21, Corollary 3.6].

The main result of this paper is an upgrade of the free entropy dimension estimate (1) to the following theorem:

Theorem (See Theorem 4.3 and Corollary 4.4)**.** For each $N \geq 3$, $\mathcal{L}(\mathbb{F}O_N)$ is a strongly 1-bounded von Neumann algebra. In particular, $\mathcal{L}(\mathbb{F}O_N)$ is never isomorphic to an interpolated free group factor.

Note that II$_1$-factors which have property Gamma, or have a Cartan subalgebra, or are tensor products of infinite dimensional factors, are automatically strongly 1-bounded by [21]. This is not the case of $\mathcal{L}(\mathbb{F}O_N)$. Instead, our proof of strong 1-boundedness relies on and is heavily inspired by recent works of Jung [22] and Shlyakhtenko [30].

If F is an l-tuple of non-commutative polynomials over m variables, one can compute Voiculescu's free derivative ∂F which yields by evaluation an operator $\partial F(X) \in M \otimes M^{op} \otimes B(\mathbb{C}^m, \mathbb{C}^l)$. In [22], Jung showed that if (M, τ) is a finite von Neumann algebra, $X \in M_{sa}^m$ is an m-tuple satisfying the polynomial relations $F(X) = 0$, then X is α-bounded with $\alpha = m - \operatorname{rank}(\partial F(X))$, *provided* that $\partial F(X)^* \partial F(X)$ has a non-zero modified Lück-Fuglede-Kadison determinant. See Section 2 for any undefined notation and terms here.

In [30], Shlyakhtenko gave another proof of Jung's result above using non-microstates free entropy techniques, and moreover used this result to show that whenever Γ is an infinite, finitely generated and finitely presented sofic group with vanishing first L^2-Betti number, then $\mathcal{L}(\Gamma)$ is strongly 1-bounded. The key idea here being that there always exists a canonical system of generators $X \in \mathbb{Q}[\Gamma]_{sa}^m \subset \mathcal{L}(\Gamma)_{sa}^m$ and rational-polynomial relations $F(X) = 0$, where

(1) $m - \operatorname{rank}(\partial F(X)) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$.

(2) $\partial F(X)^* \partial F(X)$ has a non-zero modified Lück-Fuglede-Kadison determinant.

[1]These values are also conjectured to hold for $N = 3$ but in that case Connes embeddability of $\mathcal{L}(\mathbb{F}O_N)$ is open and therefore only the inequality $-\infty \leq \delta_0(u) \leq \delta^*(u) \leq 1$ is known.

Note that the first condition above holds for any finitely generated finitely presented group, whereas the second, typically very difficult to check condition comes for "free" for sofic groups – thanks to Elek and Szabó's solution to Lück's determinant conjecture for sofic groups [14].

Returning to the quantum groups $\mathbb{F}O_N$, it is very natural to view these objects as quantum analogues of finitely generated, finitely presented sofic groups with vanishing first L^2-Betti number. Indeed, $\mathbb{F}O_N$ is hyperlinear in the sense of [11], and even residually finite in the sense that the underlying Hopf $*$-algebra $\mathbb{C}[\mathbb{F}O_N]$ is residually finite-dimensional [12]. However discrete quantum groups are much more linear in nature than ordinary discrete groups and it is not clear whether there is a quantum analogue of soficity that would allow one to prove Lück's determinant conjecture for discrete quantum group rings.

Our strategy in this paper for proving our strong 1-boundedness theorem, which now can be seen as a quantum analogue of Shlyakhtenko's sofic group result, is to first take the canonical system of generators $X = u = (u_{ij})_{1\leq i,j\leq N}$ and form the natural vector of quadratic relations $F(X) = 0$ associated to the defining orthogonality relations of $\mathbb{F}O_N$. We then proceed to show conditions (1) and (2) from above for this choice of F and X. Establishing (1) turns out to be a relatively straightforward adaptation of the results in the group case (see Lemma 4.1).

On the other hand, establishing (2) directly turns out to be much more involved and constitutes the main technical component of the paper. Without the analogue of Elek-Szabó's results in this setting, we must check the determinant condition for $D = \partial F(X)^*\partial F(X)$ explicitly. This amounts to proving the integrability of the function $\log_+ : [0,\infty) \to \mathbb{R}$ with respect to the spectral measure of D, where $\log_+(t) = \log(t)$ if $t > 0$ and $\log_+(0) = 0$.

This integrability condition is established by proving an identification of D, up to amplification and unitary equivalence, with the operator $2(1 + \mathrm{Re}(\Theta))$, where $\Theta \in B(K)$ is the so-called *edge-reversing operator* of the quantum Cayley tree associated to the quantum group $\mathbb{F}O_N$. Here, K denotes the *edge Hilbert space* associated to the quantum Cayley tree. Quantum Cayley graphs where introduced by the second author in [35] and studied further in [36], where they were a key ingredient to prove the vanishing of the first L^2-Betti number of $\mathbb{F}O_N$. More specifically, a large part of [35, 36] was devoted to the study of the eigenspaces $K_g^{\pm} = \mathrm{Ker}(\Theta \pm \mathrm{id})$.

In the quantum case, Θ is not involutive and the understanding of its behavior on the orthogonal complement of $K_g^+ \oplus K_g^-$ is essential for the study of the integrability condition of D. In the present article, we unveil a shift structure for the action of $\mathrm{Re}(\Theta)$ on the orthogonal complement of $K_g^+ \oplus K_g^-$, reducing the initial problem to an integrability question for real parts of weighted shifts.

Finally, let us conclude this introduction with the following natural question: Although we now know that $\mathcal{L}(\mathbb{F}O_N)$ is not isomorphic to a free group factor, could it still be possible that $\mathcal{L}(\mathbb{F}O_N)$ is isomorphic to $\mathcal{L}(\Gamma)$ for some other classical discrete group Γ? In particular, what about Γ being an ICC lattice in $SL(2,\mathbb{C})$? For such Γ, it is known that $\mathcal{L}(\Gamma)$ is a full, strongly solid, strongly 1-bounded II_1-factor which has the HAP and the CMAP. Note also that [18] provides other examples of groups Γ such that $\mathcal{L}(\Gamma)$ satisfies the same properties.

The remainder of the paper is organized as follows. In Section 2 we introduce some basic notation and preliminaries about discrete quantum groups and free entropy dimension. In Section 3 we proceed to the spectral analysis of the reversing operator, reducing the determinant class question to the case of weighted shifts. In Section 4 we study the relations in $\mathbb{F}O_N$ from the point of view of free entropy dimension and we prove the main 1-boundedness result. Finally the Appendix summarizes some background results from [35, 36] on quantum Cayley graphs used in Section 3.

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2. Notation and Preliminaries

Scalar products are linear on the right. We denote by $\otimes$ the tensor product of Hilbert spaces and the minimal tensor product of C^*-algebras. We use the leg numbering notation for elements of multiple tensor products. The flip operator on Hilbert spaces is denoted $\Sigma : H \otimes K \to K \otimes H$. For example, if H, K, L are Hilbert spaces, $T \in B(H \otimes K)$, $S \in B(K)$, then $T_{12} \in B(H \otimes K \otimes L)$, $S_2 \in B(H \otimes K \otimes L)$, $T_{32} \in B(L \otimes K \otimes H)$ are given by $T \otimes \mathrm{id}$, $\mathrm{id} \otimes S \otimes \mathrm{id}$, and $(\mathrm{id} \otimes \Sigma)(\mathrm{id} \otimes T)(\mathrm{id} \otimes \Sigma)$, respectively.

Let us denote $\log_+(t) = \log(t)$ for $t > 0$ and $\log_+(0) = 0$. The function $\log_+$ can be applied to positive operators using Borel functional calculus. If M is a von Neumann algebra with finite faithful normal trace τ, Lück's modified Fuglede-Kadison determinant of $x \in M$ is $\Delta_\tau^+(x) = \exp(\tau(\log_+(|x|))) \in [0, \infty)$. We will say that $x \in (M, \tau)$ is of *determinant class* if $\Delta_\tau^+(x) > 0$, i.e. $\tau(\log_+(|x|)) > -\infty$. Here, the quantity $\tau(\log_+(|x|))$ is computed via the Lebesgue integral

$$\tau(\log_+(|x|)) = \int_{(0,\infty)} \log(\lambda) \mathrm{d}\mu(\lambda) = \lim_{\epsilon \to 0^+} \int_\epsilon^{\|x\|} \log(\lambda) \mathrm{d}\mu(\lambda) \in [-\infty, \infty[$$

where μ denotes the spectral distribution of $|x|$ induced by τ.

We denote $L^2(M, \tau)$ the GNS space, equipped with the natural left and right M-module structures $x\hat{y} = \widehat{xy}$ and $\hat{y}x = \widehat{yx} = Jx^*J\hat{y}$, where $\hat{x}$ denotes the image in $L^2(M, \tau)$ of $x \in M$. We denote M° the opposite von Neumann algebra, $L^2(M^\circ, \tau)$ the corresponding GNS space with left and right actions of M° denoted $\hat{y}x = \widehat{yx}$, $x\hat{y} = \widehat{xy}$, where we use the product of M.

2.1. **Discrete quantum groups.** We use the setting of *Woronowicz C^*-algebras* [43], i.e. unital C^*-algebras A equipped with a $*$-homomorphism $\Delta : A \to A \otimes A$ such that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and $\Delta(A)(1 \otimes A)$, $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$. Woronowicz proved the existence and uniqueness of a state $h \in A^*$ such that $(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = h(\cdot)1$, called the Haar state [42]. The Woronowicz C^*-algebra (A, Δ) is called *reduced* if the GNS representation π_h associated with h is faithful. Note that $(\pi_h \otimes \pi_h)\Delta$ factors through π_h and in this way $\pi_h(A)$ is naturally a reduced Woronowicz C^*-algebra.

If Γ is a discrete group, the full and reduced C^*-algebras $C_\mathrm{f}^*(\Gamma)$, $C_\mathrm{r}^*(\Gamma)$ are Woronowicz C^*-algebras with respect to the coproducts given by $\Delta(g) = g \otimes g$, where group elements $g \in \Gamma$ are identified with the corresponding unitary elements in $C_\mathrm{f}^*(\Gamma)$, $C_\mathrm{r}^*(\Gamma)$. In general we shall interpret Woronowicz C^*-algebras as *discrete quantum group* C^*-algebras and denote $(A, \Delta) = (C^*(\Gamma), \Delta)$, where Γ is the discrete quantum group associated with (A, Δ). There is always a reduced version $C_\mathrm{r}^*(\Gamma)$ of $C^*(\Gamma)$, as above, as well as a full version $C_\mathrm{f}^*(\Gamma)$. The von Neumann algebra of Γ is $\mathcal{L}(\Gamma) = C_\mathrm{r}^*(\Gamma)'' \subset B(\ell^2(\Gamma))$, where $\ell^2(\Gamma)$ is the GNS space of h.

The main class of examples for the present article are the orthogonal free quantum groups $\mathbb{F}O(Q)$ [41, 34, 3], where $Q \in GL_N(\mathbb{C})$ is a matrix such that $Q\bar{Q} \in \mathbb{C}I_N$. The corresponding full Woronowicz C*-algebras are defined by generators and relations:

$$C_\mathrm{f}^*(\mathbb{F}O(Q)) = \langle u_{ij}, 1 \leq i, j \leq n \mid u \text{ unitary}, Q\bar{u}Q^{-1} = u \rangle$$

where $\bar{u} = (u_{ij}^*)_{ij}$, with the coproduct given on generators by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. In the case $Q = I_N$ we denote $\mathbb{F}O(Q) = \mathbb{F}O_N$.

Remark 2.1. In the literature, another commonly used (dual) notation for the C*-algebra $C_\mathrm{f}^*(\mathbb{F}O_N)$ is $C^u(O_N^+)$, or sometimes $A_o(N)$. The notation $C^u(O_N^+)$ refers to the fact that this C*-algebra can be viewed as a free analogue of the C*-algebra of continuous functions on the real orthogonal group O_N. In terms of Pontryagin duality for quantum groups, $O_N^+ = \widehat{\mathbb{F}O_N}$ is the compact dual of the discrete quantum group $\mathbb{F}O_N$ and the Fourier transform [28] induces the identifications $C^u(O_N^+) = C_\mathrm{f}^*(\mathbb{F}O_N)$ and $L^\infty(O_N^+) = \pi_h(C^u(O_N^+))'' = \mathcal{L}(\mathbb{F}O_N)$. Since our perspective is to view our objects as quantum analogues of discrete groups, we stick to the notation $\mathbb{F}O_N$.

Denote by $\pi_h : C^*(\Gamma) \to B(\ell^2(\Gamma))$ the GNS representation associated with the Haar state, with canonical cyclic vector $\xi_0 \in \ell^2(\Gamma)$. The multiplicative unitary [2] of Γ is the unitary operator V acting on $\ell^2(\Gamma) \otimes \ell^2(\Gamma)$ and given by the formula $V(x\xi_0 \otimes y\xi_0) = \Delta(x)(1 \otimes y)(\xi_0 \otimes \xi_0)$ for x, $y \in C^*(\Gamma)$. It satisfies the so-called pentagonal equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$. The reduced algebra $C_\mathrm{r}^*(\Gamma) \subset B(\ell^2(\Gamma))$ can be recovered as the closed linear span of the slices $(\varphi \otimes \mathrm{id})(V)$, $\varphi \in B(\ell^2(\Gamma))_*$, with coproduct $\Delta(x) = V(x \otimes 1)V^*$. Another useful operator is the polar part of the antipode. This is the involutive unitary $U \in B(\ell^2(\Gamma))$ given by $U(x\xi_0) = R(x)\xi_0$ $(x \in C^*(\Gamma))$, where $R : C^*(\Gamma) \to C^*(\Gamma)^\circ$ is the unitary antipode.

The *dual algebra* $c_0(\Gamma)$ can be defined as the closed linear span, in $B(\ell^2(\Gamma))$, of the slices $(\mathrm{id} \otimes \varphi)(V)$, $\varphi \in B(\ell^2(\Gamma))_*$, and equipped with the coproduct $\Delta : c_0(\Gamma) \to M(c_0(\Gamma) \otimes c_0(\Gamma))$, $a \mapsto$

$V^*(1 \otimes a)V$ (following [2]). It is a (not necessarily unital) Hopf-C^*-algebra [33]. We have then $V \in M(c_0(\Gamma) \otimes C^*_{\mathrm{r}}(\Gamma))$. We denote $p_0 = (\mathrm{id} \otimes h)(V) \in c_0(\Gamma)$, which is also the orthogonal projection onto $\mathbb{C}\xi_0 \subset H$.

In the "classical case", when Γ is a real discrete group, on can check that $V = \sum_{g\in\Gamma} \delta_g \otimes \pi_h(g)$, where δ_g is the characteristic function of $\{g\}$ acting by pointwise multiplication on $\ell^2(\Gamma)$ and $\pi_h(g)$ is the operator of left translation by g. In particular $c_0(\Gamma)$ identifies with the C^*-algebra of functions on Γ vanishing at infinity, as the notation suggests.

2.2. **Quantum Cayley graphs.** Let $p_1 \in Z(M(c_0(\Gamma)))$ be a central projection such that $Up_1 = p_1U$ and $p_0p_1 = 0$. The *quantum Cayley graph* X [35] associated to (Γ, p_1) is given by

- the vertex and edge Hilbert spaces $\ell^2(X^{(0)}) = \ell^2(\Gamma)$ and $\ell^2(X^{(1)}) = \ell^2(\Gamma) \otimes p_1\ell^2(\Gamma)$,
- the vertex and edge C^*-algebras $c_0(X^{(0)}) = c_0(\Gamma)$ and $c_0(X^{(1)}) = c_0(\Gamma) \otimes p_1 c_0(\Gamma)$, naturally represented on the corresponding Hilbert spaces,
- the reversing operator $\Theta = \Sigma(1 \otimes U)V(U \otimes U)\Sigma \in B(\ell^2(X^{(1)}))$,
- the boundary operator $E = V \in B(\ell^2(X^{(1)}), \ell^2(X^{(0)}) \otimes \ell^2(X^{(0)}))$.

For brevity we denote $\ell^2(X^{(0)}) = \ell^2(\Gamma) = H$ and $\ell^2(X^{(1)}) = H \otimes p_1H = K$. The fact that p_1 commutes with U ensures that K is stable under Θ and Θ^*. Using the densely defined "augmentation form" $\epsilon : H \to \mathbb{C}$ induced by the co-unit of $C^*_{\mathrm{f}}(\Gamma)$, one can also consider source and target maps $E_1 = (\mathrm{id} \otimes \epsilon)E$, $E_2 : (\epsilon \otimes \mathrm{id})E : K \to H$. When p_1 has finite rank, these are in fact bounded operators.

In the classical case, p_1 is the characteristic function of a subset $S \subset \Gamma$ such that $S^{-1} = S$ and $e \notin S$. Denoting by $(e_g)_{g\in\Gamma}$ the canonical Hilbertian basis of $\ell^2(\Gamma)$, it is easy to compute $\Theta(e_g \otimes e_h) = e_{gh} \otimes e_{h^{-1}}$ and $E(e_g \otimes e_h) = e_g \otimes e_{gh}$. Hence the operators Θ, E encode the graph structure of the usual Cayley graph associated with (Γ, S), with edges given by "source, direction" pairs $(g, h) \in \Gamma \times S$. Note that in the quantum case, Θ is always unitary, but not necessarily involutive. More details about quantum Cayley graphs, especially in the case of trees, are given in the Appendix.

If Γ is a discrete group, the unitaries $v = g \in C^*_{\mathrm{f}}(\Gamma)$ or $C^*_{\mathrm{r}}(\Gamma)$ corresponding to group elements can be recovered as those unitaries v which are *group-like*, i.e. satisfy the relation $\Delta(v) = v \otimes v$. More generally, a *unitary corepresentation* of a Woronowicz C^*-algebra $C^*(\Gamma)$ on a Hilbert space H is a unitary element $v \in M(K(H) \otimes C^*(\Gamma)))$ such that $(\mathrm{id} \otimes \Delta)(v) = v_{12}v_{13} \in M(K(H) \otimes C^*(\Gamma) \otimes C^*(\Gamma))$. Here, $K(H)$ denotes the C*-algebra of compact operators on the Hilbert space H. Applying $\mathrm{id} \otimes \pi_h$ yields a bijection between corepresentations of $C^*(\Gamma)$ and $C^*_{\mathrm{r}}(\Gamma)$, hence one can speak of corepresentations of the discrete quantum group Γ.

We denote by $\mathrm{Corep}(\Gamma)$ the category of finite dimensional corepresentations of Γ. It is a rigid tensor C^*-category, with direct sum $v \oplus w$, and tensor product $v \otimes w = v_{13}w_{23}$. The space of $v \in \mathrm{Corep}(\Gamma)$ is denoted H_v and we put $\dim v = \dim H_v$. We write $v \subset w$ (resp. $v \simeq w$) if $\mathrm{Hom}(v, w)$ contains an injective (resp. bijective) map, and we choose a set $\mathrm{Irr}(\Gamma)$ of representatives of irreducible corepresentations up to equivalence. Any corepresentation dual to v will be denoted $\bar{v}$, and the quantum (or intrinsic) dimension of v is denoted $\mathrm{qdim}\, v$. See e.g. [26] for more details.

The structure of $c_0(\Gamma)$ can be described using the theory of corepresentations. More precisely, there is a canonical dense subspace of H that can be identified with $\bigoplus_{\alpha\in\mathrm{Irr}\,\Gamma} B(H_\alpha)$ in such a way that $c_0(\Gamma) \subset B(H)$ identifies with $c_0 - \bigoplus_{\alpha\in\mathrm{Irr}\,\Gamma} B(H_\alpha)$ acting on the dense subspace by left multiplication. Moreover this gives a decomposition of the multiplicative unitary V (which is also a unitary corepresentation): $V = \sum_{\alpha\in\mathrm{Irr}\,\Gamma} \alpha \in M(c_0(\Gamma) \otimes C^*_{\mathrm{r}}(\Gamma))$. We denote $p_\alpha \in c_0(\Gamma) \subset B(H)$ the minimal central projection corresponding to the block $B(H_\alpha)$, so that $H = \bigoplus p_\alpha H$ and $p_\alpha H \simeq B(H_\alpha)$. For the trivial corepresentation $\tau = \mathrm{id}_{\mathbb{C}} \otimes 1$ we have $p_\tau = p_0$.

2.3. **Free entropy dimension.** There are two main approaches to free entropy dimension, based respectively on microstates and conjugate variables. The tools that we are going to use in this article are more closely related to the second one, although the invariance of strong 1-boundedness under von Neumann algebra isomorphisms is proved by Jung in the first framework.

For a tuple of indeterminates $x = (x_1, \dots, x_m)$, we denote $\mathbb{C}\langle x\rangle$ the corresponding algebra of noncommutative polynomials. The free difference quotient ∂_i is the unique derivation $\partial_i : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle$ such that $\partial_i x_j = \delta_{ij}(1 \otimes 1)$, where $\mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle$ is equipped with the bimodule structure $P \cdot (R \otimes S) \cdot Q = PR \otimes SQ$. We denote $\partial P = \sum \partial_i P \otimes e_i^* \in \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle \otimes (\mathbb{C}^m)^*$ and, if $P = (P_1, \ldots, P_l) \in \mathbb{C}\langle x\rangle^l$, $\partial P = \sum \partial_i P_j \otimes e_j \otimes e_i^* \in \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle \otimes B(\mathbb{C}^m, \mathbb{C}^l)$.

Fix a tuple $X = (X_1, \ldots, X_m)$ of self-adjoint elements in a von Neumann algebra M with faithful finite normal trace τ, and denote $W^*(X) \subset M$ the von Neumann subalgebra generated by X. We say that $\xi_i \in L^2(M, \tau)$ is the (necessarily unique) conjugate variable of X_i if $\xi_i \in L^2(W^*(X), \tau)$ and $\langle \xi_i, P(X)\rangle = (\tau \otimes \tau)((\partial_i P)(X))$ for all $P \in \mathbb{C}\langle x\rangle$. The free Fisher information of X is $\Phi^*(X) = \sum_i \|\xi_i\|_2^2$ if all conjugate variables exist, and $+\infty$ otherwise.

Replacing M by a free product if necessary, one can assume that M contains a free family $S = (S_1, \ldots, S_m)$ of elements with $(0, 1)$-semicircular law with respect to τ, which is also freely independent from X. The non-microstates free entropy [39] is defined by

$$\chi^*(X) = \frac{1}{2} \int_0^{+\infty} \left(\frac{m}{1+t} - \Phi^*(X + \sqrt{t}S) \right) \mathrm{d}t + \frac{m}{2} \log(2\pi e) \in [-\infty, +\infty[\,,$$

and the non-microstates free entropy dimension is

$$\delta^*(X) = m - \liminf_{\epsilon \to 0} \frac{\chi^*(X + \sqrt{\epsilon}S)}{\log \sqrt{\epsilon}}.$$

The (modified) microstates free entropy dimension $\delta_0(X)$ is defined by the very same formula, using the relative microstates free entropy $\chi(X + \sqrt{\epsilon}S : S)$ instead of $\chi^*(X + \sqrt{\epsilon}S)$ [38].

One can observe that we have $\delta_0(X) \le \alpha$ **iff** $\chi(X + \sqrt{\epsilon}S : S) \le (\alpha - m)|\log \sqrt{\epsilon}| + o(\log \sqrt{\epsilon})$ as $\epsilon \to 0$. Following Jung [21], one says that X is *α-bounded* (for δ_0) if it satisfies the slightly stronger condition $\chi(X + \sqrt{\epsilon}S : S) \le (\alpha - m)|\log \sqrt{\epsilon}| + K$ for small $\epsilon > 0$ and some K independent of ϵ. Similarly, one can say that X is α-bounded for δ^* if $\chi^*(X + \sqrt{\epsilon}S) \le (\alpha - m)|\log \sqrt{\epsilon}| + K$.

Recall that it is a major open question in free probability theory to decide whether $\delta_0(X)$ is an invariant of $W^*(X)$. Indeed, $\mathcal{L}(F_m)$ admits a tuple of generators X such that $\delta_0(X) = m$ [40], and therefore the W*-isomorphism invariance of δ_0 would provide a solution to the celebrated free group factor isomorphism problem. Jung proved the following very strong result: if X is 1-bounded and $\chi(X_i) > -\infty$ for at least one i, then any other tuple X' of self-adjoint generators of $W^*(X)$ is 1-bounded [21]. In particular, in that case one cannot have $W^*(X) \simeq \mathcal{L}(F_m)$ for $m \ge 2$. Let us also record the following deep result comparing the two versions of free entropy: we always have $\chi(X) \le \chi^*(X)$ [8]. In particular $\chi(X + \sqrt{\epsilon}S : S) \le \chi^*(X + \sqrt{\epsilon}S)$ so that 1-boundedness for δ^* implies 1-boundedness for δ_0.

Our main tool in this article is the following result, originally proved by Jung in the microstates framework [22], and reproved by Shlyakhtenko using non-microstates free entropy [30]. As above, for any $P \in \mathbb{C}\langle x\rangle^l$ and $X \in M_{sa}^m$ one can consider $\partial P \in \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle \otimes B(\mathbb{C}^m, \mathbb{C}^l)$ and $\partial P(X) \in M \otimes M^\circ \otimes B(\mathbb{C}^m, \mathbb{C}^l)$ is a bounded operator from $L^2(M, \tau) \otimes L^2(M^\circ, \tau) \otimes \mathbb{C}^m$ to $L^2(M, \tau) \otimes L^2(M^\circ, \tau) \otimes \mathbb{C}^l$. The operator $\partial P(X)$ moreover respects the right $M \otimes M^\circ$-module structures given by $(\zeta \otimes \xi \otimes \eta) \cdot (x \otimes y) = \zeta x \otimes y\xi \otimes \eta$. We denote by $\mathrm{rank}(\partial P(X))$ the Murray-von Neumann dimension over $M \otimes M^\circ$ of the closure of $\mathrm{Im}(\partial P(X))$ in $L^2(M, \tau) \otimes L^2(M^\circ, \tau) \otimes \mathbb{C}^l$.

Theorem 2.2. ([21, Thm. 6.9] and [30, Thm. 2.5]) *Suppose that $X \in M_{sa}^m$ satisfies the identity $F(X) = 0$ for $F \in \mathbb{C}\langle x\rangle^l$. Assume moreover that $\partial F(X)$ is of determinant class. Then X is α-bounded for δ_0 and δ^*, with $\alpha = m - \mathrm{rank}(\partial F(X))$.*

3. Regularity of the reversing operator

In Section 4 we will prove that $\mathcal{L}(\mathbb{F}O_N)$ is strongly 1-bounded by applying Theorem 2.2 to the tuple X of canonical generators and a specific vector of relations F. It will turn out that the real part of the operator $\partial F(X)$ is closely related to the real part of the reversing operator Θ of the quantum Cayley graph of $\mathbb{F}O_N$ with its canonical generators. In this section we prove the crucial technical result that $1 + \mathrm{Re}\,\Theta$ is of determinant class — which is a regularity property for the spectral measure of $\mathrm{Re}\,\Theta$ at the edge of the spectrum. This result can be seen as further evidence that the quantum groups $\mathbb{F}O_N$ should be somehow regarded as quantum analogues of sofic or determinant class groups.

Note that all results in this section hold also in the non-Kac case, that is, for all discrete quantum groups $\mathbb{F}O(Q)$ with $Q \in GL_N(\mathbb{C})$, $N \geq 2$, $Q\bar{Q} \in \mathbb{C}I_N$, except the ones isomorphic to the duals of $SU_{\pm 1}(2)$ — which corresponds to the assumption $\mathrm{qdim}\, u > 2$.

Our study relies heavily on results about quantum Cayley graphs proved in [35, 36], which we recall in the Appendix. Note that the eigenspace $K_g^+ = \mathrm{Ker}(\Theta + \mathrm{id})$ — and, by symmetry $K_g^- = \mathrm{Ker}(\Theta - \mathrm{id})$ —, were the main subject of study in [35, 36]. These stable subspaces behave trivially with respect to the determinant class issue. Note also that in the classical case, they span the whole of the ambient edge Hilbert space K, but not in the case of $\mathbb{F}O_N$. Hence our main concern in the present article is the behavior of Θ on $K_g^{+\perp} \cap K_g^{-\perp}$.

Recall the definition A.9 of the reflection operator W, which is isometric and involutive. The study of K_g^+ in [35] shows that W restricts to the identity on K_g^+ and K_g^-. More precisely, the proof of [35, Theorem 5.3] shows that any vector $\xi \in K_g^+$ can be written $\xi = \zeta - (1+W)\eta + p_{--}\Theta(\eta - \zeta)$ with $\zeta \in K_{++}$ and $\eta \in K_{+-}$, and W restricts to the identity on K_{++} and K_{--} by definition.

Definition 3.1. We denote $K_s = \mathrm{Ker}(W-1)$, $K_a = \mathrm{Ker}(W+1)$ and $L = K_s \cap K_g^{+\perp} \cap K_g^{-\perp}$. We have then an orthogonal decomposition $K = K_g^+ \oplus K_g^- \oplus K_a \oplus L$.

The structure of K_a and the behavior of $\Theta + \Theta^*$ on K_a are quite simple and we describe them in the next Proposition. We use the notation for the left/right ascending/descending subspaces, e.g. $K_{+-} = p_{+-}K$, which is recalled in the Appendix.

Proposition 3.2. *We have $K_a \subset K_{+-} \oplus K_{-+}$ and the orthogonal projection onto K_{+-} restricts to an isomorphism $K_a \simeq K_{+-}$ (up to a constant $\sqrt{2}$). Moreover K_a is $(\Theta + \Theta^*)$-stable and in the isomorphism with K_{+-} the operator $\Theta + \Theta^*$ corresponds to $-(r + r^*)$, where $r = -p_{+-}\Theta p_{+-}$.*

Proof. Since by definition W restricts to the identity on K_{++} and K_{--} and switches K_{+-} and K_{-+} in an involutive and isometric way, the first two assertions are clear. The identity $W\Theta W = \Theta^*$ implies $[W, \Theta + \Theta^*] = 0$ hence K_a and K_s are $(\Theta + \Theta^*)$-stable. Due to this stability and the inclusion $K_a \subset K_{+-} \oplus K_{-+}$ we have $\Theta + \Theta^* = (p_{+-} + p_{-+})(\Theta + \Theta^*)(p_{+-} + p_{-+})$ on K_a. Since $p_{+-}\Theta p_{-+} = p_{-+}\Theta p_{+-} = 0$ by A.6 this yields

$$\Theta + \Theta^* = p_{+-}(\Theta + \Theta^*)p_{+-} + p_{-+}(\Theta + \Theta^*)p_{-+} \text{ on } K_a,$$

and the last assertion follows. □

Note that the operator r on K_{+-} was studied in [35], and it is an infinite direct sum of right shifts with explicit weights converging to 1. Note however that we will be interested in vector states corresponding to vectors in K_{++} whereas $K_a \perp K_{++}$, so that the behavior of $\Theta + \Theta^*$ on K_a is not relevant for our precise analytical issue.

Now we turn to the study of $\Theta + \Theta^*$ on L. It turns out that it also behaves like the real part of a shift, but the study is slightly more involved. Recall the shorthand notation $r = -p_{+-}\Theta p_{+-}$, $s = p_{+-}\Theta p_{++}$ and $s' = p_{+-}\Theta^* p_{--}$.

Proposition 3.3. *Consider the map $\Lambda = (1+W)(r - r^*) + 2(s^* - s'^*) : K_{+-} \to K$. Then Λ is injective, $\overline{\mathrm{Im}\,\Lambda} = L$ and $\Lambda^*\Lambda = 8 - 2(r + r^*)^2$.*

Proof. We note that $s^* = p_{++}\Theta^* p_{+-}$ is injective on K_{+-}: indeed the weights $s_{k,l}$ appearing in A.11 vanish only for $l = 0$, and $q_0 K_{+-} = \{0\}$. In particular $p_{++}\Lambda = 2s^*$ is injective, hence Λ is injective.

It is clear from the definitions that L and $\overline{\mathrm{Im}\,\Lambda}$ are subspaces of K_s. Hence we have $\overline{\mathrm{Im}\,\Lambda} = L$ **iff** $\mathrm{Ker}\,\Lambda^* \cap K_s = L^\perp \cap K_s = K_g^+ \oplus K_g^-$. But we have $K_g^+ \oplus K_g^- \subset K_s$ and $K_g^+ \oplus K_g^- = \mathrm{Ker}(\Theta - \mathrm{id}) \oplus \mathrm{Ker}(\Theta + \mathrm{id}) = \mathrm{Ker}(\Theta^2 - \mathrm{id}) = \mathrm{Ker}(\Theta - \Theta^*)$. Hence it suffices to prove that $\Lambda^*(\zeta) = 0$ $\Leftrightarrow$ $\Theta\zeta = \Theta^*\zeta$ for $\zeta \in K_s$. The second identity is equivalent to the four equations obtained by applying p_{++}, p_{+-}, p_{-+} and p_{--}.

Since $\zeta = W\zeta$, the equations $p_{++}\Theta\zeta = p_{++}\Theta^*\zeta$ and $p_{--}\Theta\zeta = p_{--}\Theta^*\zeta$ are trivial — indeed we have e.g. for the first one:

$$\begin{aligned} p_{++}\Theta\zeta &= p_{++}p_{\star+}\Theta\zeta = p_{++}\Theta p_{-+}\zeta + p_{++}\Theta p_{--}\zeta \quad \text{by Proposition A.6} \\ &= p_{++}\Theta p_{-+}W\zeta + p_{++}\Theta p_{--}\zeta \\ &= p_{++}\Theta^* p_{+-}\zeta + p_{++}\Theta^* p_{--}\zeta \quad \text{by Propositions A.9 and A.7} \\ &= p_{++}\Theta^* p_{\star-}\zeta = p_{++}\Theta^*\zeta \quad \text{by Proposition A.6.} \end{aligned}$$

Moreover the equations $p_{+-}\Theta\zeta = p_{+-}\Theta^*\zeta$ and $p_{-+}\Theta\zeta = p_{-+}\Theta^*\zeta$ are equivalent because $p_{+-}\Theta W\zeta = Wp_{-+}\Theta^*\zeta$ and $p_{+-}\Theta^* W\zeta = Wp_{-+}\Theta\zeta$. Finally the equation $p_{+-}\Theta\zeta = p_{+-}\Theta^*\zeta$ reads $p_{+-}\Theta p_{+-}\zeta + p_{+-}\Theta p_{++}\zeta = p_{+-}\Theta^* p_{+-}\zeta + p_{+-}\Theta^* p_{--}\zeta$, i.e. $-r\zeta + s\zeta = -r^*\zeta + s'\zeta$, which is equivalent to $\Lambda^*\zeta = 0$ since $\zeta = W\zeta$.

Finally we can compute, using Equations (A.5) and (A.6) which read respectively $ss^* + rr^* = p_{+-}$ and $s's'^* + r^*r = p_{+-}$:

$$\begin{aligned}\Lambda^*\Lambda &= \Lambda^* p_{++}\Lambda + \Lambda^* p_{+-}\Lambda + \Lambda^* p_{-+}\Lambda + \Lambda^* p_{--}\Lambda \\ &= 4ss^* + (r^* - r)(r - r^*) + (r^* - r)(r - r^*) + 4s's'^* \\ &= 2(r^* - r)(r - r^*) + 4(\mathrm{id} - rr^*) + 4(\mathrm{id} - r^*r) = 8 - 2(r + r^*)^2.\end{aligned}$$

□

Recall that r is a direct sum of right shifts with weights $c_{k,l} \in [0,1]$ converging to 1 as $k \to \infty$. In particular one sees that $\|r + r^*\| = 2$ so that $0 \in \mathrm{Sp}(\Lambda^*\Lambda)$ and the image of Λ is not closed. Denoting $\mathcal{K}$ the "canonical dense subspace of K", i.e. the algebraic direct sum of the subspaces $p_n K$, we clearly have $\Lambda(K_{+-} \cap \mathcal{K}) \subset \mathcal{K}$ hence $L \cap \mathcal{K}$ is a dense subspace of L.

Proposition 3.4. *There exists an isomorphism* $\Upsilon : K_{+-} \to L$ *and vectors* $e_i \in q_1 p_1 K_{+-}$ *such that* $\Upsilon^*(\Theta + \Theta^*)\Upsilon = -(r + r^*)$ *and* $(h \otimes \mathrm{Tr})(\Upsilon T \Upsilon^*) = \sum (f_i | T f_i)$*, where* $f_i = (8 - 2(r + r^*)^2)^{-1/2} e_i$.

Proof. We first show that $(\Theta + \Theta^*)\Lambda = -\Lambda(r + r^*)$. Since $W\Lambda = \Lambda$ we have $p_{++}(\Theta + \Theta^*)\Lambda = 2p_{++}\Theta\Lambda$ and we compute, using the identity (A.4):

$$\begin{aligned}p_{++}\Theta\Lambda &= p_{++}\Theta p_{-+}\Lambda + p_{++}\Theta p_{--}\Lambda = p_{++}\Theta p_{-+} W(r - r^*) - 2p_{++}\Theta p_{--} s'^* \\ &= s^*(r - r^*) - 2s^* r = -s^*(r + r^*) = -\tfrac{1}{2} p_{++}\Lambda(r + r^*).\end{aligned}$$

If we knew that p_{++} is injective on L, this would suffice to obtain the desired relation because we already know that $(\Theta + \Theta^*)(L) \subset L$. This is true but not completely obvious since $\mathrm{Im}\,\Lambda$ is only dense in L. So we check the other components. We have, using again (A.5) and (A.6):

$$\begin{aligned}p_{+-}\Theta\Lambda &= p_{+-}\Theta p_{+-}\Lambda + p_{+-}\Theta p_{++}\Lambda = -r(r - r^*) + 2ss^* \text{ and} \\ p_{+-}\Theta^*\Lambda &= p_{+-}\Theta^* p_{+-}\Lambda + p_{+-}\Theta^* p_{--}\Lambda = -r^*(r - r^*) - 2s's'^* \text{ hence} \\ p_{+-}(\Theta + \Theta^*)\Lambda &= (-r^2 - rr^* + r^*r + r^{*2}) = -(r - r^*)(r + r^*) = -p_{+-}\Lambda(r + r^*).\end{aligned}$$

Applying W to both sides we obtain $p_{-+}(\Theta + \Theta^*)\Lambda = -p_{-+}\Lambda(r + r^*)$. Finally we have using (A.3):

$$\begin{aligned}p_{--}(\Theta + \Theta^*)\Lambda &= 2p_{--}\Theta\Lambda = 2p_{--}\Theta p_{+-}\Lambda + 2p_{--}\Theta p_{++}\Lambda = 2s'^*(r - r^*) + 4p_{--}\Theta p_{++} s^* \\ &= 2s'^*(r - r^*) + 4s'^* r^* = 2s'^*(r + r^*) = -p_{--}\Lambda(r + r^*).\end{aligned}$$

Then we perform the polar decomposition of Λ as $\Lambda = \Upsilon|\Lambda|$, with $|\Lambda| = \sqrt{\Lambda^*\Lambda} \in B(K_{+-})$. Since Λ has dense image in L, $\Upsilon \in B(K_{+-}, L)$ is a surjective isometry. Since $(\Theta + \Theta^*)$, $(r + r^*)$ are self-adjoint, the identity $(\Theta + \Theta^*)\Lambda = -\Lambda(r + r^*)$ implies $(\Theta + \Theta^*)\Upsilon = -\Upsilon(r + r^*)$.

To compute $h \otimes \mathrm{Tr}$ we fix an ONB $(\zeta_i)_i$ of $p_1 H$, so that we have $(h \otimes \mathrm{Tr})(X) = \sum_i (\xi_0 \otimes \zeta_i \mid X(\xi_0 \otimes \zeta_i))$. Observe that $\xi_0 \otimes p_1 H = p_0 K = q_0 p_0 K \oplus q_1 p_0 K$, and we can assume that the one-dimensional subspace $q_0 p_0 K$ is spanned by $\xi_0 \otimes \zeta_1$. Since r, s, s', W commute with the projections q_l, it is also the case for Λ. In particular the property $q_0 K_{+-} = \{0\}$ implies $q_0 L = \{0\}$, hence $\xi_0 \otimes \zeta_1 \perp L$. On the other hand the vectors $\xi_0 \otimes \zeta_i$, $i \geq 2$, form a basis of $q_1 p_0 K$.

We have then $(h \otimes \mathrm{Tr})(\Upsilon T \Upsilon^*) = \sum_{i \geq 2} (\Upsilon^*(\xi_0 \otimes \zeta_i) \mid T\Upsilon^*(\xi_0 \otimes \zeta_i))$. We obtain the formula of the statement by putting $f_i = \Upsilon^*(\xi_0 \otimes \zeta_i)$. Note moreover that $|\Lambda|$ is injective and $|\Lambda| = (8 - 2(r + r^*)^2)^{1/2}$ by Proposition 3.3. Hence f_i has the required form if we define $e_i = |\Lambda|(f_i) = \Lambda^*(\xi_0 \otimes \zeta_i) = 2s^*(\xi_0 \otimes \zeta_i) \in p_1 K_{+-}$. Since $\xi_0 \otimes \zeta_i \in q_1 K$ we have $e_i \in q_1 p_1 K_{+-}$ as claimed. □

Theorem 3.5. *The element* $1 + \mathrm{Re}\,\Theta \in U\mathcal{L}(\mathbb{F}O_n)U \otimes B(p_1 H)$ *is of determinant class with respect to the functional* $(h \otimes \mathrm{Tr})$.

Proof. Denote p_g^+, p_g^-, p_a, p_L the orthogonal projections onto K_g^+, K_g^-, K_a and L respectively. Since they commute with $\Theta + \Theta^*$, we have to prove that $(h \otimes \mathrm{Tr})(\log_+(q(1 + \mathrm{Re}\,\Theta)))$ is finite for each projection $q = p_g^+, p_g^-, p_a, p_L$ separately. This is clear for p_g^+, p_g^- since $1 + \mathrm{Re}\,\Theta = 0$ and 2 on the corresponding subspaces. The term with p_a vanishes since $(h \otimes \mathrm{Tr})$ is a sum of vector states associated to vectors in $p_0 K = p_0 K_{++}$ which is orthogonal to $K_a \subset K_{+-} \oplus K_{-+}$.

Hence we are left with the term corresponding to $p_L = \Upsilon\Upsilon^*$, which according to Proposition 3.4 is equal to:

$$\begin{aligned}(h\otimes \mathrm{Tr})(\Upsilon\Upsilon^* \log_+(1+\mathrm{Re}\,\Theta)\Upsilon\Upsilon^*) &= (h\otimes \mathrm{Tr})(\Upsilon \log_+(1-\mathrm{Re}\,r)\Upsilon^*)\\ &= \textstyle\sum_i (f_i \mid \log_+(1-\mathrm{Re}\,r) f_i)\\ &= \tfrac{1}{8}\textstyle\sum_i (e_i \mid (1-(\mathrm{Re}\,r)^2)^{-1}\log_+(1-\mathrm{Re}\,r)e_i).\end{aligned}$$

We fix i and we put $\eta_0 = e_i/\|e_i\| \in q_1p_1K_{+-}$. According to A.11 the map r maps $q_1p_kK_{+-}$ isometrically to $q_1p_{k+1}K_{+-}$, up to the scalar $c_{k+1,1} \in]0,1]$, and we have $r^*(\eta_0) = 0$. If we define recursively $\eta_{k+1} = r\eta_k/\|r\eta_k\|$, this shows that we can identify the restriction of r to $\overline{C^*(r)\eta_0}$ with a weighted unilateral shift on $\ell^2(\mathbb{N}) \simeq \overline{\mathrm{Span}}\{\eta_k\}$. Observe moreover that η_0 lies in the range of $\sqrt{1-(\mathrm{Re}\,r)^2}$, since $e_i = 2\sqrt{2}\sqrt{1-(\mathrm{Re}\,r)^2} f_i$. The result now follows from the following Lemma. □

Lemma 3.6. *Let R be a weighted unilateral shift on $\ell^2(\mathbb{N})$ with weights $c_k \in]0,1]$ — in other words $R\delta_k = c_{k+1}\delta_{k+1}$ where $(\delta_k)_k$ is the canonical basis of $\ell^2(\mathbb{N})$. We assume that δ_0 is in the range of $\sqrt{1-(\mathrm{Re}\,R)^2}$ and we denote ω the vector state associated to δ_0. Then $\omega((1-(\mathrm{Re}\,R)^2)^{-1}\log_+(1-\mathrm{Re}\,R))$ is finite.*

Remark 3.7. Denote by μ the spectral measure of $\mathrm{Re}(R)$ with respect to ω, which is supported on $[-1,1]$. Then we have $\omega(f(\mathrm{Re}\,R)) = (\delta_0|f(\mathrm{Re}\,R)\delta_0) = \int_{-1}^1 f(t)\mathrm{d}\mu(t)$ for any $f \in L^\infty([-1,1])$, and if $f : [-1,1] \to \mathbb{R}$ is any Borel map we say that $\omega(f(\mathrm{Re}\,R))$ is finite if f is integrable with respect to μ. In the Lemma above we take $f(t) = (1-t^2)^{-1}\log_+(1-t)$ and the finiteness of $\omega(f(\mathrm{Re}\,R))$ is equivalent to the convergence, at 1 and -1, of the integral

$$\int_{-1}^1 \frac{\log_+(1-t)}{1-t^2}\mathrm{d}\mu(t). \tag{2}$$

Proof. This kind of result is perhaps well-known to experts in operator theory. However we provide an elementary proof for the convenience of the reader.

We proceed by comparison with the standard unilateral shift $R_0 : \delta_k \to \delta_{k+1}$. Recall that the moments $m_k(\mathrm{Re}(R_0)) = \omega((\mathrm{Re}\,R_0)^k)$ are given in terms of the Catalan numbers $C_k = \frac{1}{k+1}\binom{2k}{k}$ by $m_{2k+1} = 0$, $m_{2k} = 4^{-k}C_k$ [27, Corollary 2.14]. Recall also that the Catalan numbers are counting the number of Dyck paths $\pi \in D_k$ of length $2k$, as can be seen by expanding $(R_0 + R_0^*)^{2k}\delta_0$ and looking for the δ_0 component. See [27, Propositions 2.11 and 2.13]. In the case of a general R, we still have $m_{2k+1} = 0$ because R is odd with respect to the natural $\mathbb{Z}_2$-grading. Moreover, still by expanding $(R+R^*)^{2k}\delta_0$ one sees that the even moments $m_{2k}(\mathrm{Re}\,R)$ are given by a sum over Dyck paths, $m_{2k}(\mathrm{Re}\,R) = 4^{-k}\sum_{\pi\in D_k} c_\pi$, where the contributions c_π are products of weights c_k. In particular we have $c_\pi \in]0,1]$ and $0 \le m_{2k}(\mathrm{Re}\,R) \le 4^{-k}\#D_k = m_{2k}(\mathrm{Re}\,R_0)$.

As above, denote by μ, μ_0 the spectral measures of $\mathrm{Re}(R)$ and $\mathrm{Re}(R_0)$ with respect to ω, which are both supported on $[-1,1]$. Note that $f : t \mapsto 1/(1-t^2)$ is μ-integrable because δ_0 lies in the range of $\sqrt{1-(\mathrm{Re}\,R)^2}$: indeed, approximating f by $f_C : t \mapsto \min(f(t), C)$ and writing $\delta_0 = g(\mathrm{Re}\,R)\zeta$ with $\zeta \in \ell^2(\mathbb{N})$, $g : t \mapsto \sqrt{1-t^2}$, we have $|f_C(t)g(t)^2| \le 1$ hence $\int_{-1}^1 f_C(t)\mathrm{d}\mu(t) = (\zeta|(f_Cg^2)(\mathrm{Re}\,R)\zeta) \le \|\zeta\|^2$ for all C and $\int_{-1}^1 f(t)\mathrm{d}\mu(t) \le \|\zeta\|^2$ by monotone convergence.

In particular the integral (2) converges **iff** the corresponding integral over $[0,1]$ is finite. Adding the finite quantity $\int_0^1 \log_+(1+t)/(1-t^2)\mathrm{d}\mu(t)$ to this new integral, we conclude that the convergence of (2) is equivalent to

$$\int_0^1 \frac{\log_+(1-t^2)}{1-t^2}\mathrm{d}\mu(t) = \frac{1}{2}\int_{-1}^1 \frac{\log_+(1-t^2)}{1-t^2}\mathrm{d}\mu(t) > -\infty,$$

where in the right-hand integral we have switched back to integrating over $[-1,1]$ using the fact that μ is symmetric.

We then perform the power series expansion $\log_+(1-t^2)/(1-t^2) = \sum a_k t^{2k}$ on $]-1,1[$: the convergence of (2) is equivalent to the finiteness of

$$\int_{-1}^1 \frac{\log_+(1-t^2)}{1-t^2}\mathrm{d}\mu(t) = \int_{-1}^1 \sum_{k\in\mathbb{N}} a_k t^{2k}\mathrm{d}\mu(t).$$

Since it is readily seen that all coefficients a_k are non-positive and t^{2k} is non-negative, one can permute the sum and the integral and compare to R_0:

$$\int_{-1}^{1} \frac{\log_+(1-t^2)}{1-t^2} \mathrm{d}\mu(t) = \sum_{k\in\mathbb{N}} a_k m_{2k}(\operatorname{Re} R) \geq \sum_{k\in\mathbb{N}} a_k m_{2k}(\operatorname{Re} R_0) = \int_{-1}^{1} \frac{\log_+(1-t^2)}{1-t^2} \mathrm{d}\mu_0(t).$$

Now we can conclude because the spectral measure of R_0 with respect to ω is well-known: it is the semicircular law $\mathrm{d}\mu_0(t) = \frac{1}{\pi}\sqrt{1-t^2}\mathrm{d}t$ [27, Proposition 2.15]. Hence we are led to the following Bertrand integral, which is well-known to be finite:

$$\int_{-1}^{1} \frac{\log(1-t^2)}{\sqrt{1-t^2}} \mathrm{d}t > -\infty.$$

□

4. Free entropy and relations in $\mathbb{F}O_N$

In this section we will apply Jung and Shlyakhtenko's Theorem 2.2 to $M = \mathcal{L}(\mathbb{F}O_N) \subset B(H)$. We fix the tuple of standard generators $u = (u_{ij})_{ij}$, which we now consider as elements of the *reduced* C^*-algebra $C^*_{\mathrm{r}}(\mathbb{F}O_N)$. We consider the "canonical" vector of relations $F = (F_1, F_2) \in \mathbb{C}\langle x_{ij}\rangle \otimes (M_N(\mathbb{C}) \oplus M_N(\mathbb{C}))$ given by $F_1 = x^t x - 1$ and $F_2 = x x^t - 1$, with $x = (x_{kl})_{kl} \in \mathbb{C}\langle x_{ij}\rangle \otimes M_N(\mathbb{C})$. Note that we have $m = N^2$ and $l = 2N^2$ with the notation of Section 2.

Recall from Section 2 that $\operatorname{rank} \partial F(u)$ is the Murray-von Neumann dimension of $\overline{\operatorname{Im}}\,\partial F(u)$ in the right $M \otimes M^\circ$-module $H \otimes H^\circ \otimes (M_N(\mathbb{C}) \oplus M_N(\mathbb{C}))$. The following Lemma is a straightforward adaptation of [30, Lemma 3.1] and its proof, and relies heavily on the computation of the first L^2-Betti number of $\mathbb{F}O_N$ in [36].

Lemma 4.1 ([30, Lemma 3.1])**.** *We have* $\operatorname{rank} \partial F(u) = N^2 - 1$.

Proof. By definition, $\mathbb{C}[\mathbb{F}O_N]$ is the quotient of $\mathbb{C}\langle x_{ij}\rangle$ by the ideal generated by the polynomials F_{pkl}, $p = 1, 2$, k, $l = 1, \dots N$. Recall that $H \otimes H^\circ$ is equipped with the M, M-bimodule structure corresponding to the left action of M (resp. M°) on itself. We make it into a $\mathbb{C}\langle x_{ij}\rangle, \mathbb{C}\langle x_{ij}\rangle$-bimodule by evaluating polynomials at $x_{ij} = u_{ij}$, so that $P \cdot (\zeta \otimes \xi) \cdot Q = P(u)\zeta \otimes \xi Q(u)$.

A derivation $\delta : \mathbb{C}\langle x_{ij}\rangle \to H \otimes H^\circ$ factors through $\mathbb{C}[\mathbb{F}O_N]$ **iff** we have $\delta(F_{pkl}) = 0$ for all p, k, l — indeed by Leibniz' rule and the fact that $F_{pkl}(u) = 0$ this implies $\delta(P F_{pkl} Q) = 0$ for all P, $Q \in \mathbb{C}\langle x_{ij}\rangle$. Now derivations $\delta : \mathbb{C}\langle x_{ij}\rangle \to H \otimes H^\circ$ are in 1-1 correspondence with the tuples of values $\zeta_{ij} = \delta(x_{ij}) \in H \otimes H^\circ$, the derivation corresponding to $(\zeta_{ij})_{ij}$ being given by $\delta(P) = \partial P \# \zeta = \sum \partial_{ij} P \# \zeta_{ij}$. Then δ factors through $\mathbb{C}[\mathbb{F}O_N]$ **iff** $\partial F \# \zeta = (\partial F_{pkl} \# \zeta)_{pkl} = 0$. Here we use the notation $(R \otimes S) \# \xi = R \cdot \xi \cdot S$.

This shows that the space of derivations $\operatorname{Der}(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ)$ is isomorphic as a right $M \otimes M^\circ$-module to $\operatorname{Ker} \partial F(u) \subset (H \otimes H^\circ)^m$, where $m = N^2$. Taking von Neumann dimensions, we obtain

$$\operatorname{rank} \partial F(u) = N^2 - \dim_{M\otimes M^\circ} \operatorname{Der}(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ).$$

On the other hand there are general exact sequences for Hochschild cohomology:

$$0 \to H^0(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ) \to H \otimes H^\circ \to \operatorname{Inn}(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ) \to 0,$$

$$0 \to \operatorname{Inn}(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ) \to \operatorname{Der}(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ) \to H^1(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ) \to 0.$$

If one uses the definition $\beta_k^{(2)}(\mathbb{F}O_N) = \dim_{M\otimes M^\circ} H^k(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ)$ for the L^2-Betti numbers (see [23, 31]), the additivity of Lück-von Neumann dimension readily yields

$$\dim_{M\otimes M^\circ} \operatorname{Der}(\mathbb{C}[\mathbb{F}O_N], H \otimes H^\circ) = \beta_1^{(2)}(\mathbb{F}O_N) - \beta_0^{(2)}(\mathbb{F}O_N) + 1.$$

By [36, Corollary 5.3] this is equal to 1, which concludes the proof. □

Recall that $\partial F(u) = \partial(F_1, F_2)(u)$ is an operator in $B(H) \otimes B(H) \otimes B(M_N(\mathbb{C}), M_N(\mathbb{C}) \oplus M_N(\mathbb{C}))$. In the next Lemma we identify $M_N(\mathbb{C})$ with $p_1 H$ and we make the connection with the reversing operator studied in Section 3.

Lemma 4.2. *We have* $\partial(F_1, F_2)(u)^* \partial(F_1, F_2)(u) = 2\partial F_1(u)^* \partial F_1(u) \in B(H \otimes H \otimes M_N(\mathbb{C}))$ *and* $\partial F_1(u)^* \partial F_1(u)$ *is unitarily conjugated to* $2 + 2\operatorname{Re}(\Theta \otimes 1) \in B(H \otimes p_1 H \otimes H)$. *Moreover the state* $(h \otimes h \otimes \operatorname{Tr})$ *is transformed into* $(h \otimes \operatorname{Tr} \otimes h)(V_{23}^* \cdot V_{23})$ *under the same conjugation.*

Proof. We first compute the free derivatives. We have $F_{1kl} = \sum_p x_{pk}x_{pl} - \delta_{kl}$ hence $\partial_{ij}F_{1kl} = \delta_{kj}(1 \otimes x_{il}) + \delta_{lj}(x_{ik} \otimes 1)$. Using the matrix units E_{ij} as a basis of $M_N(\mathbb{C})$, this yields

$$\begin{aligned}\partial F_1(E_{ij}) &= \textstyle\sum_{kl}\delta_{jk}(1 \otimes x_{il} \otimes E_{kl}) + \sum_{kl}\delta_{jl}(x_{ik} \otimes 1 \otimes E_{kl}) \\ &= \textstyle\sum_l (1 \otimes x_{il} \otimes E_{jl}) + \sum_k (x_{ik} \otimes 1 \otimes E_{kj})\end{aligned}$$

so that $\partial F_1 = \sum_{il}(1 \otimes x_{il} \otimes T\lambda(E_{li})) + \sum_{ik}(x_{ik} \otimes 1 \otimes \lambda(E_{ki}))$ where $\lambda(M) \in B(M_N(\mathbb{C}))$ is the map of left multiplication by M, and $T \in B(M_N(\mathbb{C}))$ is the transpose map. Similarly for $F_2 = (\sum x_{kp}x_{lp} - \delta_{kl})_{kl}$ we have

$$\begin{aligned}\partial F_2(E_{ij}) &= \textstyle\sum_{kl}\delta_{ik}(1 \otimes x_{lj} \otimes E_{kl}) + \sum_{kl}\delta_{il}(x_{kj} \otimes 1 \otimes E_{kl}) \\ &= \textstyle\sum_l (1 \otimes x_{lj} \otimes E_{il}) + \sum_k (x_{kj} \otimes 1 \otimes E_{ki})\end{aligned}$$

and $\partial F_2 = \sum_l (1 \otimes x_{lj} \otimes T\lambda(E_{lj})T) + \sum_k (x_{kj} \otimes 1 \otimes \lambda(E_{kj})T)$.

Then we evaluate at $x_{ij} = u_{ij}$. Recall that $C^*_{\mathrm{r}}(\mathbb{F}O_N) \otimes C^*_{\mathrm{r}}(\mathbb{F}O_N)^\circ$ acts on $H \otimes H$ by $\mathrm{id} \otimes \rho$, where $\rho(x)$ is the map of right multiplication by x, which can also be written $\rho(x) = US(x)U$ in the Kac case. Here S is the antipode and we have in particular $S(u_{ij}) = u_{ji}$. We obtain in $B(H \otimes H \otimes M_N(\mathbb{C}))$:

$$\begin{aligned}\partial F_1(u) &= \textstyle\sum (1 \otimes U u_{li} U \otimes T\lambda(E_{li})) + \sum (u_{ik} \otimes 1 \otimes \lambda(E^*_{ik})) \\ &= (1 \otimes U \otimes T)(\mathrm{id} \otimes \lambda)(u_{32})(1 \otimes U \otimes 1) + (\mathrm{id} \otimes \lambda)(u^*_{31}),\end{aligned}$$

where $u \in M_N(C^*_{\mathrm{r}}(\mathbb{F}O_N)) \simeq M_N(\mathbb{C}) \otimes C^*_{\mathrm{r}}(\mathbb{F}O_N)$.

Now we identify $M_N(\mathbb{C}) = B(\mathbb{C}^N)$ with p_1H using the decomposition of the multiplicative unitary recalled in Section 2 — in particular we have then $(\mathrm{id} \otimes \lambda)(u_{21}) = u_{21} \in C^*_{\mathrm{r}}(\Gamma) \otimes p_1 c_0(\Gamma) \subset B(H \otimes p_1H)$. Moreover in this identification T corresponds with the restriction of U to p_1H. Hence we have finally $\partial F_1(u) = (1 \otimes U \otimes U)u_{32}(1 \otimes U \otimes 1) + u^*_{31}$, which is also the restriction of $(1 \otimes U \otimes U)V_{32}(1 \otimes U \otimes 1) + V^*_{31}$ to $H \otimes H \otimes p_1H$. This is a sum of two unitaries and we obtain in particular $\partial F_1(u)^* \partial F_1(u) = 2 + 2\,\mathrm{Re}\, W$ on $H \otimes H \otimes p_1 H$, where $W = V_{31}(1 \otimes U \otimes U)V_{32}(1 \otimes U \otimes 1)$.

Proceeding similarly with F_2 we obtain

$$\begin{aligned}\partial F_2(u) &= \textstyle\sum (1 \otimes U u_{jl} U \otimes T\lambda(E_{lj})T) + \sum (u_{kj} \otimes 1 \otimes \lambda(E_{kj})T) \\ &= (1 \otimes U \otimes T)(\mathrm{id} \otimes \lambda)(u^*_{32})(1 \otimes U \otimes T) + (\mathrm{id} \otimes \lambda)(u_{31})(1 \otimes 1 \otimes T) \\ &= (1 \otimes U \otimes U)V^*_{32}(1 \otimes U \otimes U) + V_{31}(1 \otimes 1 \otimes U) \text{ on } H \otimes H \otimes p_1H,\end{aligned}$$

and $\partial F_2(u)^* \partial F_2(u) = 2 + 2\,\mathrm{Re}(1 \otimes U \otimes U)V_{32}(1 \otimes U \otimes U)V_{31}(1 \otimes 1 \otimes U)$. We moreover observe that $(1 \otimes U \otimes U)V_{32}(1 \otimes U \otimes U) \in 1 \otimes B(H) \otimes Uc_0(\mathbb{F}O_N)U$ and $V_{31} \in B(H) \otimes 1 \otimes c_0(\mathbb{F}O_N)$. Since $[c_0(\Gamma), Uc_0(\Gamma)U] = 0$, we can permute both terms and we obtain $\partial F_2(u)^* \partial F_2(u) = 2 + 2\,\mathrm{Re}\, W = \partial F_1(u)^* \partial F_1(u)$.

Now we perform unitary conjugations to "simplify" W. We first use $U_2 \Sigma_{23}$ which yields the symmetric form $W \sim_u V_{21}U_2V_{23} \in B(H \otimes p_1H \otimes H)$. Conjugating further by U_1 we obtain $W \sim_u \tilde{V}_{12}U_2V_{23}$, where $\tilde{V} = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma$. Finally we conjugate by $V_{13}V_{23}$ and we use the formula $V_{13}V_{23}\tilde{V}_{12} = \tilde{V}_{12}V_{13}$ from [2, Proposition 6.1]:

$$W \sim_u V_{13}V_{23}\tilde{V}_{12}U_2V^*_{13} = \tilde{V}_{12}V_{13}U_2V^*_{13} = \tilde{V}_{12}U_2 = \Theta \otimes 1.$$

Notice at last that $h \otimes h \otimes \mathrm{Tr}$ is a sum of vector states associated to vectors of the form $\xi_0 \otimes \xi_0 \otimes \zeta$. We have $U\xi_0 = \xi_0$ and $V(\xi_0 \otimes 1) = \xi_0 \otimes 1$. Applying the various unitaries used to transform W we obtain

$$V_{13}V_{23}U_1\Sigma_{23}U_2(\xi_0 \otimes \xi_0 \otimes \zeta) = V_{13}V_{23}(\xi_0 \otimes \zeta \otimes \xi_0) = V_{23}(\xi_0 \otimes \zeta \otimes \xi_0)$$

and the last claim follows. ∎

Thanks to Theorem 3.5 we can finally prove our main theorem:

Theorem 4.3. *The von Neumann algebra $\mathcal{L}(\mathbb{F}O_N)$ is strongly 1-bounded for all $N \geq 3$.*

Proof. The 1-boundedness of the tuple of generators $u = (u_{ij})$ of $\mathcal{L}(\mathbb{F}O_N)$ is a straightforward consequence of Jung's and Shlyakhtenko's Theorem 2.2, applied to u and to the relations $F = (F_{pkl})$ introduced at the beginning of this section. Note that on the matrix algebra $B(p_1H) \otimes 1$, any positive functional, and in particular $(\mathrm{Tr} \otimes h)(V^* \cdot V)$, is dominated by a multiple of the standard trace $\mathrm{Tr} \otimes h$. Then $\partial F(u)$ is of determinant class with respect to $(h \otimes h \otimes \mathrm{Tr})$ by Lemma 4.2 and Theorem 3.5. Moreover $N^2 - \mathrm{rank}\, \partial F(u) = 1$ by Lemma 4.1.

Strong 1-boundedness of $\mathcal{L}(\mathbb{F}O_N)$ will now follow if at least one of the generators u_{ij} has finite free entropy. Recall that for a single self-adjoint variable $X = X_1$ in a finite von Neumann algebra (M, τ) with law μ, we have the formula [37, Proposition 4.5]

$$\chi(X) = \iint \log |s-t| \mathrm{d}\mu(s) \mathrm{d}\mu(t) + C.$$

In particular if μ has an essentially bounded density with respect to the Lebesgue measure (and is compactly supported), then $\chi(X)$ is evidently finite. This is indeed the case for all generators u_{ij} of $\mathcal{L}(\mathbb{F}O_N)$ — according to [6, Theorem 5.3] the density is even continuous. □

Corollary 4.4. *For $N \geq 3$ the von Neumann algebra $\mathcal{L}(\mathbb{F}O_N)$ is not isomorphic to any finite von Neumann algebra (with separable predual) which admits a tuple of self-adjoint generators X with $\delta_0(X) > 1$. In particular it is not isomorphic to any free group factor $\mathcal{L}(F_n)$.*

More generally, $\mathcal{L}(\mathbb{F}O_N)$ is not isomorphic to any interpolated free group factor $\mathcal{L}(F_r)$, nor to any group von Neumann algebra $\mathcal{L}(\Gamma)$ where $\Gamma = *_{i=1}^k \mathbb{Z}/n_i\mathbb{Z}$ is a free product of cyclic groups, for instance $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{*N}$. Indeed these von Neumann algebras admit tuples of self-adjoint generators with $\delta_0 = r$, $\delta_0 = k - \sum_{i=1}^k n_i^{-1}$ respectively and these values are strictly bigger than 1 in the non amenable cases. According to [21, Lemma 3.7], $\mathcal{L}(\mathbb{F}O_N)$ is not isomorphic either to any free product of R^ω-embeddable diffuse finite von Neumann algebras.

Appendix A. Computation rules in quantum Cayley trees

In this appendix we recall definitions and results from [35, 36] about quantum Cayley graphs for discrete quantum groups. We use the notation about discrete quantum groups recalled in Section 2.

Definition A.1 ([35, Definition 3.1])**.** Let Γ be a discrete quantum group, and fix a central projection $p_1 \in Z(M(c_0(\Gamma)))$ such that $Up_1 = p_1U$ and $p_0p_1 = 0$. The *quantum Cayley graph X* [35] associated to (Γ, p_1) is given by

- the vertex and edge Hilbert spaces $\ell^2(X^{(0)}) = \ell^2(\Gamma)$ and $\ell^2(X^{(1)}) = \ell^2(\Gamma) \otimes p_1\ell^2(\Gamma)$,
- the vertex and edge C^*-algebras $c_0(X^{(0)}) = c_0(\Gamma)$ and $c_0(X^{(1)}) = c_0(\Gamma) \otimes p_1 c_0(\Gamma)$, naturally represented on the corresponding Hilbert spaces,
- the antilinear involutions $J^{(0)} = J$ and $J^{(1)} = J \otimes J$ on $\ell^2(X^{(0)})$ and $\ell^2(X^{(1)})$,
- the boundary operator $E = V \in B(\ell^2(X^{(1)}), \ell^2(X^{(0)}) \otimes \ell^2(X^{(0)}))$.
- the reversing operator $\Theta = \Sigma(1 \otimes U)V(U \otimes U)\Sigma \in B(\ell^2(X^{(1)}))$,

We denote $\ell^2(X^{(0)}) = \ell^2(\Gamma) = H$ and $\ell^2(X^{(1)}) = H \otimes p_1 H = K$. We also consider the source and target operators $E_1 = (\mathrm{id} \otimes \epsilon)E$, $E_2 : (\epsilon \otimes \mathrm{id})E : K \to H$, which are *a priori* only densely defined.

Recall that H is the GNS space for the Haar state $h \in C^*(\Gamma)^*$, with canonical cyclic vector ξ_0. Denote $\mathbb{C}[\Gamma]$ the canonical dense Hopf $*$-subalgebra of $C^*(\Gamma)$. The following Proposition computes the structure maps of the quantum Cayley graph in terms of the Hopf algebra structure of $\mathbb{C}[\Gamma]$.

Proposition A.2 ([35, Lemma 3.5, Proposition 3.6])**.** *For any x, $y \in \mathbb{C}[\Gamma]$ we have*

- $\Theta(x\xi_0 \otimes y\xi_0) = (\mathrm{id} \otimes S)((x \otimes 1)\Delta(y))(\xi_0 \otimes \xi_0)$,
- $E_1(x\xi_0 \otimes y\xi_0) = \epsilon(y)x\xi_0$ *and* $E_2(x\xi_0 \otimes y\xi_0) = xy\xi_0$.

Moreover $E_1\Theta = E_2$ and $E_2\Theta = E_1$. If $p_1 \in c_0(\Gamma)$, then E_1 and E_2 are bounded.

A key feature of the quantum case is that the reversing operator need not be involutive — in fact if p_1 is "generating" Θ is involutive **iff** Γ is a genuine discrete group [35, Proposition 3.4]. Hence the study of the following eigenspaces becomes non trivial, important, and turns out to be useful for applications:

Definition A.3 ([35, Definition 3.1])**.** The space of antisymmetric (or geometric) edges is $K_g^+ = \mathrm{Ker}(\Theta + \mathrm{id})$. The space of symmetric edges is $K_g^- = \mathrm{Ker}(\Theta - \mathrm{id})$.

Recall the isomorphism $c_0(\Gamma) = c_0\text{-}\bigoplus_{\alpha \in \mathrm{Irr}(\Gamma)} p_\alpha c_0(\Gamma)$ with $p_\alpha c_0(\Gamma) \simeq B(H_\alpha)$. The classical Cayley graph introduced in the next Definition is an essential tool for the study of the quantum Cayley graph.

Definition A.4 ([35, Definition 3.1, Lemma 4.4])**.** Denote $\mathcal{D} \subset \mathrm{Irr}(\Gamma)$ the subset such that $p_1 = \sum_{\alpha \in \mathcal{D}} p_\alpha$. The *classical Cayley graph* G associated to (Γ, p_1) is given by

– the vertex set $G^{(0)} = \mathrm{Irr}(\Gamma)$,
– the edge set $G^{(1)} = \{(\alpha, \beta, \gamma, i)\ ;\ \alpha, \beta \in \mathrm{Irr}(\Gamma), \gamma \in \mathcal{D}, 1 \leq i \leq \dim \mathrm{Hom}(\beta, \alpha \otimes \gamma)\}$,
– the boundary map $e : G^{(1)} \to G^{(0)} \times G^{(0)}, (\alpha, \beta, \gamma, i) \mapsto (\alpha, \beta)$,
– the reversing map $\theta : G^{(1)} \to G^{(1)}, (\alpha, \beta, \gamma, i) \mapsto (\beta, \alpha, \bar{\gamma}, i)$.

The component γ of an edge is called its *direction.* If e is injective, and in particular if the classical Cayley graph G is a tree, $G^{(1)}$ can and will be identified with $\{(\alpha, \beta) \in \mathrm{Irr}(\Gamma)^2 \mid \exists \gamma \in \mathcal{D}\ \beta \subset \alpha \otimes \gamma\}$. The *origin* of the classical Cayley graph G is the trivial corepresentation τ and if G is connected we denote $|\alpha| = d(\tau, \alpha)$ the distance of a vertex α to the origin.

To any subset $A \subset G^{(0)}$ one can associate the projection $p = \sum_{\alpha \in A} p_\alpha \in M(c_0(X^{(0)}))$. One can proceed similarly with $G^{(1)}$ and $c_0(X^{(1)})$, but one can also associate to $B \subset G^{(0)} \times G^{(0)}$ the projection $p = \sum_{(\alpha,\beta) \in B} E^*(p_\alpha \otimes p_\beta)E \in M(c_0(X^{(1)}))$, whose image corresponds to the space of edges going from α to β. This motivates the following definition.

Definition A.5. Assume that the classical Cayley graph G associated to (Γ, p_1) is a tree. For any $n \in \mathbb{N}$ one puts $p_n = \sum_{|\alpha|=n} p_\alpha \in B(H)$ and $p_n = \sum_{|\alpha|=n} p_\alpha \otimes p_1 \in B(K)$. The left and right projections onto ascending edges are $p_{\star+} = \sum_n E^*(p_n \otimes p_{n+1})E \in B(K)$ and $p_{+\star} = J^{(1)} p_{\star+} J^{(1)} \in B(K)$. The projections onto descending edges are $p_{\star-} = \sum_n E^*(p_n \otimes p_{n-1})E$ and $p_{-\star} = J^{(1)} p_{\star-} J^{(1)}$. Finally one puts $p_{++} = p_{+\star} p_{\star+}$, $p_{+-} = p_{+\star} p_{\star-}$, $p_{-+} = p_{-\star} p_{\star+}$, $p_{--} = p_{-\star} p_{\star-}$ and $K_{++} = p_{++}K$, $K_{+-} = p_{+-}K$, $K_{-+} = p_{-+}K$, $K_{--} = p_{--}K$.

In the classical case one has $p_{+\star} = p_{\star+}$ so that $p_{+-} = p_{-+} = 0$. However in the quantum case p_{+-} and p_{-+} are in general non-zero, and this is related to the non involutivity of the reversing operator. The following proposition shows indeed that $p_{+-}\Theta p_{+-}$ acts as a right shift in the decomposition $K_{+-} = \bigoplus p_n K_{+-}$.

Proposition A.6 ([35, Proposition 4.3])**.** *Assume that the classical Cayley graph associated to* (Γ, p_1) *is a tree. Then we have*

– $p_{\star+} + p_{\star-} = \mathrm{id}$ *and* $p_{+\star} + p_{-\star} = \mathrm{id}$, $[p_{\star\pm}, p_{\pm\star}] = 0$, $[p_{\pm\star}, p_n] = [p_{\star\pm}, p_n] = 0$,
– $\Theta p_{+\star} p_n = p_{n+1} p_{\star-} \Theta$ *and* $\Theta p_{-\star} p_n = p_{n-1} p_{\star+} \Theta$, $p_{\star-} = \Theta p_{+\star} \Theta^*$ *and* $p_{-\star} = \Theta^* p_{\star+} \Theta$,
– $E_2 p_{+-} = E_2 p_{-+} = 0$ *and* $p_n E_2 = E_2 p_{n-1} p_{++} + E_2 p_{n+1} p_{--}$.

We denote $r = -p_{+-}\Theta p_{+-}$ *and* $s = p_{+-}\Theta p_{++}$.

In the rest of the Appendix we consider the case of a free product of orthogonal and unitary free quantum groups: $\Gamma = F = \mathbb{F}O(Q_1) * \cdots * \mathbb{F}O(Q_k) * \mathbb{F}U(R_1) * \cdots * \mathbb{F}U(R_l)$, endowed with the projection $p_1 = \sum_{\alpha \in \mathcal{D}} p_\alpha$ associated to the set $\mathcal{D}$ of fundamental representations of the factors $\mathbb{F}O(Q_i)$, $\mathbb{F}U(R_j)$, together with their duals. This is a little bit stronger than requiring the classical Cayley graph G to be a tree, see [35, Proposition 4.5]. In that case we have the following useful facts. Note that the identities $(p_{++} + p_{--})\Theta^n(p_{++} + p_{--}) = (p_{++} + p_{--})\Theta^{-n}(p_{++} + p_{--})$ can be interpreted as a weak involutivity property.

Proposition A.7 ([35, Proposition 4.7, Proposition 5.1])**.** *Consider the quantum Cayley graph of* (F, p_1)*. Then the restriction* $E_2 : K_{++} \to H$ *is injective and we have* $(p_{++} + p_{--})\Theta^n(p_{++} + p_{--}) = (p_{++} + p_{--})\Theta^{-n}(p_{++} + p_{--})$ *for all* n*.*

Here is an example of computation using the rules above:

Lemma A.8. *In the quantum Cayley graph of* (F, p_1) *we have*

$$p_{--}\Theta p_{++}\Theta^* p_{+-} = -p_{--}\Theta p_{+-}\Theta^* p_{+-} \tag{A.3}$$

$$p_{++}\Theta p_{--}\Theta p_{+-} = -p_{++}\Theta^* p_{+-}\Theta p_{+-} \tag{A.4}$$

$$p_{+-}\Theta p_{++}\Theta^* p_{+-} = p_{+-} - p_{+-}\Theta p_{+-}\Theta^* p_{+-} \tag{A.5}$$

$$p_{+-}\Theta^* p_{--}\Theta p_{+-} = p_{+-} - p_{+-}\Theta^* p_{+-}\Theta p_{+-}. \tag{A.6}$$

Proof. For the first identity it suffices to use A.6 and write $p_{--}\Theta p_{++}\Theta^* p_{+-} + p_{--}\Theta p_{+-}\Theta^* p_{+-} = p_{--}\Theta p_{\star+}\Theta^* p_{+-} + p_{--}\Theta p_{\star-}\Theta^* p_{+-} = p_{--}\Theta\Theta^* p_{+-} = 0$. The second identity is proved similarly after replacing $p_{++}\Theta p_{--}$ with $p_{++}\Theta^* p_{--}$ on the left-hand side thanks to A.7. The third identity appears already in the proof of [35, Proposition 6.2]. For the last one we note that, according to A.6, $p_{+-}\Theta^* p_{--}\Theta p_{+-} = p_{+-}\Theta^* p_{-\star}\Theta p_{+-}$ and $p_{+-}\Theta^* p_{+-}\Theta p_{+-} = p_{+-}\Theta^* p_{+\star}\Theta p_{+-}$. Adding both operators we obtain $p_{+-}\Theta^*\Theta p_{+-} = p_{+-}$. □

The subspaces K_{+-} and K_{-+} are strongly connected to each other through the *reflection operator* W introduced as follows.

Proposition A.9 ([35, Lemma 5.2])**.** *Consider the quantum Cayley graph of (F, p_1). There exists a unique unitary operator $w : K_{+-} \to K_{-+}$ such that $w(p_{+-}\Theta)^n p_{++} = (p_{-+}\Theta^{-1})^n p_{++}$ for all $n \geq 1$. We denote $W = wp_{+-} + w^* p_{-+} + p_{++} + p_{--} \in B(K)$: this is an involutive unitary operator such that $W\Theta W = \Theta^*$, $Wp_{++} = p_{++}$, $Wp_{--} = p_{--}$ and $Wp_{+-} = p_{-+}W$.*

The families of projections $\{p_n\}$, $\{p_{++}, p_{+-}, p_{-+}, p_{--}\}$ form two commuting partitions of the unit in $B(K)$. To perform the most precise analysis of Θ one needs to further decompose the space K. Observe that there are 4 commuting representations of $c_0(\Gamma)$ on K, given by $\pi_4 : c_0(\Gamma)^{\otimes 4} \to B(K) = B(H \otimes p_1 H), x \otimes y \otimes y' \otimes x' \mapsto (x \otimes y)(Ux'U \otimes Uy'U)$. In the case of $\mathbb{F}O_N$, the subspaces $p_n K \simeq B(H_n \otimes H_1)$ are irreducible with respect to π_4, and the subspaces $p_{\pm\pm} p_n K \simeq B(H_{n\pm1}, H_{n\pm1})$ are irreducible with respect to the representation $\pi_4 \circ (\Delta \otimes \Delta)$ of $c_0(\Gamma) \otimes c_0(\Gamma)$. The reversing operator Θ does not commute to these representations, but it does commute to $\pi_4 \circ \Delta^3$ [35, Proposition 3.7] and we consider:

Definition A.10. Denote $q_l = \pi_4 \Delta^3(p_{2l}) \in B(K)$. We have $\sum q_l = \mathrm{id}$ and $[q_l, p_k] = [q_l, p_{\pm\pm}] = 0$. We have $p_{++} q_l p_k \neq 0$ **iff** $0 \leq l \leq k+1$, $p_{\pm\mp} q_l p_k \neq 0$ **iff** $1 \leq l \leq k$ and $p_{--} q_l p_k \neq 0$ **iff** $0 \leq l \leq k-1$.

The subspace $q_0 K$ is the "classical subspace" of the space of edges, see [35, Remarks 6.4] and [36, Reminder 4.3]. In particular $\Theta^2 = \mathrm{id}$ on $q_0 K$.

Proposition A.11 ([35, Lemma 6.3])**.** *Consider the quantum Cayley graph of $(\mathbb{F}O_N, p_1)$. For all choices of signs the operator $p_{\pm\pm}\Theta p_{\pm\pm}$ is a multiple of an isometry on each subspace $q_l p_{\pm\pm} p_n K$ and we have, for μ, ν, μ', $\nu' \in \{+, -\}$ and $p_{\mu'\nu'} p_k q_l \neq 0$:*

$$\|p_{\mu\nu}\Theta p_{\mu'\nu'} p_k q_l\| = \begin{cases} c_{k+1,l} & \text{if } \mu\nu = \mu'\nu' \text{ and } \mu' = + \\ s_{k+1,l} & \text{if } \mu\nu \neq \mu'\nu' \text{ and } \mu' = + \end{cases}$$

$$\|p_{\mu\nu}\Theta p_{\mu'\nu'} p_k q_l\| = \begin{cases} c_{k,l} & \text{if } \mu\nu = \mu'\nu' \text{ and } \mu' = - \\ s_{k,l} & \text{if } \mu\nu \neq \mu'\nu' \text{ and } \mu' = - \end{cases}$$

where $s_{k,l}^2 = \dfrac{\mathrm{qdim}(\alpha_l)\,\mathrm{qdim}(\alpha_{l-1})}{\mathrm{qdim}(\alpha_k)\,\mathrm{qdim}(\alpha_{k-1})}$ *and* $c_{k,l}^2 + s_{k,l}^2 = 1$*, with the convention that* $\mathrm{qdim}(\alpha_{-1}) = 0$.

Note that this statement corrects [35, Lemma 6.3] in the case when $\mu' = -$, which is not used in that article.

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